TRANSFORMATIONS OF DOMAINS IN THE PLANE AND APPLICATIONS IN THE THEORY OF FUNCTIONS

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In this paper we shall consider a family of transformations $S_n$ ($n = 1, 2, \ldots$) operating on open or closed sets in the complex plane $z$. $S_n$ is defined relatively to a fixed point called the center of transformation, and it transforms an open set into a starlike domain which, for $n > 1$, is also $n$-fold symmetric with respect to this point. Therefore, for $n > 1$, $S_n$ may be classified as a method of symmetrization.

This method of symmetrization was already defined by Szegö [4] for domains which are starlike with respect to the center of transformation.

The definition of $S_n$ will be extended (in the way usually used for symmetrizations) so that $S_n$ will operate also on a certain class of functions and a family of condensers, in the plane. It will be proved that $S_n$ diminishes the capacity of a condenser and this result will be used in order to obtain certain theorems in the theory of functions.

1. Definitions and notations. The transformations $S_n$ are defined as follows.

**DEFINITION 1.** Let $\Omega$ be an open set in the plane $z$, which does not contain the point at infinity, and let $z_0$ be a point of $\Omega$. If $|z - z_0| < \rho$, $(0 < \rho)$, is a circle contained in $\Omega$, we define:

\[
L_\rho(\varphi) = \int_E \frac{dr}{r},
\]

where $|z - z_0| = r$ and

\[
E = \{z | z \in \Omega, |z - z_0| > \rho, \arg (z - z_0) = \varphi\};
\]

\[
L_\rho^{(n)}(\varphi) = \frac{1}{n} \sum_{k=0}^{n-1} L_\rho(\varphi + \frac{2\pi k}{n});
\]

\[
R(\varphi) = \rho \exp \{L_\rho(\varphi)\};
\]

\[
R^{(n)}(\varphi) = \left[\prod_{k=0}^{n-1} R\left(\varphi + \frac{2\pi k}{n}\right)\right]^{1/n} = \rho \exp \{L_\rho^{(n)}(\varphi)\}.
\]

Evidently, $R^{(n)}(\varphi)$ does not depend on $\rho$. 

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Now, the set obtained from $\Omega$ by the transformation $S_n = S_n(z_0)$, with center $z_0$ is defined as follows:

\begin{equation}
S_n\Omega = \{z \mid z - z_0 = re^{i\varphi}, 0 \leq r < R^{(n)}(\varphi), 0 \leq \varphi < 2\pi\}.
\end{equation}

If instead of $\Omega$ we have a compact set $H$, which has an interior point $z_0$, we define:

\begin{equation}
S_nH = \{z \mid z - z_0 = re^{i\varphi}, 0 \leq r \leq R^{(n)}(\varphi), 0 \leq \varphi < 2\pi\}.
\end{equation}

It is easily verified that $S_n\Omega$ is a simply-connected domain and that $S_nH$ is a connected compact set. Both sets are starlike with respect to $z_0$.

We shall extend the definition of $S_n$ over a family of functions $\mathcal{G}$ which will now be defined. A non-constant real function $g(z)$ belongs to $\mathcal{G}$ if it is continuous over the extended plane $z$, if it takes its maximum value at infinity and if its minimum is assumed on a set of points, the interior of which is not empty. Let $g(z)$ be a function of $\mathcal{G}$ and let $m$ and $M$ be its minimum and maximum values, respectively. We define the following sets:

\begin{equation}
\begin{aligned}
G_m &= \{z \mid g(z) = m\}, \\
G_c &= \{z \mid g(z) < c\},
\end{aligned}
\end{equation}

for $m < c \leq M$.

$G_c$ (for $m < c < M$) is an open bounded set while $G_m$ is a compact set. Let $z_0$ be an interior point of $G_m$ and suppose that the circle $|z - z_0| = \rho$, $(0 < \rho)$, is contained in $G_m$. Denote by $L_{\rho}(c, \varphi), L^{(n)}_{\rho}(c, \varphi), R^{(n)}(c, \varphi)$ the functions defined by (1), (2), (3) with $G_c$ replacing $\Omega$. Clearly, for a fixed $\varphi$, $L_{\rho}(c, \varphi)$ is strictly monotonic increasing, for $m \leq c \leq M$. We also have:

\begin{equation}
\begin{aligned}
\lim_{c \to d^-} L_{\rho}(c, \varphi) &= L_{\rho}(d, \varphi), & m < d \leq M; \\
\lim_{c \to m} L_{\rho}(c, \varphi) &= L_{\rho}(m, \rho).
\end{aligned}
\end{equation}

Let $S_n = S_n(z_0)$. From these properties of $L_{\rho}(c, \varphi)$, it follows that:

\begin{equation}
S_nG_c \subset S_nG_d, \quad \text{for } m \leq c < d \leq M;
\end{equation}

\begin{equation}
S_nG_c = \bigcup_{m \leq d < c} S_nG_d, \quad \text{for } m < c \leq M;
\end{equation}

\begin{equation}
S_nG_m = \bigcap_{m < d \leq M} S_nG_d.
\end{equation}

Since $\bar{G}_c \subset \bigcap_{c < d \leq M} G_d$ we also have:

\begin{equation}
S_n\bar{G}_c \subset \bigcap_{c < d \leq M} S_nG_d, \quad m \leq c < M.
\end{equation}

