ANOTHER CHARACTERIZATION OF THE $n$-SPHERE AND RELATED RESULTS

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R. F. DICKMAN, L. R. RUBIN AND P. M. SWINGLE

In [5] we defined an irreducible $B(J)$-cartesian membrane and an excluded middle membrane property $EM$, and used these to characterize the $n$-sphere. There the class $B(J)$ was of $(n-1)$-spheres contained in a compact metric space $S$. Since part of the proof does not depend upon the fact that elements of $B(J)$ are $(n-1)$-spheres, we consider the possibility of other entries in the class $B(J)$. Recent developments in this direction have been made by Bing in [2] and by Andrews and Curtis in [1]. In [3] and [4] Bing constructed a space $B$ not homeomorphic with $E^3$, which has been called the dogbone space. By Theorem 6 of [2], the sum of two cones over the one point compactification $\bar{B}$ of $B$ is homeomorphic with $S^4$. This sum of two cones over a common base $X$ is called the suspension of $X$.

In [1] Andrews and Curtis showed that if $\alpha$ is a wild arc in $S^n$ that the decomposition space $S^n/\alpha$ is not homeomorphic with $S^n$. They proved, however, that the suspension of $S^n/\alpha$ is always homeomorphic with $S^{n+1}$ for any arc $\alpha \subset S^n$. The reader will easily see that a class $B$ of $S^n/\alpha$ as described will satisfy the conditions for a class $B(J)$ for which an $n$-sphere will have property $EM$.

The results below were obtained in considering such spaces, and Theorem 1 below is a weaker characterization of the $n$-sphere than is Theorem 2 of [5]. We find it difficult to determine the properties $J \in B(J)$ must have for $S$ to have Property $EM$, as is shown by our Theorem 4 below.

I. Definition and basic properties. Let $S$ always be a compact metric space and let $B(J)$ be a class of mutually homeomorphic subcontinua of $S$. We put conditions on this general class $B(J)$ in our theorems below.

We define a $B(J)$-cartesian membrane as we did in [5] and [6]. Let $F$ be a compact subset of $S$ containing $J \in B(J)$. Let $M$ be a subcontinuum of $F$, $b \in M$ and $C$ be homeomorphic to $J$. Denote by $(C \times M, b)$ the decomposition space [10: pp 273–274] of the upper semi-continuous decomposition of the cartesian product $C \times M$, where the only nondegenerate element is taken to be $C \times b$ (intuitively the decomposition space is a sort of generalized cone with vertex at the point $C \times b$). With this notation we give:

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DEFINITION 1. We say that $F$ is a $B(J)$-cartesian membrane from $b$ to $J$ (or for brevity with base $J$) if and only if there is a homeomorphism $h$ from $(C \times M, b)$ onto $F$ for some $M$ such that:

(i) for some $a \in M - b$, $J = h(C \times a)$,
(ii) for all $q \in M - b$, $h(C \times q) \in B(J)$, and
(iii) $h(C \times b) = b$.

If $M$ is irreducible from $a$ to $b$, then we prefix the above definition by "irreducible." Whenever $F$ is a $B(J)$-cartesian membrane and $F = h(C \times m, b)$, $h$ is assumed to be a homeomorphism from $(C \times M, b)$ onto $F$ with properties (i), (ii) and (iii). We say $b$ is the vertex of $F$ and $J$ is the base of $F$.

The definition of $B(J)$-cartesian membrane is rather general; for example, a point or any continuum can be taken as a $B(J)$-cartesian membrane. We shall place restrictions on the space $S$ to limit possibilities such as these when the need arises. The excluded middle membrane property of Theorem 2 in [5] is the following:

Property EM. We say that the space $S$ has Property EM with respect to the class $B(J)$ if the following hold:

1. The class $B(J)$ is not empty;
2. For each $J \in B(J)$, $S = F'_1 + F'_2$ where $F'_1$ and $F'_2$ are irreducible $B(J)$-cartesian membranes with base $J$, such that $F'_1 \not\subset F'_2$ and $F'_2 \not\subset F'_1$ and whenever $S$ is such a union and $F'_3$ is any other $B(J)$-cartesian membrane containing $J$, then $F'_3$ contains $F'_1$ or $F'_2$ but not both; and
3. If $J \in B(J)$ and $p \in S - J$, then there exists a $B(J)$-cartesian membrane from $p$ to $J$.

Below $F$, $F'$, $F'_1$ and $F'_2$ are always irreducible $B(J)$-cartesian membranes.

We proved in [5] that when $B(J)$ is a class of $(n - 1)$-spheres and $n > 1$ that:

(A) A necessary and sufficient condition that $S$ be an $n$-sphere is that $S$ have Property EM.

