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Introduction. Throughout this paper we consider analytic mappings $f(z)$ of an arbitrary open Riemann surface R into an arbitrary Riemann surface S. Heins [3] introduced the class of Lindelöfian maps when R is hyperbolic, and defined them in terms of Green's functions; further contributions have been made by Rao [4], [5]. In the case of planar regions these maps are the classical functions of bounded characteristic.

Sario [6], [7], has utilized principal functions [1] on the range surface to obtain generalizations of the main theorems for mappings of R into S. In this paper a different first main theorem is obtained in which the proximity function is a generalization of Nevanlinna's proximity function by means of the substitution of a principal function for the logarithmic function. It is shown that the resulting class of functions of bounded characteristic are the Lindelöfian maps, and that an extremal decomposition characterization of these functions can be obtained as in the classical case.

1. An auxiliary family of functions. Analytic mappings from an arbitrary open surface R into an arbitrary surface S can be considered in terms of families $\mathscr{T}$ of LH functions, i.e., harmonic functions, with isolated logarithmic singularities having integral coefficients. For the purposes of this paper we slightly generalize the term, parametric disk: $\Delta = (Q, \mu)$ is a parametric disk if Q is a classical parametric disk, and there is defined on it a metric μ that is a real scalar multiple of the induced metric.

We let ζ be the local variable on S, and fix $\sigma \in S$ and a parametric disk at σ. If S is closed we define $t(\zeta, \sigma, \alpha)$ for $\alpha \in S \backslash \sigma$ (set difference) as the LH function on S which has singularities $\log|\zeta - \alpha|$ and $-\log|\zeta - \sigma|$ and is normalised by

$$\lim_{\zeta \to \sigma} (t(\zeta, \sigma, \alpha) + \log|\zeta - \sigma|) = 0$$

in terms of the fixed parametric disk. At α a parametric disk is fixed such that

$$\lim_{\zeta \to \alpha} (t(\zeta, \sigma, \alpha) - \log|\zeta - \alpha|) = 0$$

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in terms of it. We define successively

$$t(\zeta, \alpha, \sigma) = -t(\zeta, \sigma, \alpha) , \qquad \alpha \in S\backslash\sigma ,$$
$$t(\zeta, \alpha, \delta) = t(\zeta, \alpha, \sigma) + t(\zeta, \sigma, \delta) , \qquad \alpha, \delta \in S\backslash\sigma .$$

These functions form the family $\mathscr{T}$.

If S is open, more than one family can usually be formed. We consider an exhaustion of S by regular regions Ω that contain σ and α, and define on Ω the function $t_\Omega(\zeta, \sigma, \alpha)$ which satisfies the above conditions for $t(\zeta, \sigma, \alpha)$ as well as one of the following:

(a) the normal derivative of $t_\Omega(\zeta, \sigma, \alpha)$ vanishes on the boundary $\partial\Omega$ of Ω,

(b) a consistent partition of the boundaries of the regions Ω is given, and $t_\Omega(\zeta, \sigma, \alpha)$ has constant value and vanishing flux over each part of $\partial\Omega$ ([1] pp. 87–90).

By the theory of normal operators ([1] pp. 152 ff.) $t(\zeta, \sigma, \alpha)$ is defined as the directed limit of $t_\Omega(\zeta, \sigma, \alpha)$ as S is exhausted by the regions Ω. $t(\zeta, \alpha, \sigma)$ and $t(\zeta, \alpha, \delta)$ are then defined as in the case of closed surfaces S. Each condition in (a) and (b) determines a family $\mathscr{T}$. It will be represented by $\mathscr{T}_0$ if (a) is satisfied and by $\mathscr{T}_1(P)$ if (b) is satisfied for a partition P; if P is the identity partition I, we write $\mathscr{T}_1(I)$.

