

Pacific Journal of Mathematics

AN APPROXIMATE GAUSS MEAN VALUE THEOREM

WATSON BRYAN FULKS

AN APPROXIMATE GAUSS MEAN VALUE THEOREM

W. FULKS

1. **Introduction.** The mean value theorem of Gauss, and its converse, due to Koebe, have long been known to characterize harmonic functions. Since any second order homogeneous elliptic operator L can, by an appropriate linear change of variables, be reduced (at a given point) to the Laplacian, it seems reasonable to expect that solutions of $Lu = 0$ should, when averaged over appropriate small ellipsoids, satisfy an approximate Gauss-type theorem, and one could hope that such a mean value property would characterize the solutions of the equation.

It turns out that this is the case. In fact the operator need not be elliptic, but may be parabolic, or of mixed elliptic and parabolic type. While the methods used here do not permit the weak smoothness conditions on the solutions admitted by Koebe's theorem, the result is stronger than might be expected in that no smoothness, not even measurability, is required of the coefficients of L : they need only be defined.

Since the result applies to parabolic equations, it seems of interest to examine the heat equation, for it can be cast in the required form. This leads to a characterization of its solutions in terms of averages over parabolic arcs.

2. **The basic theorem.** In the following $D_i = \partial/\partial y_i$, $D_{ij} = \partial^2/\partial y_i \partial y_j$, $u_{,ij} = D_{ij}u$, and ∇_y is the gradient operator with respect to the components of y .

It is convenient to consider equations of the form $Lu = f$, where f need only be defined, and may depend on u and any of its derivatives.

LEMMA. *Let $A = [a_{ij}]$ be an $n \times n$ constant nonnegative definite symmetric matrix, and denote by $B = [b_{ij}]$ the unique nonnegative definite symmetric square root of A . Let u be defined in a neighborhood of a point y in E_n , and be twice differentiable at y . For this y define the quadratic function q of x by*

$$q(x) \equiv (Bx \cdot \nabla_y)^2 u(y).$$

Then the sum of the coefficients of the squared terms of $q(x)$ is $\sum_{i,j} a_{ij} u_{,ij}(y)$.

Received July 24, 1963. This paper was prepared at Oregon State University under an NSF grant.

Proof. We have

$$q(x) = (Bx \cdot \nabla)^2 u = \left(\sum_{i,m} b_{im} x_m D_i \right) \left(\sum_{j,k} b_{jk} x_k D_j \right) u = \sum_{k,m} \left(\sum_{ij} b_{im} b_{jk} u_{,ij} \right) x_k x_m .$$

The sum of the coefficients of the squared terms is then

$$\sum_k \left(\sum_{ij} b_{ik} b_{jk} \right) u_{,ij} = \sum_{i,j} \left(\sum_k b_{ik} b_{kj} \right) u_{,ij} = \sum_{i,j} a_{ij} u_{,ij} .$$

THEOREM. Let $L = \sum_{i,j} a_{ij}(y) D_{ij}$ be a well defined symmetric differential operator with a nonnegative definite matrix $A(y) = [a_{ij}(y)]$ in an open region R in E_n . Let $B(y) = [b_{ij}(y)]$ be the unique nonnegative definite square root of A , and for $y \in R$ and r sufficiently small, define

$$(1) \quad u_r(y) \equiv \frac{1}{\Omega_r} \int_{|x|=r} u(y + B(y)x) d\Omega_r$$

where Ω_r is the area of the sphere $\{|x| = r\}$. Let u be a function defined in a neighborhood of a point $y_0 \in R$, which is twice differentiable at y_0 . Then for u to be a solution of $Lu = f$ at y_0 it is necessary and sufficient that

$$(2) \quad u_r(y_0) = u(y_0) + C_n r^2 f(y_0) + o(r^2) \quad \text{as } r \rightarrow 0 ,$$

where C_n is a certain constant depending only on n , in fact it is easily verified that

$$C_n = \frac{n-1}{2n} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)} .$$

Proof. Denote the constant matrices $A(y_0)$, $B(y_0)$ by A and B respectively. Since u is twice differentiable at y_0 we have

$$(3) \quad u(y_0 + B(y_0)x) = u(y_0 + Bx) = u(y_0) + (Bx \cdot \nabla_y) u(y) |_{y_0} \\ + \frac{1}{2} (Bx \cdot \nabla_y)^2 u(y) |_{y_0} + o(|Bx|^2) .$$

But $|Bx| \leq \|B\| |x|$. Thus on $\{|x| = r\}$, (3) becomes

$$(4) \quad u(y_0 + B(y_0)x) = u(y_0) + (Bx \cdot \nabla_y) u(y) |_{y_0} \\ + \frac{1}{2} (Bx \cdot \nabla_y)^2 u(y) |_{y_0} + o(r^2) .$$

Dividing (4) by Ω_r and integrating over $\{|x| = r\}$ we get

$$(5) \quad u_r(y_0) = u(y_0) + \frac{1}{2\Omega_r} \int_{|x|=r} (Bx \cdot \nabla_y)^2 u(y) |_{y_0} d\Omega_r + o(r^2) .$$

We next observe

$$\frac{1}{2\Omega_r} \int_{|x|=r} x_i x_j d\Omega_r = C_n r^2 \delta_{ij}$$

where C_n is a constant depending only on n . Thus (5) becomes, by the lemma,

$$(6) \quad u_r(y_0) = u(y_0) + C_n r^2 \sum_{i,j} a_{ij}(y_0) u_{,ij}(y_0) + o(r^2).$$

But (6) is compatible with (2) if and only if $Lu = f$ at y_0 .