We observed in our proof of (A) that if $S$ had Property EM with respect to a class of mutually homeomorphic continua, we were able to prove:

(B) That whenever $S = F'_1 + F'_2$ where $F'_1$ and $F'_2$ have base $J$, $F'_1 \cdot F'_2 = J$;
(C) If $F = h(C \times M, b)$ was an irreducible $B(J)$-cartesian membrane, then $M$ was always a simple continuous arc with $b$ as endpoint; and
(D) If $S = F'_1 + F'_2$ where $F'_1$ and $F'_2$ have base $J$ and $F'_3$ is any other irreducible $B(J)$-cartesian membrane with base $J$, then $F'_1 = F'_3$ or $F'_2 = F'_3$. 

In the first paragraph of the proof of Theorem 2 of [5], (D) appeared easily as result (R_1); then by a long proof we showed that $F_1 \cap F_2 = J$, which is (B) above, and we note this long proof only depends upon $J$ being a continuum, not on $J$ being an $(n-1)$-sphere. Finally, the following argument show that (C) holds. Let $S = F_1 + F_2$, where $F_1$ and $F_2$ are irreducible $B(J)$-cartesian membranes with base $J$. By (B) $F_1 \cdot F_2 = J$, and so every element of $B(J)$ separates $S$. Then if $F_1 = h(C \times M, b)$ where $M$ is irreducible from $a$ to $b$, and if $z \in M - a - b$, $h(C \times z) \in B(J)$ by (ii) of Definition 1 above. Hence $h(C \times z)$ separates $S$, and therefore separates $F_1$. This implies $z$ separates $M$, and so $M$ is a simple continuous arc, as desired in (C).

II. Characterization of the $n$-sphere, for $n > 1$. We give now several lemmas that will enable us to characterize the $n$-sphere.

**NOTATION.** For a subset $K$ of $S$, we will use $\text{cl}(K)$ to denote the closure of $K$ in $S$, and for an open subset $U$ of $S$, we will use $\text{Fr}(U)$ to denote the set $\text{cl}(U) - U$.

**Lemma 1.** If $S$ has Property EM, then $S$ is homogeneous.

**Proof.** Let $x, y \in S$, $x \neq y$, and let $J$ be an element of $B(J)$ such that $J \subset S - x - y$. By (3) of Property EM there exists an irreducible $B(J)$-cartesian membrane $F = h(C \times M, x)$ from $x$ to $J$ and by (D) and (2) of Property EM, $S = F + F''$, where $F''$ has base $J$. Now by (B) each $J' \in B(J)$ separates $S$, hence by (ii) of Definition 1, some $J_0 = h(C \times q)$ separates $x$ from $y$. Then by (2) of Property EM, $S = F_1 + F_2$ where $F_1$ and $F_2$ have base $J_0$. From (D) and (3) of Property EM there exists $h_1$ and $h_2$ such that $F_1 = h_1(C \times M_1, x)$ and $F_2 = h_2(C \times M_2, y)$. From (C) $M_1$ and $M_2$ are simple continuous arcs and $x$ and $y$ are endpoints of $M_1$ and $M_2$, respectively. Hence from (B) there exists a homeomorphism from $S$ onto $S$ that carries $x$ onto $y$; therefore $S$ is homogeneous [10: p 378].

A topological space $X$ is invertible [7] if for each nonempty open set $U$ in $X$ there is a homeomorphism $h$ of $X$ onto itself such that $h(X - U)$ lies in $U$.

**Lemma 2.** If $S$ has Property EM then $S$ is invertible.

**Proof.** For any open set $U$ in $S$ and any point $x \in U$, some $J \in B(J)$ separates $x$ from $\text{Fr}(U)$; then if $S = F_1 + F_2$ where $F_1$ and $F_2$ have base $J$, we can find a homeomorphism as in Lemma 1, that maps $S$ onto $S$ such that $F_1$ maps onto $F_1$ and $F_2$ maps onto $F_2$, hence $(S - U)$ into $U$. 
THEOREM 1. Let \( n > 1 \) and let each element of \( B(J) \) contain a point at which it is locally euclidean of dimension \((n - 1)\). Then \( S \) is an \( n \)-sphere if and only if \( S \) has Property EM.