**Definition 2.** Let $g(z) \in \mathcal{G}$. Using the notations introduced...
above, we define the function $g^{(n)}(z)$ obtained from $g(z)$ by the transformation $S_n = S_n(z_0)$, as follows:

\[
S_n g \equiv g^{(n)}(z) = \begin{cases} 
\inf \{ c \mid z \in S_n G_c \} , & \text{for } z \in S_n G_M , \\
M , & \text{otherwise} . 
\end{cases}
\]  

(11)

From (8) and (9) we now conclude:

\[
\{ S_n G_c = \{ z \mid g^{(n)}(z) < c \} , \text{ for } m < c \leq M , \}
\]

(12)

2. A lemma concerning the function $g^{(n)}(z)$.

**Lemma 1.** The function $g^{(n)}(z)$ is continuous over the extended plane $z$. If moreover $g(z)$ is Lip on every compact subset of $G_M$\(^1\) then $g^{(n)}(z)$ is Lip on every compact subset of $S_n G_M$.

**Proof.** We begin by proving the continuity of $g^{(n)}(z)$. If $z^* \in S_n G_m$ and $g^{(n)}(z^*) = d > m$ then by (10) and (12), the set $S_n G_{d+\varepsilon} - S_n \overline{G}_{d-\varepsilon}$ (where $m < d^* - \varepsilon < d^* + \varepsilon < M$) is an open neighbourhood of $z^*$ in which $|g^{(n)}(z) - g^{(n)}(z^*)| \leq \varepsilon$. If $z^*$ belongs to $S_n G_m$ or $z^*$ belongs to the complement of $S_n G_M$, then the set $S_n G_{m+\varepsilon}(m < m + \varepsilon < M)$, and the complement of $S_n \overline{G}_{M-\varepsilon}(m < M - \varepsilon < M)$ respectively, are open neighbourhoods of $z^*$ in which $|g^{(n)}(z) - g^{(n)}(z^*)| \leq \varepsilon$.

In order to prove the second assertion of the lemma it is sufficient to show that $g^{(n)}(z)$ is Lip on every set $S_n G_c(m < c < M)$. Without loss of generality we may suppose that $z_0 = 0$ and that $\rho = 1$. (And in this case we shall write $L^{(n)}(c, \varphi)$ instead of $L^{(n)}(c, \varphi)$.) We now map the $z$ plane, cut along the positive real axis from zero to infinity, by a branch of $w = \log z$, $(w = u + iv)$, onto the strip $0 < v < 2\pi$. (The points of the positive real axis will be mapped both on $v = 0$ and $v = 2\pi$). We denote by $H_c$ and $H_M^*$ the images of $G_c$ and $S_n G_c$ by this mapping, and we put $h(w) = g(e^w)$ and $h^{(n)}(w) = g^{(n)}(e^w)$.

Let $c$ be a fixed number in the open interval $(m, M)$. Since $g(z)$ is Lip on $G_c$, the function $h(w)$ is Lip on $H_c$, and if it is shown that $h^{(n)}(w)$ is Lip on $H_c^*$, the required result follows.

Since $h(w)$ is Lip on $H_c$, there exists a number $p > 0$ such that:

\[
|h(w_1) - h(w_2)| \leq p |w_1 - w_2| , \text{ for any } w_1, w_2 \in H_c .
\]

We need the following assertion:

If $\delta$ is a positive number and $v_1, v_2, c_1, c_2$ are real numbers such that:

\[
|v_1 - v_2| < \delta , m < c_1 < c_2 - \rho \delta < c < c + \rho \delta ,
\]

1 A function $g(z)$ is Lip on a set $E$ if there exists a constant $p$, such that for any two points $z_1, z_2 \in E$, we have $|g(z_1) - g(z_2)| \leq p |z_1 - z_2|$. 

then

\[ L^{(n)}(c_2, v_2) \geq L^{(n)}(c_1, v_1) + [\delta^2 - (v_1 - v_2)^2]^{1/2}. \]

Because of the definition of \( L^{(n)}(c, v) \), it is enough to prove (14) for \( n = 1 \). Without loss of generality we may suppose that \( 0 \leq v_k < 2\pi \), \((k = 1, 2)\).

Denote by \( J_k \) the intersection of the half line \( \text{Im} w = v_k, \text{Re} w \geq 0 \), with the set \( H_{c_k} \), for \( k = 1, 2 \). The Lebesgue measure of \( J_k \) is \( L(c_k, v_k) \).

Using (5) and (13) the following is easily verified:

Let \( w \in J_1 \). If \( w = u_2 + iv_2, \ u_2 \geq 0 \) and \( |w_1 - w_2| \leq \delta \), then \( w_2 \in J_2 \). From this and the fact that \( J_1 \) is bounded on the right, (14) follows for \( n = 1 \).

It will now be shown that

\[ |h^{(n)}(w') - h^{(n)}(w'')| \leq p |w' - w''|, \] for any \( w', w'' \in H^n \).

