

Pacific Journal of Mathematics

ON SOME FINITE GROUPS AND THEIR COHOMOLOGY

KUNG-WEI YANG

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The purposes of this paper are: (I) to characterize the finite groups whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group and which have periodic cohomology of period 4, (II) to show that all possible cohomologies of such a group G can be realized by direct sums of G -modules which belong to a specific finite family of G -modules.

The author wishes to express his deep gratitude to Professor G. Whaples and Dr. K. Grant for many helpful suggestions and continual encouragement.

The reader is referred to [1, Ch. XII] for basic notions, definitions and notations concerning cohomology of finite groups. The only departure from [1, Ch. XII] is the following: we shall say that a finite group G has periodic cohomology of period k if k is the *least* positive integer such that $\hat{H}^k(G, Z)$ contains a maximal generator [1, pp. 260–261]. And to avoid typographical difficulties we will denote by $Z(l)$ the cyclic group of order l .

PROPOSITION I. *Let G be a finite group whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group. Then G has periodic cohomology of period 4 if and only if G has a presentation*

$$G = \{\sigma, \tau: \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\}, \text{ with the conditions}$$

- (i) s is an odd integer > 1 ,
- (ii) t is a positive even integer prime to s .

Proof. Let G be a finite group whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group and which has periodic cohomology of period 4. It is well-known [1, Theorem 11.6, p. 262] that if a finite group has periodic cohomology (of finite period) every Sylow subgroup of the group is either cyclic or is a generalized quaternion group. Since we assume that the 2-Sylow subgroups of G are not isomorphic to a generalized quaternion group, every Sylow subgroup of G is cyclic. It is also well-known [6, Theorem 11, p. 175] that a finite group G containing only cyclic Sylow subgroups is metacyclic and has a presentation

$$G = \{\sigma, \tau: \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^r\}, \text{ with the conditions}$$

- (1) $0 < s$, (st = the order of the group G),
 (2) $((r - 1)t, s) = 1$
 (3) $r^t \equiv 1 \pmod{s}$, and conversely.

We observe that if $s = 1$ or $t = 1$ or $r = 1$ the finite group G is cyclic and G has periodic cohomology of period 2 (or 0). These cases are therefore excluded. On the other hand, once these exceptional cases are excluded G is no more a cyclic group and it will have periodic cohomology of period ≥ 4 .

Notice that (1), (2) and (3) imply (i)

Let H be the subgroup of G generated by the element σ . H is clearly a cyclic normal subgroup of order s . And G/H is cyclic of order t . By condition (2), s and t are relatively prime to each other. We can therefore apply the decomposition theorem of Hochschild-Serre [2, Theorem 1, p. 127] and obtain

$$(4) \quad \hat{H}^k(G, K) \cong \hat{H}^k(G/H, K^H) \oplus (\hat{H}^k(H, K))^{G/H},$$

for all $k > 0$ and for all G -module K . (For $k > 0$, $\hat{H}^k(G, K) = H^k(G, K)$). In particular, we have

$$\hat{H}^k(G, Z) \cong \hat{H}^k(G/H, Z) \oplus (\hat{H}^k(H, Z))^{G/H},$$

for $k > 0$. The G/H -operators on $\hat{H}^k(H, K)$ are explicitly described in [2, p. 117]. In particular, G/H -operators on $\hat{H}^k(H, Z)$ are induced by the automorphisms of H which are themselves induced, on H , by inner automorphisms of G . In the present situation, all such automorphisms of H are generated by the automorphism $f(\rho) = \rho^r (= \tau\rho\tau^{-1})$, where $\rho \in H$. The automorphism $f: H \rightarrow H$ induces an automorphism f^* of $\hat{H}^k(H, Z)$ [4, Lemma 3, p. 343] such that if $g_{2k} \in \hat{H}^{2k}(H, Z)$, then $f^*(g_{2k}) = r^k g_{2k}$. Therefore $\hat{H}^k(G, Z)$ has a maximal generator, i.e. G has periodic cohomology of period ≤ 4 if and only if $f^*(g) = g$ for all $g \in \hat{H}^k(H, Z)$. This is equivalent to

$$(5) \quad r^2 \equiv 1 \pmod{s}.$$

(We recall that $r = 1$ we excluded). An elementary number theoretic calculation shows that the only solution for r in (2) and (5) is $r \equiv -1 \pmod{s}$. Therefore the number t in (3) is an even positive integer (if it is negative, we can present G by letting $\tau' = \tau^{-1}$). This shows that the finite group G has a presentation as mentioned above.

The converse of the proposition is obvious.