Since each function t is a principal function ([1] p. 169), a family $\mathscr{T}$ will be called a principal family. We note that a change in the fixed parametric disk at σ changes every function $t(\zeta, \sigma, \alpha)$ by the same constant but leaves $t(\zeta, \alpha, \delta)$ unaltered. Further, in view of our definition of parametric disk, for any given $\mathscr{T}$ and constant k there exists a family $\mathscr{T}'$ such that for all m,

$$\{\zeta \mid t(\zeta, \sigma, \alpha) = m\} = \{\zeta \mid t'(\zeta, \sigma, \alpha) = m + k\}, \; t \in \mathscr{T}, \; t' \in \mathscr{T}' .$$

We consider functions belonging to any principal family. If $\alpha, \delta \in S\backslash\sigma$, these functions have the following four obvious properties.

$$\begin{gathered} t(\zeta, \alpha, \alpha) = 0 , \\ t(\zeta, \alpha, \delta) + t(\zeta, \delta, \alpha) = 0 , \\ \lim_{\zeta \to \delta} (t(\zeta, \delta, \alpha) + \log|\zeta - \delta| = t(\delta, \sigma, \alpha) ,, \\ t(\sigma, \alpha, \delta) = 0 . \end{gathered} \tag{1}$$

LEMMA 1.1. *$t(\alpha, \gamma, \delta) + t(\gamma, \delta, \alpha) + t(\delta, \alpha, \gamma) = 0$ when α, γ, δ are distinct points in S.*

Proof. If S is open we let $\Omega \subseteqq S$ be a regular region containing α, γ, and δ, and consider functions t_Ω defined on Ω. We remove small closed disks in Ω that contain α, γ, δ and apply Green's formula to

$t_\Omega(\zeta, \alpha, \gamma)$ and $t_\Omega(\zeta, \delta, \gamma)$ over the remaining region. On letting the disks shrink to points we obtain

$$-t_\Omega(\alpha, \delta, \gamma) + t_\Omega(\delta, \alpha, \gamma) + R(\gamma) = 0$$

where

$$R(\gamma) = \lim_{\zeta \to \gamma} (t_\Omega(\zeta, \delta, \gamma) - t_\Omega(\zeta, \alpha, \gamma) = t_\Omega(\gamma, \delta, \alpha))$$

and the lemma follows by letting $\Omega \to S$.

If S is closed the same method is applied to S instead of to Ω.

COROLLARY 1.2. $t(\alpha, \delta, \sigma) = t(\delta, \alpha, \sigma)$, $\alpha, \delta \in S \backslash \sigma$.

Proof. This is obvious when α and δ are identical; if they are distinct it follows from replacing γ by σ in the lemma and applying (1).

COROLLARY 1.3. *If α is distinct from σ and γ, then $t(\alpha, \zeta, \gamma)$ is of class LH on $S \backslash \sigma$ with singularities at α and σ.*

Proof. If $\gamma = \sigma$ this is implied by Corollary 1.2. Otherwise

$$t(\alpha, \zeta, \gamma) + t(\zeta, \sigma, \alpha) = t(\alpha, \sigma, \gamma) ,$$

which is constant.

LEMMA 1.4. *$\psi: S \backslash \sigma \times S \backslash \sigma \to [-\infty, \infty] \mid \psi(\gamma, \alpha) = t(\delta, \gamma, \alpha)$ is continuous for every fixed δ.*

Proof. If $\delta = \sigma$ then ψ is identically zero; if not,

$$\psi(\gamma, \alpha) = t(\delta, \gamma, \sigma) + t(\delta, \sigma, \alpha)$$

and each term is continuous by Corollary 1.2.

Sario [8] proves that if $E \subseteqq S \backslash \sigma$ is compact and Q is an open set containing E and σ, then $t(\gamma, \alpha, \sigma) \in \mathscr{T}_0$ is uniformly bounded for $\alpha \in E$, $\gamma \in S \backslash Q$. The same proof holds for $t(\gamma, \alpha, \sigma) \in \mathscr{T}_1(P)$. From the harmonicity of $t(\gamma, \alpha, \sigma)$ in γ and in α, and from its uniform boundedness, it follows by a lemma of Heins ([2] p. 445) that

LEMMA 1.5. *If $S' = S \times S \backslash ((\sigma, \sigma) \cup \{(\sigma, \zeta)\} \cup \{(\zeta, \sigma)\} \cup \{(\zeta, \zeta)\})$, then $\phi: S' \to (-\infty, \infty) \mid \phi(\gamma, \alpha) = t(\gamma, \alpha, \sigma)$ is continuous.*

LEMMA 1.6. *If $S' = S \times S \backslash (\sigma, \sigma)$ then $\phi: S' \to [-\infty, \infty] \mid \phi(\gamma, \alpha) = t(\gamma, \alpha, \sigma)$ is continuous.*

