

# Pacific Journal of Mathematics

**A RESULT CONCERNING INTEGRAL BINARY QUADRATIC  
FORMS**

WILLIAM EDWARD CHRISTILLES

# A RESULT CONCERNING INTEGRAL BINARY QUADRATIC FORMS

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This paper contains an extension of an earlier work by Dickson ([1], p. 95), in which the following theorem was proven:

**THEOREM 1.** (Dickson's Theorem). *If a number is represented properly by a form  $[a, b, c]$  of discriminant  $D = 4ac - b^2$ , then any divisor of that number is represented by some form of the same discriminant  $D$ .*

**DEFINITION.** ([1], p. 68). A positive form  $[a, b, c]$  is called reduced if  $-a < b \leq a, c \geq a$ , with  $b \geq 0$  if  $c = a$ .

As a consequence of the above definition it follows that  $4a^2 \leq 4ac = D + b^2 \leq D + a^2, 3a^2 \leq D$ , and finally  $a \leq \sqrt{(1/3)D}$

**THEOREM 2.** *Let  $M$  be properly represented by the integral positive definite quadratic form  $ax^2 + bxy + cy^2$  of discriminant  $D = 4ac - b^2$ . If  $M \leq 3D/16$  and  $(D, M) = 1$ , then in every factorization of  $M$  one of the factors is  $a_i$ , one of the minimal values of a primitive quadratic form of discriminant  $D$ . In other words,  $M = M_1M_2$  where  $M_1$  is a unit or a prime and  $M_2$  is the product of no more than two  $a_i$ .*

*Proof.* According to the remark following the definition  $a_i \leq \sqrt{D/3}$ , where equality for a primitive reduced form is possible only if  $a_i = b_i = c_i = 1$  and hence  $D = 3$  so that the inequality  $0 < M \leq 3D/16$  cannot be satisfied. Thus  $a_i < \sqrt{D/3}$ .

Now assume  $M = r_1r_2$ . Then according to Theorem 1 it follows that

$$r_1 = a_i\alpha_i^2 + b_i\alpha_i\gamma_i + c_i\gamma_i^2, \quad r_2 = a_j\alpha_j^2 + b_j\alpha_j\gamma_j + c_j\gamma_j^2$$

where the two quadratic forms are primitive reduced forms of discriminant  $D$ . Hence

$$\begin{aligned} (4a_i r_1)(4a_j r_2) &= [(2a_i\alpha_i + b_i\gamma_i)^2 + D\gamma_i^2] [(2a_j\alpha_j + b_j\gamma_j)^2 + D\gamma_j^2] \\ &= (\beta_i^2 + D\gamma_i^2)(\beta_j^2 + D\gamma_j^2) = 16a_i a_j M \\ &< 16(D/3)M \leq (16D/3)(3D/16) = D^2, \end{aligned}$$

where  $\beta_i = (2a_i\alpha_i + b_i\gamma_i)$  and  $\beta_j = (2a_j\alpha_j + b_j\gamma_j)$ . This implies that  $\gamma_i\gamma_j = 0$ , say  $\gamma_i = 0$ , and therefore  $r_1 = a_i$ .

To prove the final statement of the theorem, assume  $M \neq a_i$  and

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let  $r_2$  be a minimal factor of  $M$  so that  $r_2 \neq a_j$ . If  $M_1$  is any prime factor of  $r_2$ , then  $M = M_1 M_2$  where  $M_2 = (M/r_2)(r_2/M_1) = a_i a_j$ .

#### REFERENCE

1. L. E. Dickson, *Introduction to the Theory of Numbers*, Dover Publications, Inc., New York, 1929.

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