Suppose that there are two points \( w_1, w_2 \) in \( H^s \) for which this inequality does not hold, and let \( \delta \) be a number such that:

\[ |h^{(n)}(w_1) - h^{(n)}(w_2)| > p\delta > p |w_1 - w_2|. \]

Let \( h^{(n)}(w_1) < h^{(n)}(w_2) \). Then we can find numbers \( c_1, c_2 \) such that:

\[ m \leq h^{(n)}(w_1) < c_1 < c_2 - p\delta < h^{(n)}(w_2) - p\delta < c - p\delta. \]

Now the numbers \( c_1, c_2, v_1 = \text{Im} w_1, v_2 = \text{Im} w_2 \) satisfy (13), and therefore inequality (14) holds. Since, for \( m < c < M \),

\[ H^s_c = \{w \mid 0 \leq \text{Im} w \leq 2\pi, h^{(n)}(w) < c\} = \{w \mid 0 \leq v \leq 2\pi, u < L^{(n)}(c, v)\}, \]

it follows (by (16)) that \( w_1 \in H^s_{c_1} \) and \( w_2 \in H^s_{c_2} \); hence \( u_1 = \text{Re} w_1 < L^{(n)}(c_1, v_1) \) and \( u_2 = \text{Re} w_2 \geq L^{(n)}(c_2, v_2) \). These inequalities together with (14) yield \( |w_1 - w_2| > \delta \), which is in contradiction to (15). This completes the proof of the lemma.

**Remark.** The following is a consequence of the second part of the lemma: If \( g(z) \) is Lip on every compact subset of \( G_{M} - G_{m} \), then \( g^{(n)}(z) \) is Lip on every compact subset of \( S_n G_{M} - S_n G_{m} \).

3. **On a class of functions** \((C, z_0)\). Let \( C = (D, E_0, E_i) \) be a condenser in the complex plane \( z \), i.e. a system consisting of a domain \( D \) and two disjoint closed sets \( E_0 \) and \( E_i \), such that \( D \) does not contain the point at infinity, \( E_0 \) is bounded, \( E_i \) is unbounded and the union of \( E_0 \) and \( E_i \) is equal to the complement of \( D \).

Suppose that \( E_0 \) contains an interior point \( z_0 \), let \( z - z_0 = re^{i\varphi} \) and denote by \( S_{\varphi} \) the ray \( \text{arg} (z - z_0) = \varphi \). Then a subclass \((C, z_0)\) of \( \mathcal{C} \) is defined as follows.
A real function \( g(z) \), continuous over the extended plane \( z \), belongs to \((C, z_0)\) if:

(i) \( g(z) \) possesses continuous first partial derivatives, in \( D \).

(ii) \( g(z) \equiv 0 \) in \( E_0 \), \( g(z) \equiv 1 \) in \( E_1 \) and \( 0 < g(z) < 1 \) in \( D \).

(iii) The set of points on the ray \( S_\phi \), at which \( g(z) \) assumes a given value \( c \) \((0 < c < 1)\), is finite.

(iv) Any compact set of points on \( S_\phi \), which is contained in \( D \), contains only a finite number of points (possibly zero) at which \( \partial g(r, \phi)/\partial r = 0 \).

Suppose that the Dirichlet problem of the equation \( \Delta u = 0 \), with continuous boundary values, always has a solution in \( D \). Then there exists a real function \( \omega(z) \), continuous over the extended plane \( z \), which is harmonic in \( D \), vanishes on \( E_0 \) and assumes the value 1 on \( E_1 \). This function is the potential functions of \( C \). Evidently, it belongs to \((C, z_0)\).

Let \( g(z) \in (C, z_0) \). Using property (iii) we find that (6) may be replaced by

\[
\lim_{c \to c_0} L_\phi(c, \phi) = L_\phi(c_0, \phi), \quad \text{for } 0 \leq c_0 \leq 1. \tag{17}
\]

Therefore in this case, the function \( g^{(n)}(z) \equiv S_n(z_0)g \) may be defined in the following way:

\[
g^{(n)}(z) = g^{(n)}(r, \phi) = \begin{cases} 0, & \text{for } r \leq R^{(n)}(0, \phi), \\ c, & \text{for } r = R^{(n)}(c, \phi), 0 < c < 1, \\ 1, & \text{for } r \geq R^{(n)}(1, \phi). \end{cases} \tag{18}
\]

Since, for a fixed \( \phi \), \( g^{(n)}(r, \phi) \) is a strictly monotonic increasing function of \( r \) in the interval \( R^{(n)}(0, \phi) < r < R^{(n)}(1, \phi) \) and since \( g^{(n)}(r, \phi) \) is continuous over the entire plane, it follows that \( R^{(n)}(c, \phi) \) is continuous in both variables for \( 0 < c < 1, 0 \leq \phi < 2\pi \).

The following definition extends the transformation \( S_n \) over a family of condensers \( \{C\} \).

**Definition 3.** Let \( C = (D, E_0, E_1) \) be a condenser in the complex plane \( z \), such that \( E_0 \) contains an interior point \( z_0 \). Put \( G_1 = D \cup E_0 \) and suppose that \( S_n G_1 \) (with \( S_n = S_n(z_0) \)) does not contain the entire open plane. Then, the condenser \( C^{(n)} \) obtained from \( C \) by the transformation \( S_n = S_n(z_0) \) is defined as follows:

\[
C^{(n)} = (D^{(n)}, E_0^{(n)}, E_1^{(n)}),
\]

where \( D^{(n)} = S_n G_1 - S_n E_0, E_0^{(n)} = S_n E_0 \) and \( E_1^{(n)} = \) the complement of \( S_n G_1 \).
4. A theorem concerning the Dirichlet integral of functions belonging to \((C, z_0)\).

