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ON SOME FINITE GROUPS AND THEIR COHOMOLOGY

KUNG-WEI YANG

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The purposes of this paper are: (I) to characterize the finite groups whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group and which have periodic cohomology of period 4, (II) to show that all possible cohomologies of such a group G can be realized by direct sums of G -modules which belong to a specific finite family of G -modules.

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The reader is referred to [1, Ch. XII] for basic notions, definitions and notations concerning cohomology of finite groups. The only departure from [1, Ch. XII] is the following: we shall say that a finite group G has periodic cohomology of period k if k is the *least* positive integer such that $\hat{H}^k(G, Z)$ contains a maximal generator [1, pp. 260–261]. And to avoid typographical difficulties we will denote by $Z(l)$ the cyclic group of order l .

PROPOSITION I. *Let G be a finite group whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group. Then G has periodic cohomology of period 4 if and only if G has a presentation*

$$G = \{\sigma, \tau: \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\}, \text{ with the conditions}$$

- (i) s is an odd integer > 1 ,
- (ii) t is a positive even integer prime to s .

Proof. Let G be a finite group whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group and which has periodic cohomology of period 4. It is well-known [1, Theorem 11.6, p. 262] that if a finite group has periodic cohomology (of finite period) every Sylow subgroup of the group is either cyclic or is a generalized quaternion group. Since we assume that the 2-Sylow subgroups of G are not isomorphic to a generalized quaternion group, every Sylow subgroup of G is cyclic. It is also well-known [6, Theorem 11, p. 175] that a finite group G containing only cyclic Sylow subgroups is meta-cyclic and has a presentation

$$G = \{\sigma, \tau: \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^r\}, \text{ with the conditions}$$

- (1) $0 < s$, ($st =$ the order of the group G),
 (2) $((r - 1)t, s) = 1$
 (3) $r^t \equiv 1 \pmod{s}$, and conversely.

We observe that if $s = 1$ or $t = 1$ or $r = 1$ the finite group G is cyclic and G has periodic cohomology of period 2 (or 0). These cases are therefore excluded. On the other hand, once these exceptional cases are excluded G is no more a cyclic group and it will have periodic cohomology of period ≥ 4 .

Notice that (1), (2) and (3) imply (i)

Let H be the subgroup of G generated by the element σ . H is clearly a cyclic normal subgroup of order s . And G/H is cyclic of order t . By condition (2), s and t are relatively prime to each other. We can therefore apply the decomposition theorem of Hochschild-Serre [2, Theorem 1, p. 127] and obtain

$$(4) \quad \hat{H}^k(G, K) \cong \hat{H}^k(G/H, K^H) \oplus (\hat{H}^k(H, K))^{e_1^H},$$

for all $k > 0$ and for all G -module K . (For $k > 0$, $\hat{H}^k(G, K) = H^k(G, K)$). In particular, we have

$$\hat{H}^k(G, Z) \cong \hat{H}^k(G/H, Z) \oplus (\hat{H}^k(H, Z))^{e_1^H},$$

for $k > 0$. The G/H -operators on $\hat{H}^k(H, K)$ are explicitly described in [2, p. 117]. In particular, G/H -operators on $\hat{H}^k(H, Z)$ are induced by the automorphisms of H which are themselves induced, on H , by inner automorphisms of G . In the present situation, all such automorphisms of H are generated by the automorphism $f(\rho) = \rho^\tau (= \tau\rho\tau^{-1})$, where $\rho \in H$. The automorphism $f: H \rightarrow H$ induces an automorphism f^* of $\hat{H}^k(H, Z)$ [4, Lemma 3, p. 343] such that if $g_{2k} \in \hat{H}^{2k}(H, Z)$, then $f^*(g_{2k}) = r^k g_{2k}$. Therefore $\hat{H}^4(G, Z)$ has a maximal generator, i.e. G has periodic cohomology of period ≤ 4 if and only if $f^*(g) = g$ for all $g \in \hat{H}^4(H, Z)$. This is equivalent to

$$(5) \quad r^2 \equiv 1 \pmod{s}.$$

(We recall that $r = 1$ we excluded). An elementary number theoretic calculation shows that the only solution for r in (2) and (5) is $r \equiv -1 \pmod{s}$. Therefore the number t in (3) is an even positive integer (if it is negative, we can present G by letting $\tau' = \tau^{-1}$). This shows that the finite group G has a presentation as mentioned above.

The converse of the proposition is obvious.

We know that if l is the order of the group G then for any G -module K all the cohomology groups $\hat{H}^k(G, K)$ ($-\infty < k < \infty$) are of exponent l —that is, for all $g \in \hat{H}^k(G, K)$, $lg = 0$. Let

$$s = p_1^{a_1} \cdots p_h^{a_h}, P_1 = \{p_1, \dots, p_h\} \quad \text{and} \quad t = q_1^{b_1} \cdots q_e^{b_e}, P_2 = \{q_1, \dots, q_e\}$$

be decompositions of s and t into products of prime powers (where

$q_1 = 2$ and $v_1 \geq 1$). It is obvious from (4) that a group with periodic cohomology of period 4 has P_2 -period [1, Exercise 11, p. 265] equal to 2. Conversely, we have

PROPOSITION II. *Let G be a group having a presentation*

$$G = \{\sigma, \tau: \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\} \text{ with the conditions}$$

- (i) s is an odd integer > 1 .
- (ii) t is a positive even integer prime to s .

