UNIMODULAR GROUP MATRICES WITH RATIONAL INTEGERS AS ELEMENTS

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1. Introduction. Let $G$ be a finite group of order $n$ with elements $g_1, g_2, \cdots, g_n$. Let

$$x_{ij}, \quad 1 \leq i, j \leq n$$

be variables in one-to-one correspondence with the elements of $G$. The $n \times n$ matrix

$$X = (x_{ij})_{1 \leq i, j \leq n}$$

is called the group matrix for $G$. If numerical values are substituted for the variables (1) in $X$, we say $X$ is a group matrix for $G$. In this paper we study group matrices which have rational integers as elements. Let $A'$ denote the transpose of the matrix $A$. A generalized permutation matrix is a square matrix with only 0, 1, $-1$ as elements and having exactly one nonzero element in each row and in each column. A square matrix $A$ is said to be unimodular if the determinant of $A$ is $\pm 1$. The result obtained in this paper is the following theorem.

**Theorem.** Let $G$ be a finite solvable group. Let $A$ be a unimodular matrix of rational integers such that $B = AA'$ is a group matrix for $G$. Then $A = A_1T$ where $A_1$ is a unimodular group matrix of rational integers for $G$ and $T$ is a generalized permutation matrix.

This theorem has already been proved for cyclic groups in [1] and for abelian groups in [2]. The present proof is a modification of the proof in [2].

2. Proof of the theorem. Let

$$1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_{m-1} \subset H_m = G$$

be an ascending chain of subgroups of $G$, where each $H_i$ is normal in $H_{i-1}$ with cyclic factor group $H_i/H_{i-1}$ of order $n_i$, $1 \leq i \leq m$. We let $n_0 = 1$, so that $H_i$ has order $n_in_{i-1} \cdots n_1$. In order to simplify the proof we take the elements of $G$ in a particular order. This will not affect the theorem as a reordering of the elements of $G$ changes the group matrix $X$ to $PXP'$ for $P$ a permutation matrix. Thus let

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$H_i$ be generated by the elements of $H_{i-1}$ and an element $a_i$ such that the coset $a_iH_{i-1}$ has order $n_i$. By induction we define column vectors $V_i$ of the elements of $H_i$. We let

(4) \[ V_0 = (1) \]

be the one row column vector whose only element is the identity of $G$. Suppose

(5) \[ V_{i-1} = (h_1, h_2, \cdots, h_i)' \]

with

(6) \[ t = n_0n_1 \cdots n_{i-1} \]

has been defined, where $h_1, h_2, \cdots, h_i$ are the ordered elements of $H_{i-1}$. For any $g \in G$ let

\[
V_{i-1}g = (gh_1, gh_2, \cdots, gh_i)', \\
V_i \in G = (h_1g, h_2g, \cdots, h_ig)'.
\]

Then define $V_i$ to be the column vector

(7) \[
V_i = \begin{bmatrix}
V_{i-1} \\
a_1 V_{i-1} \\
a_1^2 V_{i-1} \\
\vdots \\
a_1^{n_i-1} V_{i-1}
\end{bmatrix}
\]

For an arbitrary finite group $G$ with ordered elements $g_1, g_2, \cdots, g_n$ we define the left regular representation of $G$ by the matrix equations

\[
(gg_1, gg_2, \cdots, gg_n) = (g_1, g_2, \cdots, g_n)P^l(g), \quad g \in G.
\]

Here $P^l(g)$ is a permutation matrix depending on the element $g \in G$. It is straightforward to check that the matrix $X$ of (2) is given by

\[
X = \sum_{g \in G} x_{g} P^l(g).
\]

The set of all $P^l(g)$ for $g \in G$ is denoted by $L(G)$.

We define the right regular representation of $G$ by

\[
(gg_1, gg_2, \cdots, gg_n)' = P(g)(g_1, g_2, \cdots, g_n)', \quad g \in G.
\]

The set of all permutation matrices $P(g)$ for $g \in G$ is denoted by $R(G)$.

The group ring of the left (right) regular representation is the set of all linear combinations of the $P^l(g)$ ($P(g)$) for $g \in G$, and is denoted by $L^*(G)$ ($R^*(G)$). Thus the matrix (2) is the typical member.
of $L^*(G)$. The following two known facts are vital for the proof of our theorem:

(i) any matrix in $L^*(G)$ commutes with any matrix in $R^*(G)$;
(ii) any matrix that commutes with all the matrices in $R(G)$ is a member of $L^*(G)$.