**Theorem 1.** Let \(C = (D, E_0, E_1)\) be a condenser in the complex plane \(z\), such that \(E_0\) contains an interior point \(z_0\). Suppose that \(g(z)\) belongs to \((C, z_0)\) and that its Dirichlet integral over \(D\) is finite. If \(S_n = S_n(z_0), (n = 1, 2, 3, \cdots)\), \(g^{(n)}(z) = S_n g\), and \(D^{(n)}\) is the domain mentioned in Definition 3, then:

\[
\int_{D^{(n)}} (Fg^{(n)})^2 dxdy \leq \int_D (Fg)^2 dxdy .
\]

**Remark.** This theorem was proved by Szegö [4], for \(n = 2, 3, \cdots\), in the special case where, \(D\) is a doubly-connected domain bounded by two smooth curves which are starlike with respect to \(z_0\); \(E_0\) and \(E_1\) are connected sets; and the function \(g(z)\) is the potential function of the condenser \(C\).

**Proof.** By property (i) of \(g(z)\) and by the remark at the end of Lemma 1 it follows that \(g^{(n)}(z)\) is Lip on every compact subset of \(D^{(n)}\). Therefore the first partial derivatives of \(g^{(n)}(x, y)\) exist almost everywhere in \(D^{(n)}\) and are bounded in every compact subset of \(D^{(n)}\).

Without loss of generality we may suppose that \(z_0 = 0\) and that the circle \(|z| \leq \rho = 1\) is contained in \(E_0\). Again we shall write \(L^{(n)}(e, \varphi)\) instead of \(L^{(p)}(e, \varphi)\). We also introduce the following notations:

\[
D(a, b) = \{ z | a < g(z) < b \}, \\
D^{(n)}(a, b) = \{ z | a < g^{(n)}(z) < b \}, \quad \text{for } 0 < a < b < 1 .
\]

The sets \(D(a, b)\) and \(D^{(n)}(a, b)\) will be mapped by \(w = \log z \,(0 \leq Imw < 2\pi)\) on two sets which we denote by \(H(a, b)\) and \(H^{(n)}(a, b)\), respectively. Finally we define:

\[
h(w) = g(e^w), \quad h^{(n)}(w) = g^{(n)}(e^w) \quad \text{and} \quad \gamma_c = \{ w | 0 < Imw < 2\pi, h(w) = c \}, \quad \text{for } 0 < c < 1 .
\]

The proof of the theorem rests on the following inequality:

\[
\int_{H^{(n)}(a, b)} \left[ 1 + \varepsilon^2 (\nabla h^{(n)})^2 \right]^{1/2} dudv \leq \int_{H(a, b)} \left[ 1 + \varepsilon^2 (\nabla h)^2 \right]^{1/2} dudv ,
\]

where \(w = u + iv, \, 0 < a < b < 1 \) and \(\varepsilon > 0\).

Inequality (19) is derived from (20) by a standard argument which we shall briefly describe.

The closures of the sets \(D(a, b)\) and \(D^{(n)}(a, b)\) are compact sets contained in \(D\) and \(D^{(n)}\), respectively. Therefore the first partial derivatives of \(h(u, v)\) \((h^{(n)}(u, v))\) are bounded in \(H(a, b)\) \((H^{(n)}(a, b))\). It is evident from the definitions that the area of \(H(a, b)\) equals that
of $H^{(n)}(a, b)$. Taking into account these facts and using the binomial expansion of the integrands in (20), (for $\varepsilon$ small enough), we obtain:

$$\frac{\varepsilon^2}{2} \int_{H^{(n)}(a, b)} (\nabla h^{(n)})^2 dv + O(\varepsilon) \leq \frac{\varepsilon^2}{2} \int_{H(a, b)} (\nabla h)^2 dv + O(\varepsilon).$$

Dividing by $\varepsilon^2$ and letting $\varepsilon$ tend to zero we find that

$$\int_{H^{(n)}(a, b)} (\nabla h^{(n)})^2 dv \leq \int_{H(a, b)} (\nabla h)^2 dv .$$

Since the Dirichlet integral is invariant under a simple conformal mapping, it follows that

$$\int_{D^{(n)}(a, b)} (\nabla g^{(n)})^2 dxdy \leq \int_{D(a, b)} (\nabla g)^2 dxdy .$$

Hence, letting $a$ tend to zero and $b$ tend to one, we obtain the required inequality.

In the proof of (20) we may suppose that $\varepsilon = 1$.

The first step is the following assertion. Suppose that $w^* = u^* + iv^* \in H^{(n)}(a, b)$ and $0 < v^* < (2\pi/n)$. Put $h^{(n)}(u^*, v^*) = c^*$. If $\partial h/\partial u \neq 0$ at all the points of intersection of the set $\gamma_c$ and the lines $\text{Im } w = v^* + (2\pi m/n)$ ($m = 0, \cdots, n - 1$), then there exists a neighbourhood of $w^*$ in which $h^{(n)}(u, v) \in C^1$.