Proof of the sufficiency. Let \( J \in B(J) \) and let \( x \) be an element of \( J \) at which \( J \) is locally euclidean of dimension \((n - 1)\). Let \( U \) be an open \((n - 1)\)-cell neighborhood of \( x \) in \( J \). Let \( F = h(C \times M, b) \) have base \( J \). By (C), \( M \) is an arc, and if \( V \) is an open subinterval of \( M \) containing a point \( y \), \( h(U \times V) \) is an open \( n \)-cell neighborhood of \( h(x, y) \) in \( F \). Since \( h(U \times V) \) misses \( J \), \( h(U \times V) \) is open in \( F - J \), and hence in \( S \). By Lemma 1, \( S \) is homogeneous; hence every element of \( S \) has an open \( n \)-cell neighborhood, and so \( S \) is \( n \)-manifold. Doyle and Hocking in Theorem 1 of [7], have shown that if \( S \) is an invertible, \( n \)-manifold, then \( S \) is an \( n \)-sphere; hence by Lemma 2, \( S \) is an \( n \)-sphere.

The proof of the necessity is identical to that of Theorem 2 in [5].

Because 0-spheres are not connected the above proof does not hold for \( n = 1 \). We refer the reader to Theorem 1 of [5] for a characterization of the 1-sphere by an excluded middle membrane principle.

III. Related results.

LEMMA 3. If \( S \) has Property EM then \( S \) is locally connected.

Proof. We note that if \( F \) is an irreducible \( B(J) \)-cartesian membrane with base \( J \), then \( F - J \) is an open connected set in \( S \), and proceed as in the proof of Lemma 2.

LEMMA 4. If \( S \) has Property EM and \( J \in B(J) \) then \( J \) is locally connected.

Proof. Let \( S = F_1 + F \) where \( F_1 \) and \( F \) have base \( J \) and \( F = h(C \times M, b) \), where \( M \) is an arc from \( a \) to \( b \); and \( h(C \times a) = J \) as in (1) of Definition 1. Since \( S \) is locally connected, the open set \( F - J - b \) is locally connected. We define \( f(h(c, m)) = h(c, a) \), where \( h(c, m) \) is a point in \( F - J - b \); then \( f \) is a projection onto \( J \) and can easily be proved to be continuous and open. Since \( F - J - b \) is locally connected and local connectedness is preserved under open, continuous mappings, \( J \) is locally connected.

THEOREM 2. If \( S \) has Property EM and \( J \in B(J) \), then \( J \) contains a 1-sphere.

Proof. Let \( J \in B(J) \), and \( F = h(C \times M, b) \) have vertex \( b = h(C \times b) \) and base \( J \). Since \( J \) is locally connected, \( C \) must contain an arc \( I \);
and by (C), $M$ is an arc. Then the set $E' = h(I \times M, b)$ is a closed 2-cell contained in $F$. Let $E$ be any subset of $E'$ that is homeomorphic to euclidean 2-space $E^2$.

Let $b_i$ ($i = 1, 2, \cdots$) be a sequence converging to $b$ in $M$. Then the half open intervals $M_i = b - b_i$ form a basis of open sets in $M$ at $b$, and the sets $U_i(b) = h(C \times M, b)$ form a basis of open sets in $F$ at $b$. These open sets have the property that $Fr(U_i(b))$ is homeomorphic to $J$.

Choose $x \in E$, then $x \notin J$. By the homogeneity of $S$ there exists a basis of open sets $U_i(x)$ which have the property that their boundaries are homeomorphic to $J$. Now fix $i$ such that $U = U_i(x) \cdot E$ has a compact closure in $E$. Let $V$ be the component of $U$ that contains $x$. Since $E$ is locally connected, $V$ is open in $E$. Also $Fr(V) \subset Fr(U_i(x))$; therefore without loss of generality we can think of $Fr(V)$ as being a subset of $J$. Let $V'$ be a component of $E - \text{cl}(V)$. Then $V'$ is an open connected subset of $E$ and $Fr(V') \subset Fr(V)$. Since $Fr(V')$ is closed and $Fr(V')$ compact, $Fr(V')$ is compact. By Theorem 25 of [10: p 176], $Fr(V')$ is a continuum. Then by Theorem 28 of [10: p 178], $Fr(V')$ is not disconnected by the omission of any point.

Let $r, s \in Fr(V')$, and let $Y$ be an arc from $r$ to $s$ in $J$. Let $q \in Y - r - s$; now $q$ does not separate $r$ from $s$ in $Fr(V')$; hence $q$ does not separate $r$ from $s$ in $J$; then there exists an arc $Y'$ from $r$ to $s$ in $J$ that does not contain $q$, and $Y + Y'$ must contain a 1-sphere.

REMARK. Since $J$ is locally connected, $J$ is arcwise connected and as such cannot be an indecomposable continuum; by Theorem 2, $J$ cannot be hereditarily unicoherent. A simple proof using the Brouwer Invariance of Domain Theorem [9: p. 95] will show that $J$ cannot be a closed $n$-cell.