3. The heat equation. As an application of the main result let us consider the heat operator $Hu = u_{xx} - u_t$. If we make the change of variables given by $x = \xi$, $t = \tau - (1/2)\xi^2$ and set $u(x, t) = v(\xi, \tau)$ then we see that our operator takes the form $v_{\xi\xi} + 2\xi v_{\xi\tau} + \xi^2 v_{\tau\tau}$. In this case the matrix A is given by

$$A = \begin{pmatrix} 1 & \xi \\ \xi & \xi^2 \end{pmatrix}.$$

To compute B we observe that $A^2 = (1 + \xi^2)A$, so that $B = A/\sqrt{1 + \xi^2}$. Then

$$(7) \quad B \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \frac{1}{\sqrt{1 + \xi^2}} \begin{pmatrix} r \cos \theta + \xi r \sin \theta \\ \xi r \cos \theta + \xi^2 r \sin \theta \end{pmatrix}.$$

For each ξ , there is an α satisfying $-(\pi/2) \leq \alpha \leq (\pi/2)$ for which

$$\frac{\cos \theta + \xi \sin \theta}{\sqrt{1 + \xi^2}} = \cos(\theta - \alpha),$$

so that (7) takes the form

$$(8) \quad B \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \begin{pmatrix} r \cos(\theta - \alpha) \\ r\xi \cos(\theta - \alpha) \end{pmatrix}.$$

Then $v_r(\xi_0, \tau_0)$ becomes

$$v_r(\xi_0, \tau_0) = \frac{1}{2\pi} \int_0^{2\pi} v(\xi_0 + r \cos(\theta - \alpha), \tau_0 + r\xi_0 \cos(\theta - \alpha)) d\theta.$$

Replacing $\theta - \alpha$ by θ and using the symmetry of the cosine function this reduces to

$$v_r(\xi_0, \tau_0) = \frac{1}{\pi} \int_0^\pi v(\xi_0 + r \cos \theta, \tau_0 + r\xi_0 \cos \theta) d\theta.$$

By changing back to (x, t) coordinates and defining $x_0 = \xi_0$, $t_0 =$

$\tau_0 - (1/2) \xi_0^2$ and $u_r(x_0, t_0) \equiv v_r(\xi_0, \tau_0)$ we get

$$\begin{aligned} u_r(x_0, t_0) &= \frac{1}{\pi} \int_0^\pi u\left(x_0 + r \cos \theta, \tau_0 + r x_0 \cos \theta - \frac{1}{2}(x_0 + r \cos \theta)^2\right) d\theta, \\ &= \frac{1}{\pi} \int_0^\pi u\left(x_0 + r \cos \theta, t_0 - \frac{1}{2} r^2 \cos^2 \theta\right) d\theta, \\ &= \frac{1}{\pi} \int_{-r}^r u\left(x_0 + z, t_0 - \frac{1}{2} z^2\right) \frac{dz}{\sqrt{r^2 - z^2}} \end{aligned}$$

or finally

$$(9) \quad u_r(x_0, t_0) = \frac{1}{\pi} \int_{-1}^1 u\left(x_0 + rz, t_0 - \frac{1}{2} r^2 z^2\right) \frac{dz}{\sqrt{1 - z^2}}$$

which is easily seen to be a weighted average of u over the tip of a parabola with vertex at (x_0, t_0) , having the line $t = t_0$ as its axis and opening down.

This gives us the following theorem.

THEOREM. *If u is twice differentiable at a point (x_0, t_0) , then a necessary and sufficient condition that $Hu = f$ at (x_0, t_0) is that*

$$u_r(x_0, t_0) = u(x_0, t_0) + C_2 r^2 f + o(r^2) \quad \text{as } r \rightarrow 0,$$

where $u_r(x_0, t_0)$ is given by (9).

To study the heat equation in higher dimensions one can make similar transformations. But it is easier to guess the form the previous theorem would take and verify it directly by the methods which established our basic theorem. The result is given below where Δu is the n -dimensional Laplacian, and Ω is the area of the unit sphere in $n + 1$ dimensions.

THEOREM. *If u is twice differentiable at a point (x_0, t_0) in $n + 1$ dimensions, then a necessary and sufficient condition that $\Delta u - u_t = f$ at (x_0, t_0) is that*

$$u_r(x_0, t_0) = u(x_0, t_0) + C_{n+1} r^2 f + o(r^2) \quad \text{as } r \rightarrow 0$$

where

$$u_r(x_0, t_0) = \frac{2}{\Omega} \int_{|z| < 1} u\left(x_0 + zr, t_0 - \frac{1}{2n} z^2 r^2\right) \frac{dz}{\sqrt{1 - |z|^2}}$$

with $dz = dz_1 dz_2 \cdots dz_n$.

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50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 14, No. 2

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