In order to prove this assertion we shall show first that $L(c, v) \in C^1$ in a neighbourhood of $(c^*, v^*)$. By property (iii) the set $\gamma_c$ intersects the line $\text{Im } w = v^*$ in a finite number of points, which we denote by $w_1, \cdots, w_p$, where $\text{Re } w_1 < \text{Re } w_2 < \cdots < \text{Re } w_p$. By hypothesis, $\partial h / \partial u \neq 0$ at these points. Let $q$ be a positive number such that the circles $K_j : |w - w_j| \leq q$, $(j = 1, \cdots, p)$, are contained in $H(a, b)$ and $\partial h / \partial u \neq 0$ in them. Then the following is easily verified:

There exists a rectangle

$$P = \{(c, v) | |c - c^*| \leq \delta, |v - v^*| \leq \delta\},$$

(where $a < c^* - \delta < c^* + \delta < b$, $0 < v^* - \delta < v^* + \delta < (2\pi/n)$), such that:

(a) If $(c, v) \in P$ then $\gamma_c$ intersects the line $\text{Im } w = v$ in exactly $p$ points, one point in each circles $K_j$.

(b) The set $H(c^* - \delta, c^* + \delta)$ intersects the strip $v^* - \delta < \text{Im } w < v^* + \delta$ in exactly $p$ domains $Q_j$, where $Q_j \subset K_j$, $(j = 1, \cdots, p)$.

Solving $c = h(u, v)$ for $u$ in $Q_j$, we obtain a function $u = u_j(c, v)$. This function belongs to $C^1$ in the rectangle $P$ where

$$\frac{\partial u_j}{\partial c} = (\frac{\partial h}{\partial u})^{-1}, \quad \frac{\partial u_j}{\partial v} = -\left(\frac{\partial h}{\partial v}\right) \times (\frac{\partial h}{\partial u})^{-1}.$$
Since by definition:

\[ L(c, v) = \sum_{j=1}^{p} (-1)^{j+1} \times u_j(c, v) \]

it follows that \( L(c, v) \in C'[P] \). We observe that in \( Q \), we have \( \partial h/\partial u = (-1)^{j+1} \times |\partial h/\partial u| \) so that

\[ \frac{\partial L}{\partial c} = \sum_{j=1}^{p} \left| \frac{\partial u_j}{\partial c} \right| , \quad \text{in } P. \]

Evidently, similar results hold for any of the points \( c = c^*, v = v^* + (2\pi m/n) \), for \( m = 0, \cdots, n - 1 \). Therefore it is possible to find a positive number \( \eta(\eta \leq \delta) \) such that \( L^{(n)}(c, v) \in C^1 \) and \( (\partial L^{(n)}/\partial c) > 0 \) in the rectangle \( |c - c^*| < \eta, |v - v^*| < \eta \). By (18), for any fixed \( v, c = h^{(n)}(u, v) \) is the inverse function of \( u = L^{(n)}(c, v) \) in the interval \( 0 < c < 1 \). Hence it follows that in a certain neighbourhood of \( (u^*, v^*) \), \( h^{(n)}(u, v) \in C^1 \) and

\[ \frac{\partial h^{(n)}}{\partial u} = \left( \frac{\partial L^{(n)}}{\partial c} \right)^{-1}, \quad \frac{\partial h^{(n)}}{\partial v} = -\left( \frac{\partial L^{(n)}}{\partial v} \right) \times \left( \frac{\partial L^{(n)}}{\partial c} \right)^{-1}. \]

Denote by \( A(v) \) and \( A_n(v) \) the intersections of the line \( \text{Im } w = v \) with the sets \( H(a, b) \) and \( H^{(n)}(a, b) \) respectively. Let \( w \in A(v) \) and \( h(w) = c, (0 < v < 2\pi) \). If at one of the points of intersection of \( \gamma_c \) with the line \( \text{Im } w = v \), \( \partial h/\partial u \) vanishes then we shall say that \( w \) is a critical point of \( A(v) \). Let \( w \in A_n(v) \) and \( h^{(n)}(w) = c \). If the intersection of \( \gamma_c \) with one of the sets \( A(v + 2\pi m/n), (m = 0, \cdots, n - 1) \), contains a critical point of that set, we shall say that \( w \) is a critical point of \( A_n(v) \). By properties (iii) and (iv) the set of critical points of \( A(v) \) is finite, and consequently, the set of critical points of \( A_n(v) \) is finite.

We shall prove now that

\[ \int_{A_1(v)} \left[ 1 + (\mathcal{V} h^{(1)})^2 \right]^{1/2} du \leq \int_{A(v)} \left[ 1 + (\mathcal{V} h)^2 \right]^{1/2} du , \]

for \( 0 < v < 2\pi \). Inequality (20) for \( n = 1 \), follows from (25).

Let \( v_0 \) be a fixed point in the interval \( (0, 2\pi) \) and let \( \{c_0, \cdots, c_{k-1}\} \) be the set of values (possibly void) taken by \( h(w) \) at the critical points of \( A(v_0) \). We assume that these values are ordered as follows:

\[ a = c_0 < c_1 < \cdots < c_{k-1} < c_k = b . \]

Denote by \( B_i \) that subset of \( A(v_0) \) which consists of open segments, free from critical points, such that at the endpoints of each segment \( h(w) \) assumes the values \( c_i \) and \( c_{i+1} \). Evidently, for any \( l \) \((l = 0, \cdots, k-1)\) the set \( B_l \) is not void and \( A(v_0) = \bigcup_{l=0}^{k-1} B_l \).