Let P_1, P_2 be as defined above. Then there exists a finite family of G -modules \mathcal{F} such that given any sequence of abelian groups $A_k (-\infty < k < \infty)$ satisfying

- (a) each A_k is of exponent st ,
- (b) the sequence is periodic of period 4,
- (c) the P_2 -period of the sequence is equal to 2, then there exists a G -module M which is a direct sum of G -modules of \mathcal{F} such that $\hat{H}^k(G, M) = A_k (-\infty < k < \infty)$.

First we observe the following

LEMMA. *Let G be a finite group and let K be a G -module. Let S be a set of primes in the ring of integers Z and let $Q(S)$ be the quotient ring [5, p. 46] of Z with respect to the multiplicative system generated by S . (As usual when $Q(S)$ is considered as a G -module it is to be understood that G operates trivially on (the additive group of) $Q(S)$). Then*

$$\hat{H}^k(G, K \otimes Q(S)) \cong \hat{H}^k(G, K) \otimes Q(S) (-\infty < k < \infty),$$

where $\otimes = \otimes_Z$

The proof is immediate.

Proof of Proposition II. Let s, t, P_1, P_2 be as before. Let

$$\begin{aligned} s(i, \mu) &= s/p_i^\mu (i = 1, \dots, h, 0 \leq \mu \leq u_i), \\ t(i, \nu) &= t/q_i^\nu (i = 1, \dots, e, 0 \leq \nu \leq v_i). \end{aligned}$$

Let $K^1(i, \mu) = \sum_{j=1}^{s(i, \mu)} Zx_j^{(i, \mu)}$ (direct sum on the symbols $x_j^{(i, \mu)}$)

$$K^2(i, \nu) = \sum_{j=1}^{t(i, \nu)} Zy_j^{(i, \nu)} \text{ (direct sum on the symbols } y_j^{(i, \nu)} \text{).}$$

Define G -operators on $K^1(i, \mu)$ and $K^2(i, \nu)$ by

$$\begin{aligned} \sigma x_j^{(i, \mu)} &= x_{j+1}^{(i, \mu)} \\ \tau x_j^{(i, \mu)} &= x_{-j}^{(i, \mu)}, \end{aligned} \text{ (subscripts are modulo } s(i, \mu) \text{)}$$

$$\begin{aligned} \sigma y_j^{(i,\nu)} &= y_j^{(i,\nu)} \\ \tau y_j^{(i,\nu)} &= y_j^{(i,\nu)}. \end{aligned} \quad (\text{subscripts are modulo } t(i, \nu)).$$

Let

$$\begin{aligned} M^1(i, \mu) &= K^1(i, \mu) \otimes Q((P_1 - \{p_i\}) \cup P_2), \\ M^2(i, \nu) &= K^2(i, \nu) \otimes Q(P_1 \cup (P_2 - \{q_i\})). \end{aligned}$$

By (4), the above lemma and the fact that $(\hat{H}^{4k+2}(H, K^1(i, \mu))^{e/H} = (0)$, one shows

$$\begin{aligned} \hat{H}^{4k}(G, M^1(i, \mu)) &= Z(p_i^\mu) & \hat{H}^{4k}(G, M^2(i, \nu)) &= Z(q_i^\nu) \\ \hat{H}^{4k+1}(G, M^1(i, \mu)) &= (0) & \hat{H}^{4k+1}(G, M^2(i, \nu)) &= (0) \\ \hat{H}^{4k+2}(G, M^1(i, \mu)) &= (0) & \hat{H}^{4k+2}(G, M^2(i, \nu)) &= Z(q_i^\nu) \\ \hat{H}^{4k+3}(G, M^1(i, \mu)) &= (0) & \hat{H}^{4k+3}(G, M^2(i, \nu)) &= (0) \end{aligned}$$

The calculation is purely mechanical.

Now, let $0 \rightarrow I \rightarrow Z[G] \xrightarrow{\varepsilon} Z \rightarrow 0$, where $\varepsilon(\sum_{\sigma \in \mathcal{G}} l_\sigma \sigma) = \sum_{\sigma \in \mathcal{G}} l_\sigma$, $I = \text{Ker}(\varepsilon)$, and let \mathcal{F} consist of

$$\begin{aligned} I^k \otimes M^1(i, \mu) &(k = 0, 1, 2, 3, i = 1, \dots, h, 0 \leq \mu \leq u_i) \\ I^k \otimes M^2(i, \nu) &(k = 0, 1, i = 1, \dots, e, 0 \leq \nu \leq v_i), \end{aligned}$$

where $I^k = I \otimes \dots \otimes I$ (k times), $I^0 = Z$.