We know that if l is the order of the group G then for any G -module K all the cohomology groups $\hat{H}^k(G, K)$ ($-\infty < k < \infty$) are of exponent l —that is, for all $g \in \hat{H}^k(G, K)$, $lg = 0$. Let

$$s = p_1^{a_1} \cdots p_k^{a_k}, P_1 = \{p_1, \cdots, p_k\} \quad \text{and} \quad t = q_1^{b_1} \cdots q_r^{b_r}, P_2 = \{q_1, \cdots, q_r\}$$

be decompositions of s and t into products of prime powers (where

$q_1 = 2$ and $v_1 \geq 1$). It is obvious from (4) that a group with periodic cohomology of period 4 has P_2 -period [1, Exercise 11, p. 265] equal to 2. Conversely, we have

PROPOSITION II. *Let G be a group having a presentation*

$$G = \{\sigma, \tau: \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\} \text{ with the conditions}$$

- (i) s is an odd integer > 1 .
- (ii) t is a positive even integer prime to s .

Let P_1, P_2 be as defined above. Then there exists a finite family of G -modules \mathcal{F} such that given any sequence of abelian groups $A_k (-\infty < k < \infty)$ satisfying

- (a) each A_k is of exponent st ,
- (b) the sequence is periodic of period 4,
- (c) the P_2 -period of the sequence is equal to 2, then there exists

a G -module M which is a direct sum of G -modules of \mathcal{F} such that $\hat{H}^k(G, M) = A_k (-\infty < k < \infty)$.

First we observe the following

LEMMA. *Let G be a finite group and let K be a G -module. Let S be a set of primes in the ring of integers Z and let $Q(S)$ be the quotient ring [5, p. 46] of Z with respect to the multiplicative system generated by S . (As usual when $Q(S)$ is considered as a G -module it is to be understood that G operates trivially on (the additive group of) $Q(S)$). Then*

$$\hat{H}^k(G, K \otimes Q(S)) \cong \hat{H}^k(G, K) \otimes Q(S) (-\infty < k < \infty),$$

where $\otimes = \otimes_Z$

The proof is immediate.

Proof of Proposition II. Let s, t, P_1, P_2 be as before. Let

$$\begin{aligned} s(i, \mu) &= s/p_i^\mu (i = 1, \dots, h, 0 \leq \mu \leq u_i), \\ t(i, \nu) &= t/q_i^\nu (i = 1, \dots, e, 0 \leq \nu \leq v_i). \end{aligned}$$

Let $K^1(i, \mu) = \sum_{j=1}^{s(i, \mu)} Zx_j^{(i, \mu)}$ (direct sum on the symbols $x_j^{(i, \mu)}$)

$$K^2(i, \nu) = \sum_{j=1}^{t(i, \nu)} Zy_j^{(i, \nu)} \text{ (direct sum on the symbols } y_j^{(i, \nu)} \text{).}$$

Define G -operators on $K^1(i, \mu)$ and $K^2(i, \nu)$ by

$$\begin{aligned} \sigma x_j^{(i, \mu)} &= x_{j+1}^{(i, \mu)} \\ \tau x_j^{(i, \mu)} &= x_{-j}^{(i, \mu)}, \end{aligned} \text{ (subscripts are modulo } s(i, \mu) \text{)}$$

$$\begin{aligned} \sigma y_j^{(i, \nu)} &= y_j^{(i, \nu)} \\ \tau y_j^{(i, \nu)} &= y_j^{(i, \nu)}. \end{aligned} \quad (\text{subscripts are modulo } t(i, \nu)).$$

Let

$$\begin{aligned} M^1(i, \mu) &= K^1(i, \mu) \otimes Q((P_1 - \{p_i\}) \cup P_2), \\ M^2(i, \nu) &= K^2(i, \nu) \otimes Q(P_1 \cup (P_2 - \{q_i\})). \end{aligned}$$

By (4), the above lemma and the fact that $(\hat{H}^{4k+2}(H, K^1(i, \mu))^{G/H} = (0)$, one shows

$$\begin{aligned} \hat{H}^{4k}(G, M^1(i, \mu)) &= Z(p_i^\mu) & \hat{H}^{4k}(G, M^2(i, \nu)) &= Z(q_i^\nu) \\ \hat{H}^{4k+1}(G, M^1(i, \mu)) &= (0) & \hat{H}^{4k+1}(G, M^2(i, \nu)) &= (0) \\ \hat{H}^{4k+2}(G, M^1(i, \mu)) &= (0) & \hat{H}^{4k+2}(G, M^2(i, \nu)) &= Z(q_i^\nu) \\ \hat{H}^{4k+3}(G, M^1(i, \mu)) &= (0) & \hat{H}^{4k+3}(G, M^2(i, \nu)) &= (0) \end{aligned}$$

The calculation is purely mechanical.

Now, let $0 \rightarrow I \rightarrow Z[G] \xrightarrow{\varepsilon} Z \rightarrow 0$, where $\varepsilon(\sum_{\sigma \in G} l_\sigma \sigma) = \sum_{\sigma \in G} l_\sigma$, $I = \text{Ker}(\varepsilon)$, and let \mathcal{F} consist of

$$\begin{aligned} I^k \otimes M^1(i, \mu) &(k = 0, 1, 2, 3, i = 1, \dots, h, 0 \leq \mu \leq u_i) \\ I^k \otimes M^2(i, \nu) &(k = 0, 1, i = 1, \dots, e, 0 \leq \nu \leq v_i), \end{aligned}$$

where $I^k = I \otimes \dots \otimes I$ (k times), $I^0 = Z$.