Proof. It suffices to consider the continuity at points (γ_0, α_0), $\gamma_0 =$

$\alpha_0 \neq \sigma$, and (σ, α), $\alpha_0 \neq \sigma$. For the first we let Δ be a parametric disk at α_0 such that $\sigma \notin \bar{\Delta}$, and F be a closed connected neighborhood of α_0 that does not intersect $\partial\Delta$. For every $(\gamma, \alpha) \in F \times F$ there exists $\eta \in \partial\Delta$ such that $t(\gamma, \alpha, \sigma) > t(\eta, \alpha, \sigma)$, which is bounded by Sario's lemma for all $\eta \in \partial\Delta$, $\alpha \in F$. No generality is lost by taking $t(\eta, \alpha, \sigma) > 0$ for all $\eta \in \partial\Delta$, $\alpha \in F$.

Let ψ be a homeomorphism of Δ onto a closed disk in the plane, and g the Green's function on this disk. By its extremal property

$$t(\zeta, \alpha, \sigma) - g(\psi(\zeta), \psi(\alpha)) \geqq 0$$

for $\zeta, \alpha \in F$. Since for any n, there exists a neighborhood E_n of the origin of the disk such that $g(z, a) \geqq n$ for $z, a \in E_n$, we have $t(\gamma, \alpha, \sigma) > n$ for $\gamma, \alpha \in \psi^{-1}(E_n) \cap F$, and ϕ is continuous at (γ_0, α_0).

For the second case we let $\partial\Delta$ be the boundary of a parametric disk at α_0, and F and G be compact connected neighborhoods of σ and α_0 that do not intersect $\partial\Delta$. For $\zeta \in F$, $\alpha \in G$, there exists $\eta \in \partial\Delta$ such that

$$t(\zeta, \alpha, \alpha_0) < t(\eta, \alpha, \alpha_0) = t(\eta, \alpha, \sigma) + t(\eta, \sigma, \alpha_0) ,$$

which, by Sario's lemma, is bounded above, say by M. Hence

$$t(\zeta, \alpha, \sigma) < M + t(\zeta, \alpha_0, \sigma) ,$$

and the lemma follows, since for any n, $t(\zeta, \alpha_0, \sigma) < n$ in some neighborhood of σ.

We conclude this section by noting that the limits of $t(\zeta, \gamma, \sigma)$, $t(\zeta, \sigma, \gamma)$ and $t(\zeta, \gamma, \gamma)$ as $\gamma \to \sigma$ are ∞, $-\infty$ and 0 respectively, and that $t(\zeta, \sigma, \sigma)$ is not defined.

2. Jensen's formula. The main tool used in this paper is Jensen's formula generalized for Riemann surfaces. We let Ω be a regularly imbedded relatively compact region on the surface R and let $v(z)$ be an LH function on $\bar{\Omega}$. The positive singularities of $v(z)$ in Ω will be designated by a_i, $i = 1, \cdots, m$, and the negative singularities by b_j, $j = 1, \cdots, n$; their multiplicities will be given by μ_i and ν_j respectively.

We obtain the formula from the following proposition:

LEMMA 2.1. *If r is not a singularity of $v(z)$, then*

$$(2) \qquad v(r) = \frac{1}{2\pi} \int_{\partial\Omega} v(z) d^* p(z, r) + \sum_i \mu_i g(a_i, r) - \sum_j \nu_j g(b_j, r)$$

where $p(z, r)$ and $g(z, r)$ are the capacity and Green's functions defined on Ω with singularities at the point r, and $\partial\Omega$ is oriented counterclockwise about r.

Proof. We first take the case when $v(z)$ has no singularities on $\bar{\Omega}$. Let $\Delta \subseteqq \Omega$ be a small closed disk that contains r. On applying Green's formula to $p(z, r)$ and $v(z)$ over $\Omega \backslash \Delta$, and letting Δ shrink to r we obtain

$$v(r) = \frac{1}{2\pi} \int_{\partial\Omega} v(z) d^* p(z, r) .$$

We next take the case when $v(z)$ has a singularity $\nu \log |z - a|$, $a \in \partial\Omega$, but has none in Ω. Let $p(z, r)$ have the value k on $\partial\Omega$; there exists an $\varepsilon > 0$ such that the boundary components of $\Omega_\varepsilon = \{z \mid p(z, r) < k - \varepsilon\}$, have a natural one-to-one mapping on those of $\partial\Omega$.