**NOTATION.** We let diag $(X_1, X_2, \ldots, X_k)_k$ denote the direct sum of the square matrices $X_1, X_2, \ldots, X_k$:

$$\text{diag} (X_1, X_2, \ldots, X_k)_k = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 \\ 0 & X_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X_k \end{bmatrix}.$$ 

We set $[X_1]_k = X_1$. If $k > 1$ and $X_1, X_2, \ldots, X_k$ are square matrices of the same size, we set

$$[X_1, X_2, \ldots, X_k]_k = \begin{bmatrix} 0 & X_1 & 0 & \cdots & 0 \\ 0 & 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X_{k-1} \\ X_k & 0 & 0 & \cdots & 0 \end{bmatrix}.$$ 

We construct certain of the matrices in $R(G)$, where now the elements of $G$ are ordered according to (4), (5), (6), (7). Let $i$ be fixed, $1 \leq i \leq m$. Since $H_{i-1}$ is normal in $H_i$, $V_{i-1}a_i = a_iP_{i-1}(a_i)V_{i-1}$ where $P_{i-1}(a_i)$ is a $t \times t$ permutation matrix $(t$ as in (6)). Then, since

$$a_i^{\alpha} \in H_{i-1},$$

and because of (7), $V_i a_i = P_i(a_i)V_i$, where $P_i(a_i)$ is permutation matrix with the structure

$$P_i(a_i) = [P_{i-1}(a_i), P_{i-1}(a_i), \ldots, P_{i-1}(a_i), P_{i-1}(a_i)]_{n_i}.$$ 

In (9), $P_{i-1}(a_i)$ is another $t \times t$ permutation matrix.

Because of (7), we also have for any $g \in H_{i-1}$, that $V_i g = P_i(g)V_i$, where the permutation matrix $P_i(g)$ has the structure

$$P_i(g) = \text{diag} (P_{i-1}(g), P_{i-1}(g), \ldots, P_{i-1}(g))_{n_i}, \quad g \in H_{i-1}.$$ 

In (10), $P_i(g)$ is a block scalar matrix. The diagonal blocks $P_i(g)$ have dimensions $t \times t$. Furthermore, as $g$ runs over the elements of $H_{i-1}$, $P_{i-1}(g)$ runs over all the matrices of $R(H_{i-1})$. Since $H_i$ is generated by $H_{i-1}$ and $a_i$, the matrices $P_i(g)$ for $g \in H_{i-1}$ and $P_i(a_i)$ generate $R(H_i)$.
Because of the ordering of the elements of $G$, the following block scalar matrices:

(11) $Q(g) = \text{diag} (P_t(g), \ldots, P_t(g))_u$, \quad $g \in H_{i-1}$ or $g = a_i$,

(12) $u = n/tn_i$,

are the matrices in $R(G)$ determined by the $g \in H_{i-1}$ and by $g = a_i$. Here $Q(g)$ is $n \times n$.

We now prove our theorem by the following induction argument. Suppose for a fixed $i$, $1 \leq i \leq m$, that $B = AA'$ and that

(13) $AQ(g) = Q(g)A$, \quad for any $g \in H_{i-1}$.

(In particular this is satisfied if $i = 1$ since then the only such $Q(g)$ is $I_n$, the $n \times n$ identity matrix.) We shall then show that a generalized permutation matrix $T$ exists such that $B = (AT)(AT)'$ and such that $ATQ(g) = Q(g)AT$ for any $g \in H_{i-1}$ and for $g = a_i$, and so, in consequence, for any $g \in H_i$. Thus the induction will eventually yield a generalized permutation matrix $T_1$ such that $B = (AT_1)(AT_1)'$ and such that $AT_1Q(g) = Q(g)AT_1$ for any $g \in G$. It will now follow from (ii) that $AT_1 \in L^*(G)$, and the proof will be complete.

Hence assume $B = AA'$ where $A$ satisfies (13). Partition

(14) $A = (A_{\alpha,\beta})$, \quad $1 \leq \alpha, \beta \leq v = n/tn_i$,

into blocks of dimensions $t \times t$. As $Q(g)$ for $g \in H_{i-1}$ is a block scalar matrix with the blocks $P_t(g)$ of $R(H_{i-1})$ on the main block diagonal, it follows from (ii) and (13) that each

(15) $A_{\alpha,\beta} \in L^*(H_{i-1})$, \quad $1 \leq \alpha, \beta \leq v$.