Suppose we are given a sequence of abelian groups $A_k (-\infty < k < \infty)$ satisfying conditions (a), (b), (c). Since by (a) each A_k is of exponent st , it follows from [3, Theorem 6, p. 17] that A_k is a direct sum of cyclic groups. Let nA denote the direct sum of n copies of A , where A is either an abelian group or a G -module and n is a cardinal number. Then we can write

$$A_k = \sum_{i=1}^h \sum_{0 \leq \mu \leq u_i} m(k, i, \mu) Z(p_i^\mu) \oplus \sum_{i=1}^e \sum_{0 \leq \nu \leq v_i} n(k, i, \nu) Z(q_i^\nu),$$

where $m(k, i, \mu) = m(k + 4, i, \mu)$ ($i = 1, \dots, h, 0 \leq \mu \leq u_i$), $n(k, i, \nu) = n(k + 2, i, \nu)$ ($i = 1, \dots, e, 0 \leq \nu \leq v_i$) and $m(k, i, \mu), n(k, i, \nu)$ are cardinal numbers. Take

$$\begin{aligned} M &= \sum_{k=0}^3 \sum_{i=1}^h \sum_{0 \leq \mu \leq u_i} m(k, i, \mu) I^k \otimes M^1(i, \mu) \\ &\oplus \sum_{k=0}^1 \sum_{i=1}^e \sum_{0 \leq \nu \leq v_i} n(k, i, \nu) I^k \otimes M^2(i, \nu). \end{aligned}$$

Observe that $\hat{H}^{k-1}(G, K) \cong \hat{H}^k(G, I^1 \otimes K)$. Clearly $\hat{H}^k(G, M) = A_k$ ($-\infty < k < \infty$).

REMARK. In a similar but much simpler fashion one can show that all possible cohomology of a cyclic group G can also be realized by direct sums of G -modules of a certain finite family of G -modules \mathcal{F}' .

Addendum to the paper

“On Some Finite Groups And Their Cohomology”

(Received October 11, 1963)

Let group G have a presentation

$$(*) \quad G = \{ \sigma, \tau : \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^r \},$$

with the conditions

- (i) $0 < s$
- (ii) $((r - 1)t, s) = 1$
- (iii) $r^t \equiv 1 \pmod{s}$

(iv) there exists a positive integer n such that n is the order to which r belongs to moduli p_i ($i = 1, \dots, h$) (i.e. n is the smallest positive integer such that $r^n \equiv 1 \pmod{p_i}$), where $s = p_1^{n_1} \dots p_h^{n_h}$. Let s, t, P_1, P_2 , be as defined before (here q_1 is not necessarily $= 2$). It is clear from condition (iv) that G has P_1 -period equal to $2n$ and P_2 -period equal to 2.

PROPOSITION III. *Let G be a group having a presentation $(*)$ with the conditions (i), (ii), (iii), (iv). Then there exists a finite family of G -modules \mathcal{F} such that given any sequence of abelian groups $A_k (-\infty < k < \infty)$ satisfying the following conditions:*

- (a) *each A_k is of exponent st*
- (b) *the P_1 -period (in the obvious sense) of the sequence is $2n$*
- (c) *the P_2 -period of the sequence is 2,*

there exists a G -module M , which is a direct sum of G -modules of \mathcal{F} such that $\hat{H}^k(G, M) = A_k (-\infty < k < \infty)$.

Proof. Let $s(i, \mu), t(i, \nu), K^1(i, \mu), K^2(i, \nu)$, be as defined in Proposition II, Define G -operators on $K^1(i, \mu)$ and $K^2(i, \nu)$ by

$$\begin{aligned} \sigma x_j^{(i, \mu)} &= x_{j+1}^{(i, \mu)} && \text{(subscripts are modulo } s(i, \mu)) \\ \tau x_j^{(i, \mu)} &= x_{r \cdot j}^{(i, \mu)}, \\ \sigma y_j^{(i, \nu)} &= y_j^{(i, \nu)} && \text{(subscripts are modulo } t(i, \nu)) \\ \tau y_j^{(i, \nu)} &= y_{j+1}^{(i, \nu)}. \end{aligned}$$

By condition (iv) we have

$$\hat{H}^{2nk+i}(H, K^1(i, \mu))^{G/H} = (0) (i = 1, 2, \dots, 2n - 1).$$

The rest of the proof is parallel to that of Proposition II. \mathcal{F} consists of G -modules

$$\begin{aligned} I^k \otimes M^1(i, \mu) (k = 0, 1, \dots, 2n - 1; i = 1, \dots, h; \mu = 0, 1, \dots, u_i) \\ I^k \otimes M^2(i, \nu) (k = 0, 1; i = 1, 2, \dots, e; \nu = 0, 1, \dots, v_i). \end{aligned}$$

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