Suppose we are given a sequence of abelian groups A_k ($-\infty < k < \infty$) satisfying conditions (a), (b), (c). Since by (a) each A_k is of exponent st , it follows from [3, Theorem 6, p. 17] that A_k is a direct sum of cyclic groups. Let nA denote the direct sum of n copies of A , where A is either an abelian group or a G -module and n is a cardinal number. Then we can write

$$A_k = \sum_{i=1}^h \sum_{0 \leq \mu \leq u_i} m(k, i, \mu) Z(p_i^\mu) \oplus \sum_{i=1}^e \sum_{0 \leq \nu \leq v_i} n(k, i, \nu) Z(q_i^\nu),$$

where $m(k, i, \mu) = m(k+4, i, \mu)$ ($i = 1, \dots, h, 0 \leq \mu \leq u_i$), $n(k, i, \nu) = n(k+2, i, \nu)$ ($i = 1, \dots, e, 0 \leq \nu \leq v_i$) and $m(k, i, \mu), n(k, i, \nu)$ are cardinal numbers. Take

$$\begin{aligned} M &= \sum_{k=0}^3 \sum_{i=1}^h \sum_{0 \leq \mu \leq u_i} m(k, i, \mu) I^k \otimes M^1(i, \mu) \\ &\quad \oplus \sum_{k=0}^1 \sum_{i=1}^e \sum_{0 \leq \nu \leq v_i} n(k, i, \nu) I^k \otimes M^2(i, \nu). \end{aligned}$$

Observe that $\hat{H}^{k-l}(G, K) \cong \hat{H}^k(G, I^l \otimes K)$. Clearly $\hat{H}^k(G, M) = A_k$ ($-\infty < k < \infty$).

REMARK. In a similar but much simpler fashion one can show that all possible cohomology of a cyclic group G can also be realized by direct sums of G -modules of a certain finite family of G -modules \mathcal{F}' .

Addendum to the paper

“On Some Finite Groups And Their Cohomology”

(Received October 11, 1963)

Let group G have a presentation

$$(*) \quad G = \{ \sigma, \tau : \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^r \},$$

with the conditions

(i) $0 < s$

(ii) $((r - 1)t, s) = 1$

(iii) $r^t \equiv 1 \pmod{s}$

(iv) there exists a positive integer n such that n is the order to which r belongs to moduli p_i ($i = 1, \dots, h$) (i.e. n is the smallest positive integer such that $r^n \equiv 1 \pmod{p_i}$), where $s = p_1^{u_1} \dots p_h^{u_h}$. Let s, t, P_1, P_2 , be as defined before (here q_1 is not necessarily =2). It is clear from condition (iv) that G has P_1 -period equal to $2n$ and P_2 -period equal to 2.

PROPOSITION III. *Let G be a group having a presentation (*) with the conditions (i), (ii), (iii), (iv). Then there exists a finite family of G -modules \mathcal{F} such that given any sequence of abelian groups A_k ($-\infty < k < \infty$) satisfying the following conditions:*

(a) *each A_k is of exponent st*

(b) *the P_1 -period (in the obvious sense) of the sequence is $2n$*

(c) *the P_2 -period of the sequence is 2,*

there exists a G -module M , which is a direct sum of G -modules of \mathcal{F} such that $\hat{H}^k(G, M) = A_k$ ($-\infty < k < \infty$).

Proof. Let $s(i, \mu), t(i, \nu), K^1(i, \mu), K^2(i, \nu)$, be as defined in Proposition II, Define G -operators on $K^1(i, \mu)$ and $K^2(i, \nu)$ by

$$\sigma x_j^{(i, \mu)} = x_{j+1}^{(i, \mu)} \quad (\text{subscripts are modulo } s(i, \mu))$$

$$\tau x_j^{(i, \mu)} = x_{r \cdot j}^{(i, \mu)},$$

$$\sigma y_j^{(i, \nu)} = y_j^{(i, \nu)} \quad (\text{subscripts are modulo } t(i, \nu)).$$

$$\tau y_j^{(i, \nu)} = y_{j+1}^{(i, \nu)}$$

By condition (iv) we have

$$\hat{H}^{2nk+i}(H, K^1(i, \mu))^{G|H} = (0) \quad (i = 1, 2, \dots, 2n - 1).$$

The rest of the proof is parallel to that of Proposition II. \mathcal{F} consists of G -modules

$$I^k \otimes M^1(i, \mu) \quad (k = 0, 1, \dots, 2n - 1; i = 1, \dots, h; \mu = 0, 1, \dots, u_i)$$

$$I^k \otimes M^2(i, \nu) \quad (k = 0, 1; i = 1, 2, \dots, e; \nu = 0, 1, \dots, v_i).$$

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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