Since $B \in L^*(G)$, $BQ(a_i) = Q(a_i)B$ so that if

(16) $M = A^{-1}Q(a_i)A$,\n
then,

(17) $MM' = I_n$.

As $A$ is unimodular the elements of $M$ are integers. Hence (17) implies that $M$ is a generalized permutation matrix. Partition $A, A^{-1}, \ Q(a_i)$, and $M$ into $t \times t$ blocks. As each block of $A$ lies in $L^*(H_{i-1})$ and as $A^{-1}$ is a polynomial in $A$, each of the $t \times t$ blocks of $A$, of $A^{-1}$, and of $Q(a_i)$ is a linear combination of a finite number of $t \times t$ permutation matrices. Therefore each $t \times t$ block of $M$ is a linear combination of a finite number of $t \times t$ permutation matrices. A permutation matrix is doubly stochastic in the sense that the sums across each row and down each column all have a common value.
As linear combinations of matrices doubly stochastic in this sense remain doubly stochastic, each $t \times t$ block of $M$ is doubly stochastic. Let $M_t$ be a typical $t \times t$ block in $M$. Since $M$ is a generalized permutation matrix, $M_t$ contains at most one nonzero element in each of its rows and columns. As $M_t$ is doubly stochastic, it now follows that $M_t$, if it is not the zero matrix, is either a permutation matrix or the negative of a permutation matrix. Since $M$ is a generalized permutation matrix, it follows that, after partitioning into $t \times t$ blocks, $M$ is a "generalized permutation matrix" in that it has exactly one nonzero block in each of its block rows and in each of its block columns. Each nonzero block is $\pm$ a permutation matrix.

There exists a permutation matrix $R$ consisting of $t \times t$ blocks which are either the $t \times t$ zero matrix or $I_t$ such that $R'MR$ is a direct sum of cycles. That is, $R'MR = \text{diag}(E_1, E_2, \ldots, E_r)$, where

$$E_\delta = [E_{\delta,1}, E_{\delta,2}, \ldots, E_{\delta,t}], \quad 1 \leq \delta \leq r.$$  

Here each $E_{\delta,\omega}$ is $\pm$ a $t \times t$ permutation matrix.

Note that $RQ(g) = Q(g)R$ for any $g \in H_{i-1}$ since each such $Q(g)$ is block scalar when partitioned into $t \times t$ blocks. Thus

$$ARQ(g) = Q(g)AR, \quad \text{for any } g \in H_{i-1},$$

and

$$(AR)^{-1}Q(a_i)AR = R'MR$$

is a direct sum of $E_1, E_2, \ldots, E_r$. Thus if we change notation and replace $AR$ with $A$ and $R'MR$ with $M$, we have (13), (14), (15), (16), (18) and

$$M = \text{diag}(E_1, E_2, \ldots, E_r).$$

Our immediate goal is to prove that each $e_\delta$ is $n_i$ and that $r = u$. Because of (8)

$$M^{a_i} = A^{-1}Q(a_i^a_i)A$$
$$= A^{-1}Q(a_i)A$$
$$= Q(g)$$

for some $g \in H_{i-1}$, by (13).

Hence each cycle $E_\delta$ of $M$ has the property that

$$E_\delta^{a_i}$$

is block scalar. This is not possible if $e_\delta > n_i$. Hence each $e_\delta \leq n_i$.

Counting rows in $M$ we get $t(e_1 + e_2 + \cdots + e_r) = n$. If any $e_\delta < n_i$ we would have
Let \( A_a = (A_{a,1}, A_{a,2}, \ldots, A_{a,v}) \), \( 1 \leq \alpha \leq v \), be the block rows of \( A \). For each fixed \( d \) such that \( 0 \leq d < u \) it follows from (9), (11), and \( Q(a_i)A = AM \) that

\[
P_{i-1}(a_i)A_{dn_i+k} = A_{dn_i+k-1}M, \quad 2 \leq k \leq n_i.
\]

Let \( w_0 = 0 \) and let \( w_\delta = e_1 + e_2 + \cdots + e_\delta \) for \( 1 \leq \delta \leq r \). Then (20) implies than for \( 2 \leq k \leq n_i \) and \( 0 \leq \delta \leq r - 1 \),

\[
(A_{dn_i+k, w_\delta+1}, \ldots, A_{dn_i+k, w_\delta+1}) = P_{i-1}(a_i)^{-k}(A_{dn_i+k, w_\delta+1}, \ldots, A_{dn_i+k, w_\delta+1})E_{\delta+1}^{-k}.
\]