Now let \( m \) be a fixed integer, \( 0 \leq m \leq k - 1 \), and denote by \( \alpha_1, \cdots, \alpha_m \).
the segments contained in $B_m$, which were described above. We shall assume that $\alpha_j$ is at the left of $\alpha_{j+1}$, $(j = 1, \ldots, p - 1)$. In some neighbourhood of $\alpha_j$ it is possible to solve $c = h(u, v)$ for $u$ and thereby obtain a function $u = u_j(c, v)$. By (21) we obtain:

\[
\int_{a_j}^{b_j} \left[ 1 + \left( \frac{\partial h(u, v)}{\partial c} \right)^2 \right]^{1/2} \, du = \int_{c_{m}}^{c_{m+1}} \left[ 1 + \left( \frac{\partial u_j(c, v)}{\partial c} \right)^2 \right]^{1/2} \, dc ,
\]

for $j = 1, \ldots, p$.

Denote: $u'_j = L(c_j, v_0)$ and $w'_j = u'_j + iv_0$, $(j = 0, \ldots, k)$. Then $w'_0$ and $w'_k$ are the endpoints of $A_i(v_0)$ while $w'_1, \ldots, w'_{k-1}$ are the critical points of $A_i(v_0)$. Denote by $B'_m$ the open segment with endpoints $w'_m$, $w'_{m+1}$. By (22) and (24) (with $n = 1$) we get:

\[
\int_{B'_m} \left[ 1 + \left( \frac{\partial h^{(1)}(u, v_0)}{\partial c} \right)^2 \right]^{1/2} \, du = \int_{c_{m}}^{c_{m+1}} \left\{ 1 + \left[ \frac{\partial}{\partial v} \sum_{j=1}^{p} (-1)^{j+1} u_j(c, v_0) \right] \right\}^{1/2} \, dc .
\]

By (26), (27) and the well known inequality

\[
\left\{ \left( \sum_{j=1}^{p} x_j \right)^2 + \left( \sum_{j=1}^{p} y_j \right)^2 + \left( \sum_{j=1}^{p} t_j \right)^2 \right\}^{1/2} \leq \sum_{j=1}^{p} (x_j^2 + y_j^2 + t_j^2)^{1/2} ,
\]

\((x_j, y_j, t_j\) being real numbers) we finally obtain:

\[
\int_{B'_m} \left[ 1 + \left( \frac{\partial h^{(1)}(u, v_0)}{\partial c} \right)^2 \right]^{1/2} \, du \leq \int_{c_{m}}^{c_{m+1}} \left[ 1 + \left( \frac{\partial h(u, v_0)}{\partial c} \right)^2 \right]^{1/2} \, dc
\]

\[
= \sum_{j=1}^{p} \int_{a_j}^{b_j} \left[ 1 + \left( \frac{\partial h(u, v_0)}{\partial c} \right)^2 \right]^{1/2} \, du .
\]

Since (29) holds for any $m$, $(m = 0, \ldots, k - 1)$ inequality (25) follows.

It remains to prove inequality (20) for $n = 2, 3, \ldots$. Since this inequality is proved for $n = 1$, it is enough to show that

\[
\int_{A_n(v_0)} \left[ 1 + \left( \frac{\partial h^{(n)}(u, v_0)}{\partial c} \right)^2 \right]^{1/2} \, du \leq \sum_{j=0}^{n-1} \int_{A_j(v_0)} \left[ 1 + \left( \frac{\partial h^{(1)}(u, v_0)}{\partial c} \right)^2 \right]^{1/2} \, du ,
\]

where $0 < v_0 < (2\pi/n)$ and $v_j = v_0 + (2\pi j/n)$.

Let \{c_0^*, \ldots, c_{r-1}^*\} be the set of values (possibly void) assumed by $h^{(n)}(u)$ at the critical points of $A_n(v_0)$, these values being ordered as follows:

\[
a = c_0^* < c_1^* < \cdots < c_{r-1}^* < c_r^* = b .
\]

Put $u_m^* = L^{(n)}(c_m^*, v_0)$ and $u_m^* = L(c_m^*, v_j)$. By (24) we get:
\[
\int_{u_1}^{u_n+1} [1 + (\varphi h^{(n)}(u, v_0))^2]^{1/2} du = \int_{e_m}^{c_m} [1 + (\varphi L^{(n)}(c, v_0))^2]^{1/2} dc \\
= \frac{1}{n} \int_{e_m}^{c_m} \left[ n^2 + \left( \sum_{j=0}^{n-1} (\varphi L(c, v_j))^2 \right) \right]^{1/2} dc ;
\]

for \( m = 0, \ldots, r - 1 \) and \( j = 0, \ldots, n - 1 \). From (31) and (28), inequality (30) follows. This completes the proof of the theorem.

5. The transformation \( S_n \) diminishes the capacity of a condenser. Let \( C = (D, E_0, E_1) \) be a condenser in the complex plane \( z \), satisfying the conditions of Definition 3. It will be assumed that the Dirichlet problem for \( \varphi u = 0 \), with continuous boundary values, always has a solution in \( D \). (Sufficient conditions for the validity of this assumption are given, for example, in Hayman [2], Th. 4.2, pp. 63–64. Following Hayman’s terminology we shall say that a domain is admissible if it satisfies these conditions.) The capacity of the condenser \( C \) is defined as the Dirichlet integral over \( D \), of the potential function \( \omega(z) \) of \( C \) (see §3).