**Lemma 5**. Let $S$ be an $n$-sphere having Property EM with respect to some $B(J)$. (1) If $G$ is an $(n - 2)$-sphere in $J \in B(J)$, then $J - G$ is not connected; (2) if $E$ is a closed $(n - 2)$-cell in $J$, then $J - E$ is connected.

**Proof.** (1) Suppose $J - G$ is connected. Let $S = F_1 + F_2$ where $F_1$ and $F_2$ have base $J$; by (B) and (C) we can find $h_1$ and $h_2$ such that $F_1 = h_1(J \times M_1, b_1)$, $F_2 = h_2(J \times M_2, b_2)$ and $h_1| (J \times a) = h_2| (J \times a)$ where $M_1$ and $M_2$ are arcs from $a$ to $b_1$ and $a$ to $b_2$ respectively. Then $K = h_1((J - G) \times (M_1 - b_1)) + h_2((J - G) \times (M_2 - b_2))$ is connected. But $S - K = h_1(G \times M_1, b_1) + h_2(G \times M_2, b_2)$ is an $(n - 1)$-sphere is $S$ and must disconnect $S$ by the Jordan Separation Theorem [9: p. 101].

The proof of (2) is similar to that of (1).
THEOREM 3. A necessary and sufficient condition that $S$ be a 3-sphere is that $S$ have Property EM if and only if $B(J)$ is a collection of 2-spheres.

Proof. The sufficiency follows from Theorem 2 of [5].

By Theorem 2, every $J \in B(J)$ contains a 1-sphere, and by (1) of Lemma 5 every 1-sphere in $J$ separates $J$. By (2) of Lemma 5 no proper subcontinuum of a 1-sphere in $J$ separates $J$; and by Lemma 4, $J$ is locally connected; therefore by Zippin’s Characterization in [11: p. 88] $J$ is a 2-sphere. The rest follows from Theorem 2 of [5].

We need Hypothesis:

(\text{H} 1) If $F_c$, $F_b$ and $F''$ are irreducible $B(J_0)$-cartesian membranes with base $J_0$ then $F_c + F_b + F''$ is contained in some $E^3$;

(\text{H} 2) If $S_x = F_x + F''$ is a 2-sphere in $E^3$, $x$ is vertex of $B(J_0)$-cartesian membrane $F_x$ and $t'_a = h_a(c_a \times M'', x)$ ($c_a \in C$) is a projecting arc from $x$ to $J$ through a point $y \in \text{int}(S_x, E^3)$, (the interior of $S_x$ in $E^3$), then $t'_a - x \subset \text{int}(S_x, E^3)$; if $q \in \text{int}(S_x, E^3) \cdot J = J'$, then $q \notin \text{cl}(J - J')$.

THEOREM 4. Let $S$ have Property EM, let (\text{H} 1) and (\text{H} 2) hold and let there exist a region $R$ in $S$ such that $J \cdot R$ contains a 1-sphere $J_0$ and $R \cdot J$ is embedded in the euclidean $E^3$; let there exist $q \in J - R$. Then $J$ contains a closed 2-cell with $J_0$ as boundary.

Proof. By (2) of Property EM there exist irreducible $B(J)$-cartesian membranes such that $S = h(C \times M, b) + h'(C \times M', b')$ where $h \mid (C \times a) = h' \mid (C \times a)$ and $M, M'$ are arcs from $a$ to $b$ and $a$ to $b'$ respectively; since $J \supset J_0$, there exists $C_0 \subset C$ homeomorphic to $J_0$; let $h(C_0 \times M, b) = F_b$ and $h'(C_0 \times M', b') = F''$, where then $F_b$ and $F''$ are irreducible $B(J_0)$-cartesian membranes from $J_0$ to $b$ and $b'$ respectively. Let $S_b = F_b + F''$; by Theorem 2 of [5], $S_b$ is a 2-sphere.

By hypothesis there exists $q \in J - R$; thus $q \notin S_b$, and so by (\text{H} 2) the projecting arc from $b$ to $q$ does not contain a point of $\text{int}(S_b, E^3)$; let $c$ be an element of this projecting arc. By (3) of Property EM, there exists an irreducible $B(J)$-cartesian membrane $F_c = h_c(C \times M_c, c)$ with base $J_0$, a subset of an irreducible $B(J)$-cartesian membrane $h_c(C \times M_c, c)$ from $c$ to $J$; by the choice of $c$, $h_c(C \times M_c, c) = h(C \times M, b)$ and thus $S_c = F_c + F''$ is a 2-sphere.

