

# Pacific Journal of Mathematics

**LEAST SQUARES AND INTERPOLATION IN ROOTS OF UNITY**

J. L. WALSH AND AMBIKESHWAR SHARMA

# LEAST SQUARES AND INTERPOLATION IN ROOTS OF UNITY

J. L. WALSH AND A. SHARMA

We mention the Erdős-Turán theorem [2] that if  $F(\theta)$  is a real continuous function with period  $2\pi$ , and if  $t_n(\theta)$  is the unique trigonometric polynomial of order  $n$  that coincides with  $F(\theta)$  in  $2n + 1$  points equally spaced over an interval of length  $2\pi$ , then  $t_n(\theta)$  converges to  $F(\theta)$  on that interval in the mean of second order. It is the purpose of the present note to prove the analogue in the complex domain, and to discuss some related remarks.

**THEOREM 1.** *Let the function  $f(z)$  be analytic in  $D: |z| < 1$ , continuous in  $D + C$  ( $C: |z| = 1$ ), and let  $p_n(z)$  be the polynomial of degree  $n$  coinciding with  $f(z)$  in the  $(n + 1)$  st roots of unity. Then the sequence  $p_n(z)$  converges to  $f(z)$  on  $C$  in the mean of second order. Consequently we have*

$$(1) \quad \lim_{n \rightarrow \infty} p_n(z) = f(z) \quad \text{uniformly in } |z| \leq r (< 1).$$

If we set

$$(2) \quad I_n = \int_{\sigma} |f(z) - p_n(z)|^2 |dz|,$$

we have

$$p_n(z) \equiv \sum_{k=1}^{n+1} f(\omega^k) A_k(z),$$

$$A_k(z) \equiv \frac{\omega^k (z^{n+1} - 1)}{(n+1)(z - \omega^k)}, \quad \omega = e^{2\pi i / (n+1)},$$

and shall show

$$(3) \quad \lim_{n \rightarrow \infty} I_n = 0.$$

We introduce the notation

$$f(z) - t_n(z) \equiv \Delta(z), \quad E_n = \max [|\Delta z|, z \text{ on } C],$$

where  $t_n(z)$  is the polynomial of degree  $n$  of best Tchebycheff approximation to  $f(z)$  on  $C$ , and denote by  $P_n(z)$  the polynomial of degree  $n$  that coincides with  $\Delta(z)$  in the  $(n + 1)$  st roots of unity. Then we have  $P_n(z) \equiv p_n(z) - t_n(z)$ , whence

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$$\begin{aligned}
 I_n &= \int_{\sigma} |A(z) - P_n(z)|^2 |dz| \\
 &\leq 2 \int_{\sigma} |Az|^2 |dz| + 2 \int_{\sigma} |P_n(z)|^2 |dz| = I_n + I_n'' .
 \end{aligned}$$

There follow the relations  $I_n' \leq 4\pi E_n^2$ ,

$$\begin{aligned}
 I_n'' &= 2 \int_{\sigma} \left| \sum_{k=1}^{n+1} A(\omega^k) A_k(z) \right|^2 |dz| \\
 &\leq 2 \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} |A(\omega^k) \bar{A}(\omega^j)| \left| \int_{\sigma} A_k(z) \bar{A}_j(z) |dz| \right| \\
 &\leq 2 E_n^2 \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} \left| \int_{\sigma} A_k(z) \bar{A}_j(z) |dz| \right| .
 \end{aligned}$$

However, we have

$$\begin{aligned}
 A_k(z) &\equiv \frac{\omega^k}{n+1} (z^n + \omega^k z^{n-1} + \dots + \omega^{nk}) , \\
 \int_{\sigma} A_k(z) \bar{A}_j(z) |dz| &= \frac{2\pi\omega^{k-j}}{(n+1)^2} (1 + \omega^{k-j} + \omega^{2(k-j)} + \dots + \omega^{n(k-j)}) \\
 &= 2\pi\delta_{jk}/(n+1) ,
 \end{aligned}$$

where  $\delta_{jk}$  is the Kronecker  $\delta$ . It is well known [4, Theorem 5, p. 36] that  $E_n \rightarrow 0$  as  $n \rightarrow \infty$ , so (3) holds.

Equation (1) follows from (3) by the Cauchy integral formula

$$(4) \quad [f(z) - p_n(z)]^2 = \frac{1}{2\pi i} \int_{\sigma} \frac{[f(t) - p_n(t)]^2 dt}{t - z} , \quad |z| < 1 .$$

With the hypothesis of Theorem 1, the conclusion (1) is due to Fejér (1918). Theorem 1 is related to various other results concerning convergence of polynomials interpolating in roots of unity; for instance (Runge) if  $f(z)$  in Theorem 1 is analytic in  $|z| \leq 1$ , equation (1) holds uniformly in  $|z| \leq 1$ . Further references to the subject are given by Curtiss [1].