Let \( C^{(n)} = S_n C = (D^{(n)}, E_0^{(n)}, E_1^{(n)}) \), (where \( S_n = S_n(z_0) \)). The domain \( D^{(n)} \) is admissible so that the capacity of \( C^{(n)} \) is defined. We now prove the following:

**Theorem 2.** Let \( C \) and \( C^{(n)} \) be the condensers mentioned above and denote their capacities by \( I \) and \( I_n \) respectively. Then we have \( I_n \leq I \).

**Proof.** Let \( \omega^{(n)}(z) = S_n \omega(z) \), \( (S_n = S_n(z_0)) \). Since \( \omega(z) \in (C, z_0) \), by Theorem 1 we have

\[
\int_{D^{(n)}} (\varphi \omega^{(n)})^2 dxdy \leq \int_D (\varphi \omega)^2 dxdy = I .
\]

The function \( \omega^{(n)}(z) \) is continuous over the extended plane \( z \) and Lip in every compact subset of \( D^{(n)} \); it vanishes on \( E_0 \) and assumes the value 1 on \( E_1 \). Hence, by the Dirichlet minimum principle (see, Hayman [2], Th. 4.3, pp. 65–67) we have

\[
I_n \leq \int_{D^{(n)}} (\varphi \omega^{(n)})^2 dxdy .
\]

The required result follows from (32) and (33).

We shall apply Theorem 2 in order to obtain a result about the inner radius. Let \( D \) be a domain in the complex plane \( z \), \( z_0 \) a point...
of \( D \), and \( r(D, z_0) \) the inner radius of \( D \) at \( z_0 \). (We refer here to the definition given, for example, in Hayman [2] pp. 78-80, where the inner radius is defined without any assumptions on \( D \).) The domain \( D \) can be approximated from within by a series of bounded analytic domains \( \{D_n\} \), which contain the point \( z_0 \), such that \( \lim_{n \to \infty} r(D_n, z_0) = r(D, z_0) \). (An analytic domain is a domain bounded by a finite number of disjoint, simple closed, analytic curves.) By a well known method of Pólya and Szegö (see Pólya-Szego [3] pp. 44-45; also Hayman [2] pp. 81-84) the following theorem is obtained as a consequence of Theorem 2.

**Theorem 3.** Let \( D \) be a domain in the complex plane \( z \) and let \( z_0 \in D \). If \( S_n = S_n(z_0) \), then

\[
(34) \quad r(D, z_0) \leq r(S_n D, z_0).
\]

6. **Applications in the theory of functions.** In this section we denote by \( w = f(z) \) a function which is regular in \( |z| < 1 \) and by \( D \) the domain of all values \( w \) assumed by this function at least once in \( |z| < 1 \). It is known that

\[
(35) \quad |f'(0)| \leq r(D, f(0)),
\]

equality holding if and only if \( f(z) \) is a \((1,1)\) mapping, (see Hayman [2], Th. 4.5, p. 80).

As a consequence of Theorem 3 we obtain the following:

**Theorem 4.** Let \( S_n = S_n(f(0)) \) and suppose that \( S_n D \) does not contain the entire open plane. Let \( w = F(z) \) be a \((1,1)\) conformal mapping of \( |z| < 1 \) onto \( S_n D \), such that \( F(0) = f(0) \). Then we have

\[
|f'(0)| \leq |F'(0)|.
\]

**Proof.** By (35) we get: \( |f'(0)| \leq r(D, f(0)) \) and \( |F'(0)| = r(S_n D, F(0)) \). From these relations together with (34), the required inequality follows.

The following results are based on Theorem 4.

**Theorem 5.** Let \( f(z) = a_1 z + a_2 z^2 + \cdots \). Define \( R^{(n)}(\varphi) \) as in Definition 1, for the domain \( D \) and the point \( w = 0 \). Then,

\[
(36) \quad |a_1| \leq 4^{n/2} R^{(n)}(\varphi), \quad (0 \leq \varphi < 2\pi)
\]

and equality holds for the function

\[
w = \psi_n(z) = te^{i(\varphi+\theta)}z/(1 + e^{in\theta}z)^{2/n}, \quad (t \text{ and } \theta \text{ real numbers}).
\]
Proof. Let \( \phi_0 \) be a fixed real number and suppose that \( R^{(n)}(\phi_0) = d < \infty \). Denote by \( D_0 \) the domain containing the entire \( w \) plane, with the exception of \( n \) rays: \( \arg w = \phi_0 + (2\pi k/n) \), \( d \leq |w|, (k = 0, \ldots, n-1) \). The domain \( S_nD(S_n = S_n(0)) \) is contained in \( D_0 \). The function \( w = \sqrt[4]{4} \exp^{i\phi_0}f_n(z) \) where

\[
f_n(z) = z/(1 + z^n)^{2/n},
\]

maps \( |z| < 1 \) conformally, (1,1) onto \( D_0 \). Therefore, by the principle of subordination and Theorem 4 it follows that \( |a_1| \leq \sqrt[4]{4} d \), and inequality (36) is proved. The assertion concerning the function \( w = \psi_n(z) \) is evident.

The following theorem may be proved by the same method.

**Theorem 6.** Let \( f(z) = a_1z + a_2z^2 + \cdots \). Suppose that \( R^{(n)}(\phi) \leq M < \infty \) for \( 0 \leq \phi < 2\pi \) and that \( R^{(n)}(\phi_0) = \beta M (0 < \beta \leq 1) \). Then

\[
|a_1| \leq \beta M \cdot \sqrt[4]{4}/(1 + \beta^n)^{2/n},
\]

and equality holds for the function

\[
w = \phi_n(z) = Me^{i\phi}f_n^{-1}[qf_n(e^{i\theta}z)],
\]

where \( f_n(z) \) is defined by (37), \( 0 \leq \theta < 2\pi \) and \( q = \sqrt[4]{4} \beta/(1 + \beta^n)^{2/n} \).