For each fixed \( d, \delta \) such that \( 0 \leq d < u, \ 0 \leq \delta < r \), let \( F_{d,\delta} \) be the submatrix of \( A \) containing the blocks \( A_{a,\beta} \) with \( dn_i + 1 \leq \alpha \leq (d + 1)n_i \) and \( w_\delta + 1 \leq \beta \leq w_{\delta+1} \). Since each \( A_{a,\beta} \in L^*(H_{i-1}) \), each row of a given \( A_{a,\beta} \) is a permutation of the first row of this \( A_{a,\beta} \). Since \( P_{i-1}(a_i) \) and \( E_{\delta+1} \) are generalized permutation matrices, this fact and (21) imply that each row of \( F_{d,\delta} \) is a generalized permutation of the first row of \( F_{d,\delta} \). Thus if we add all the columns of \( F_{d,\delta} \) after the first to the first column of \( F_{d,\delta} \) we produce a new matrix \( \tilde{F}_{d,\delta} \) in which the integers in the first column of \( F_{d,\delta} \) are all equal, modulo 2. Next add the first row of \( \tilde{F}_{d,\delta} \) to all the other rows of \( \tilde{F}_{d,\delta} \) to get a new matrix \( \tilde{F}_{d,\delta} \). Then all the integers in the first column of \( \tilde{F}_{d,\delta} \) below the top element are zero, modulo 2.

Now partition \( A = (F_{d,\delta}) \) into its blocks \( F_{d,\delta} \). For each fixed \( \delta, 0 \leq \delta < r \), add to that column of \( A \) that intersects \( F_{0,\delta} \) at the extreme left of \( F_{0,\delta} \), all the other columns of \( A \) that intersect \( F_{0,\delta} \). This produces a new matrix \( \tilde{A} = (F_{d,\delta}) \). For each fixed \( d, 0 \leq d < u \), add the topmost row of \( \tilde{A} \) that intersects \( \tilde{F}_{d,0} \) to all the other rows of \( \tilde{A} \) that intersect \( \tilde{F}_{d,0} \). We get a new matrix \( \tilde{A} = (\tilde{F}_{d,\delta}) \). The \( r \) columns of \( \tilde{A} \) that intersect \( \tilde{F}_{0,\delta} \) at the extreme left of \( \tilde{F}_{0,\delta} \), \( 0 \leq \delta < r \), may now be regarded as vectors in a \( u \) dimensional vector space over the field of two elements. As \( r > u \), these vectors are dependent and so \( \tilde{A} \) (and hence \( A \)) is singular, modulo 2. This is a contradiction since the determinant of \( A \) is \( \pm 1 \).

Consequently each \( e_\delta = n_i, \ 1 \leq \delta \leq r \), and \( r = u \).

Now let \( E_{p,q} = \varphi_{p,q}E_{p,q} \) where \( \varphi_{p,q} = \pm 1 \) and \( E_{p,q} \) is a permutation matrix. Let \( \delta \) be fixed, \( 1 \leq \delta \leq u \). Suppose that \( P_{i-1}(a_i) \) has a one at position \((1, \omega)\) and let \( \tilde{E}_{\delta,1} \) have a one at position \((1, \mu)\). Let \( K_{\delta,1} \) be the permutation matrix in \( L(H_{i-1}) \) with a one at position \((\mu, \omega)\). \( K_{\delta,1} \) is the matrix in \( L(H_{i-1}) \) representing \( h_\mu h_\omega^{-1} \); see (2) and (5). Then \( \tilde{E}_{\delta,1} = \tilde{E}_{\delta,1}K_{\delta,1} \) has the same first row as \( P_{i-1}(a_i) \). Similarly, by induction, we determine \( K_{\delta,s} \) in \( L(H_{i-1}) \), \( 1 < s < n_i \), such that the
permutation matrices

\[
\tilde{E}_{s,s} = K_{s,s-1}E_{s,s}K_{s,s}, \quad 1 < s < n_i,
\]
each have the same first row as \(P_{t-i}(a_i)\). Then let