There exist numerous other results, somewhat similar to Theorem 1, where now a sequence of polynomials  $P_n(z, 1/z)$  of respective degrees  $n$  in  $z$  and  $1/z$  converges on  $C$  in the second-order mean to a given function  $f(z)$  defined merely on  $C$ . The function  $f(z)$  can be expressed on  $C$  as  $f(z) \equiv f_1(z) + f_2(z)$ , where  $f_1(z)$  is of the Hardy-Littlewood class  $H_2$  and  $f_2(z)$  of the analogous class  $G_2$  for the region  $|z| > 1$  (we suppose  $f_2(\infty) = 0$ ; compare e.g. [4, § 6. 11]). Any function of class  $H_2$  is orthogonal on  $C$  to any function of class  $G_2$ , so if we set  $P_n(z, 1/z) \equiv p_n(z) + q_n(1/z)$ , where  $p_n(z)$  and  $q_n(1/z)$  are polynomials of respective degrees  $n$  in their arguments,  $q_n(0) = 0$ , we have

$$(5) \quad \lim_{n \rightarrow \infty} p_n(z) \equiv \frac{1}{2\pi i} \int_{\sigma} \frac{f(t) dt}{t - z} \equiv f_1(z) \equiv \frac{1}{2\pi i} \int_{\sigma} \frac{f_1(t) dt}{t - z} , \quad z \text{ interior to } C ,$$

$$(6) \quad \lim_{n \rightarrow \infty} q_n(1/z) \equiv \frac{1}{2\pi i} \int_{\sigma} \frac{f(t)dt}{t-z} \equiv f_2(z) \equiv \frac{1}{2\pi i} \int_{\sigma} \frac{f_2(t)dt}{t-z},$$

$z$  exterior to  $C$ ,

with uniformity of approach for  $z$  on an arbitrary compact set in the respective regions. This remark concerning (5) and (6) applies for instance in the case of the Erdős-Turán theorem, where we set  $F(\theta) \equiv f(e^{i\theta})$  and  $t_n(\theta) \equiv p_n(e^{i\theta}, e^{-i\theta})$  on  $C$ .

A second remark concerning (5) and (6) is as follows. By the orthogonality relations we have for the second-order norms on  $C$

$$\|f - P_n\|^2 = \|f_1 - p_n\|^2 + \|f_2 - q_n\|^2.$$

Consequently the rapidity of convergence in the mean on  $C$  of  $p_n$  to  $f_1$  and of  $q_n$  to  $f_2$  is not less than that of  $P_n$  to  $f$ . If the positive numbers  $\epsilon_1, \epsilon_2, \dots$  are given and approach zero, there is a corresponding class  $K$  of functions  $f(z)$  belonging to  $L_2$  on  $C$  such that for each  $f(z)$  there exist polynomials  $P_n(z, 1/z)$  with

$$(7) \quad \|f(z) - P_n(z, 1/z)\| = O(\epsilon_n);$$

here the  $P_n(z, 1/z)$  may be taken as the partial sums of the Fourier or Laurent development of  $f(z)$  on  $C$ . It follows that every function  $f(z)$  in  $K$  can be written  $f(z) \equiv f_1(z) + f_2(z)$ , with  $f_1$  and  $f_2$  in  $H_2$  and  $G_2$  respectively, where the partial sums  $p_n(z)$  of the Taylor development of  $f_1(z)$  satisfy

$$(8) \quad \|f_1(z) - p_n(z)\| = O(\epsilon_n)$$

and the partial sums  $q_n(1/z)$  of the Laurent development of  $f_2(z)$  satisfy

$$(9) \quad \|f_2(z) - q_n(1/z)\| = O(\epsilon_n).$$

Thus  $f_1$  and  $f_2$  belong to  $K$  on  $C$ .

As a particular case of this application, we mention the class of functions  $L(2, k, \alpha)$ ,  $0 < \alpha < 1$ , namely the class of functions whose  $k$ th derivatives on  $C$  satisfy there a square-integrated Lipschitz condition of order  $\alpha$ ; this class was first studied by Hardy and Littlewood, theorems proved in detail by Quade [3]. An alternative definition of the class is (7) with  $\epsilon_n = 1/n^{k+\alpha}$ . It follows that every function  $f(z)$  of class  $L(2, k, \alpha)$ ,  $0 < \alpha < 1$ , can be expressed on  $C$  as  $f_1(z) + f_2(z)$ , where the latter two functions, of respective classes  $H_2$  and  $G_2$  satisfy (8) and (9) with the same values of  $\epsilon_n$ ; thus  $f_1(z)$  and  $f_2(z)$  likewise belong to  $L(2, k, \alpha)$  on  $C$ . The case  $\alpha = 1$  can be similarly treated, where the integrated Lipschitz conditions are replaced by the condition

$$(10) \quad \int_0^{2\pi} |F(\theta + h) + F(\theta - h) - 2F(\theta)|^2 d\theta = O(h^2),$$

and  $F(\theta) \equiv f^{(k)}(z)$  is continuous on  $C$ ; this class was introduced by Zygmund, and is characterized by (7) with  $\varepsilon_n = 1/n^{k+1}$ . We have as before  $f(z) \equiv f_1(z) + f_2(z)$  if  $f(z)$  is given, and the corresponding classes of  $f_1(z)$  and  $f_2(z)$  are characterized by (8) and (9) with the same values of  $\varepsilon_n$ , and by (10). These classes of analytic functions are studied in [5].

*Added in proof.* A second proof of Theorem 1, due to G. H. Curtiss, will appear shortly.

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