We now prove

**Theorem 7.** Let \( f(z) = a_1z + a_2z^2 + \cdots \) and define:

\[
R_0 = \exp\left[ \frac{1}{2\pi} \int_0^{2\pi} \log R(\phi) \, d\phi \right] = \exp\left[ \frac{1}{2\pi} \int_0^{2\pi} \log R^{(n)}(\phi) \, d\phi \right].
\]

Then \( |a_1| \leq R_0 \), and equality holds for \( w = a_1z \).²

Proof. First suppose that \( w = f(z) \) is regular in \( |z| \leq 1 \) and that \( f'(z) \neq 0 \) on \( |z| = 1 \). Then \( R(\phi) \) is a continuous function of \( \phi \), and we have

\[
\lim_{n \to \infty} R^{(n)}(\phi) = \lim_{n \to \infty} \exp\left[ \frac{1}{n} \sum_{k=0}^{n-1} \log R\left( \phi + \frac{2\pi k}{n} \right) \right] = R_0,
\]

for any real \( \phi \). Therefore, if a positive \( \varepsilon \) is given and \( n \) is sufficiently large, the domain \( S_nD \) (where \( S_n = S_n(0) \)) is contained in the circle \( |z| < R_0 + \varepsilon \). Hence, by Theorem 4 and the principle of subordin-

² The author obtained this result in a weaker form, with \( \overline{\tau}_n = \frac{1}{2\pi} \int_0^{2\pi} R^{(n)}(\phi) \, d\phi \) instead of \( R_0 \). (By the geometric-arithmetic mean theorem \( R_0 \leq \overline{\tau}_n \) for every \( n \)). The stronger form written above was suggested by the referee, to whom our thanks are due.
nation, we get $|a_1| \leq R_0 + \varepsilon$. In order to prove the theorem in the general case, we approximate the function $w = f(z)$ by functions $w = f(\rho z)$, with $0 < \rho < 1$.

Let $\Omega$ be an open set in the plane $z$ and let $z_0 \in \Omega$. Denote by $m(\varphi)$ the linear (Lebesgue) measure of the set $E(\varphi) = \{z | \arg(z - z_0) = \varphi, z \in \Omega\}$, and define

$$m^{(n)}(\varphi) = \frac{1}{n} \sum_{k=0}^{n-1} m\left(\varphi + \frac{2\pi k}{n}\right).$$

We shall show that Theorems 5, 6, 7, remain true if $R(\varphi)$ is replaced by $m(\varphi)$, and $R^{(n)}(\varphi)$ by $m^{(n)}(\varphi)$. This is a consequence of the following inequalities:

$$R(\varphi) \leq m(\varphi),$$
$$R^{(n)}(\varphi) \leq m^{(n)}(\varphi), \quad \text{for } 0 \leq \varphi < 2\pi.$$

If $R(\varphi)$ is finite, equality holds in (42) if and only if the set $E(\varphi)$ is contained in a segment $E^*$ such that $E^* - E(\varphi)$ is a set of measure zero. (We shall refer to this condition as the MR condition.) Inequality (42') follows from (42) by the geometric-arithmetic mean theorem. Hence, if $R^{(n)}(\varphi)$ is finite, equality holds in (42') if and only if

$$R(\varphi) = R\left(\varphi + \frac{2\pi k}{n}\right) = m(\varphi) = m\left(\varphi + \frac{2\pi k}{n}\right), \quad (k = 1, \ldots, n - 1).$$

From this it follows that when we replace $R(\varphi)$ by $m(\varphi)$ and $R^{(n)}(\varphi)$ by $m^{(n)}(\varphi)$, the functions mentioned at the end of Theorems 5, 6, 7, are in each case, the only functions for which equality holds.

In order to prove (42) we may suppose that $m(\varphi)$ is finite. In this case, for any $\varepsilon > 0$ we can find a subset $F$ of $E(\varphi)$, consisting of a finite number of segments, such that the linear measure of $E(\varphi) - F$ is smaller than $\varepsilon$. Therefore it is enough to prove (42) in the case that $E(\varphi)$ consists of a finite number of segments. Suppose that these segments are not adjacent. Then, by shifting them toward $z_0$ (so that they do not overlap), we increase $R(\varphi)$, while $m(\varphi)$ is invariant. But if the segments are adjacent we have $R(\varphi) = m(\varphi)$. Therefore (42) is proved.

Evidently, the MR condition for $E(\varphi)$ is sufficient in order that $R(\varphi) = m(\varphi)$. Suppose now that $R(\varphi)$ is finite and that $E(\varphi)$ does not satisfy the MR condition. Then it is possible to find a subset $F_1$ of $E(\varphi)$ and a subset $F_2$ of the complement of $E(\varphi)$ on the ray $\arg(z - z_0) = \varphi$, such that the two subsets have equal, positive measures and $F_2$ separates $F_1$ from $z_0$. Replacing $F_1$ by $F_2$ we increase $R(\varphi)$, but not $m(\varphi)$. Therefore we must have $R(\varphi) < m(\varphi)$. 

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