Since $c \notin \text{int}(S_b, E^3)$, there exists a region $R'$ about $c$ such that $\text{cl}(R') \cdot S_b = \emptyset$; then by Lemma 3 of [6] there exists an irreducible $B(J)$-cartesian membrane $F_{oc} = h_c(C \times M'_c, c)$, for $M'_c \subset M_c$, such that $F_c \cdot R' \supset F_{oc}$.

Let $\{t_{ac}\}$ be the class of all projecting subarcs from $c$ to $J$ which
are contained in \((S_c - (F_{oc} - J')) + \text{int} (S_c, E^3) - (F_{oc} - J')\), where \(J'\) is the base of \(F_{oc}\); that is \(t_{ac}\) is an arc from \(J\) to \(F_{oc}\) in and on \(S_c\).

Let \(Z' = \bigcup t_{ac}\) and let \(Z = Z' \cdot J\). Suppose \(Z' = Z'_1 + Z'_2\) separate \([11: \text{p. 8}]\). Since each \(t_{ac}\) is connected, each is contained wholly in \(Z\) or in \(Z'\); this is also true of \(J_0\) and so of \(F_1 - F_{oc}\); so let \(Z_1 \supseteq F_0 - F_{oc} \supseteq J_0\).

By Theorem 5.37 of \([11: \text{p. 66}]\) \(S_c\) is arcwise accessible from the embedding \(E^3\); hence there exists an arc \(cb'\) such that \(cb' - c - b' \subset \text{int} (S_c, E^3)\). But \(cb'\) contains a point of \(\text{int} (S_b, E^3)\) and a point \(c\) of \(S - \text{int} (S_b, E^3) - S_b\); hence \(cb'\) contains some \(v \in S_b\), because by the Jordan-Brouwer Separation Theorem \([11: \text{Theorem 5.23, p. 63}]\) \(S_b\) separates \(E^3\) into two domains. Hence by (2) of Property \(EM\) there exists a projecting arc from \(c\) to \(J\) through \(v\), and so some \(t_{ac} \supseteq v\) and \(Z' \supseteq t_{ac}\). Let \(Z_i = Z'_i \cdot Z(i = 1, 2)\), where by agreement \(Z_1 \supseteq J_0\). By hypothesis \(J \cdot R\) is contained in some euclidean \(E^3\), and so let \(E\) be the 2-cell bounded by \(J_0\) in this \(E^3\). Thus \(J_0 + E \supseteq Z\), and because of \(v\) above \(E \cdot Z \neq \phi\). If \(j \in J \cdot E\), by \((H2)\) the projecting arc \(cj\) is such that \(cj - c \subset \text{int} (S_c, E^3)\). Thus \(j \in Z\), and so \(Z = J_0 + J \cdot E = Z_1 + Z_2\) separate. Hence \(J = (Z_1 + (J - E)) + Z_2\) separate, which is a contradiction, since \(J\) is a continuum. Therefore \(Z\) and \(Z'\) are connected. By Lemma 4 \(J\) is locally connected, and so by \((H2)\) \(Z\) is also.

Since \(Z\) is closed, \(Z\) contains all of its boundary points in the space \(J\). By the Torhorst Theorem \([10: \text{p. 191, Theorem 42}]\), the boundary of any complementary domain of \(Z\) in \(E\) must be a 1-sphere \(J'\). Using \(J'_0\) in place of \(J_0\), one obtains a 2-sphere \(S'_0\) with poles \(c\) and \(b'\) and with \(J'_0\) as a base in \(S'_0\). Thus an arc \(bc'\) above exists such that \(bc' - c - b' \subset \text{int} (S'_0, E^3)\) and there exists a point \(v \in S'_0 \cdot cb'\); also there exists \(t_{ac}\) as above, now contained in the \(\text{int} (S'_0, E^3)\); hence an endpoint of \(t_{ac}\) is an element of \(\text{int} (J'_0, E^3)\); thus a point of \(Z\) is in the complementary domain above of \(Z\) in \(E\), which is a contradiction. Therefore \(Z = E\), and so \(J\) contains a closed 2-cell.

If \((H1)\) and \((H2)\) hold, \(J\) cannot be a plane universal curve.

**References**

3. ———, *A decomposition of \(E^3\) into points and tame arcs such that the decomposition space is topologically different from \(E^3\)*, Annals of Math., 65 (1957), 484-500.
4. ———, *The cartesian product of a certain nonmanifold and a line is \(E^4\)*, Bull. Amer. Math. Soc., 64 (1958), 82-84.
6. ———, *Irreducible continua and generalization of hereditarily unicoherent continua*
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