\[
S_{s} = \text{diag} \left( I_t, P_{s,1}K_{s,1}, P_{s,2}K_{s,2}, \ldots, \left( \prod_{j=1}^{n_i-1} P_{s,j} \right) K_{s,n_i-1} \right),
\]
and let \(S = \text{diag}(S_1, S_2, \ldots, S_u)\). Then

\[
S'MS = \text{diag} (\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_u)
\]
where

\[
(22) \quad \tilde{E}_{s,j} = [\tilde{E}_{s,1}, \tilde{E}_{s,2}, \ldots, \tilde{E}_{s,n_i-1}, \pm \tilde{E}_{s,n_i}]_{n_i}, \quad 1 \leq \delta \leq u.
\]
In (22) each \(\tilde{E}_{s,j}, 1 \leq j < n_i, 1 \leq \delta \leq u,\) is a permutation matrix with the same first row as \(P_{t-i}(a_i)\) and each

\[
\tilde{E}_{s,n_i}, \quad 1 \leq \delta \leq u,
\]
is some unknown permutation matrix.

Now \(SQ(g) = Q(g)S\) if \(g \in H_{t-i}\) since \(S\) is block diagonal with its blocks in \(L^*(H_{t-i})\) whereas \(Q(g)\) for \(g \in H_{t-i}\) is block scalar with its blocks in \(R(H_{t-i})\). Thus if we change notation again and replace \(AS\) with \(A\) and \(S'MS\) with \(M\) we retain the validity of (13) and (16) and now

\[
(23) \quad M = \text{diag} (\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_u).
\]
Since for any \(g \in H_{t-i}\), \(a^{-1}g_a_i = \bar{g} \in H_{t-i}\), it follows that for any \(g \in H_{t-i}\) there exists a \(\bar{g} \in H_{t-i}\) such that \(Q(g)Q(a_i) = Q(a_i)Q(\bar{g})\). Hence, using (9), (10), and (11), we find

\[
(24) \quad P_{t-i}(g)P_{t-i}(a_i) = P_{t-i}(a_i)P_{t-i}(\bar{g}), \quad g, \bar{g} \in H_{t-i}.
\]
If we let \(g \in H_{t-i}\) be such that \(P_{t-i}(g)\) has a one at position \((1, \omega)\) then (24) says: row \(\omega\) of \(P_{t-i}(a_i)\) is determined in terms of row one of \(P_{t-i}(a_i)\).

Now for \(g \in H_{t-i}\):

\[
Q(g)M = Q(g)A^{-1}Q(a_i)A \\
= A^{-1}Q(g)Q(a_i)A \quad \text{by (13)}, \ \
= A^{-1}Q(a_i)Q(\bar{g})A \quad \text{since } ga_i = a_i\bar{g}, \ \
= A^{-1}Q(a_i)AQ(\bar{g}) \quad \text{by (13)}, \ \
= MQ(\bar{g}).
\]
Hence, for fixed \(\delta\) and \(j, 1 \leq \delta \leq u, 1 \leq j < n_i\), it now follows
(using (10), (11), (22), and (23)) that

\[ P_{t-1}(g) \tilde{E}_{i,j} = \tilde{E}_{i,j} P_{t-1}(\bar{g}), \quad g, \bar{g} \in H_{t-1}. \]

As with (24), (25) determines each row of \( \tilde{E}_{i,j} \) in terms of the first row of \( \tilde{E}_{i,j} \). Consequently

\[ \tilde{E}_{i,j} = P_{i-1}(a_i), \quad 1 \leq \delta \leq u, 1 \leq j < n_i. \]

We also have (8), hence

\[ M^{n_i} = A^{-1}Q(a_i^n)A = Q(a_i)^{n_i} \]

by (13). Hence, for each \( \delta, 1 \leq \delta \leq u \),

\[ \tilde{E}_{\delta}^{n_i} = P_{i}(a_i)^{n_i}. \]

Each side of (27) is a block diagonal matrix. Equating the topmost diagonal blocks we get

\[ \left[ \prod_{j=1}^{n_i-1} \tilde{E}_{i,j} \right] \pm \tilde{E}_{\delta,n_i} = P_{i-1}(a_i)^{n_i-1} \tilde{P}_{i-1}(a_i). \]

Hence, by (26),

\[ \pm \tilde{E}_{\delta,n_i} = \tilde{P}_{i-1}(a_i), \quad 1 \leq \delta \leq u. \]

We have now proved that \( M = Q(a_i) \). Hence \( Q(a_i)A = AQ(a_i) \). As indicated earlier, this is enough to complete the proof.

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