Let the components of $\partial\Omega$ be $\{\gamma_i\}$, $i = 1, \cdots, n$, with $a \in \gamma_1$, and the corresponding components of $\partial\Omega_\varepsilon$ be $\{\gamma_{i\varepsilon}\}$. For $i \neq 1$, we apply Green's formula to $v(z)$ and $p(z, r)$ over each component of $\Omega \backslash \Omega_\varepsilon$ and obtain

$$\int_{\gamma_i - \gamma_{i\varepsilon}} v(z) d^* p(z, r) = \varepsilon \int_{\gamma_{i\varepsilon}} d^* v(z) .$$

For $i = 1$, we let $\hat{\Omega}$ be the double of $\bar{\Omega} \backslash \Omega_\varepsilon$. If q is the total flux of $p(z, r)$ along γ_1, the function

$$h(z) = \exp\left[\frac{2\pi}{q} (p(z, r) - k + i p^*(z, r)\right]$$

maps the first component of $\hat{\Omega}$ conformally onto an annulus, such that γ_1 is mapped onto the unit circle B_1 and $\gamma_{1\varepsilon}$ onto $B_{1\varepsilon} = \{w \mid |w| = \exp[-(2\pi/q)\varepsilon]\}$. We may assume that the point a is mapped on $w = 1$. Consequently $d\theta = 2\pi/q \, d^* p(z, r)$.

Since $\int_0^{2\pi} \log |e^{i\theta} - 1| = 0$, it follows that

$$\int_{\gamma_1 - \gamma_{I\varepsilon}} v(z) d^* p(z, r) = \frac{q}{2\pi} \int_{B_1 - B_{1\varepsilon}} (v \cdot h^{-1}(re^{i\theta}) - \nu \log |re^{i\theta} - 1|) \, d\theta .$$

By applying Green's formula to the last integrand and to $\log |re^{i\theta}|$ over the annulus between the circles, we find

$$\int_{\gamma_1 - \gamma_{1\varepsilon}} v(z) d^* p(z, r) = \varepsilon \int_{\gamma_{1\varepsilon}} d^* v(z) .$$

Summing over all the components of $\partial\Omega$ we obtain

$$(3) \qquad \frac{1}{2\pi} \int_{\partial\Omega} v(z) d^* p(z, r) = \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} v(z) d^* p((z, r) = v(r) .$$

For the general case we note that

$$v(z) - \sum_i \mu_i g(a_i, z) + \sum_j \nu_j g(b_j, z)$$

is a harmonic function on Ω. The application of (3) to this function yields (2). We immediately obtain

COROLLARY 2.2. (Generalized Jensen's formula). *If f is an analytic mapping of R into a Riemann surface S on which is defined a function $t(\zeta, \alpha, \delta)$ belonging to a principal family, and if $f(r)$, α and δ are distinct, then,*

$$(4) \qquad t(f(r), \alpha, \delta) = \frac{1}{2\pi}\int_{\partial\Omega} t(f(z), \alpha, \delta)d^*p(z, r) + \sum_i \mu_i g(a_i, r) - \sum_j \nu_j g(d_j, r) .$$

where $\{a_i\}$ and $\{b_j\}$ are the inverse images in Ω of α and δ respectively, and μ_i, ν_j are their multiplicities.

If $f(r)$ is a singularity of $t(\zeta, \alpha, \delta)$ the following proposition holds:

LEMMA 2.3. *If $f(r) = \alpha$, and if the Laurent expansion of $f(z)$ in the neighborhood of r is $f(z) = \sum_N^\infty c_i z_i$ with respect to the parametric disks at r and α fixed by $p(z, r)$ and $t(\zeta, \sigma, \alpha)$ respectively, then*

$$(5) \qquad \lim_{z\to r} (Np(z, r) + t(f(z), \sigma, \alpha)) = \log\left|\frac{1}{c_N}\right| .$$

If $f(z) = \delta$, then, with the above expansion,

$$(6) \qquad \lim_{z\to r} (-Np(z, r) + t(f(z), \sigma, \alpha)) = \log\left|\frac{1}{c_N}\right| .$$

Proof. We shall use the same symbol z for an arbitrary point on the surface and for its image under the mapping associated with the parametric disk under consideration. $t(f(z))$ and $p(z)$ will represent $t(f(z), \sigma, \alpha)$ and $p(z, r)$, and l_i, etc., constant coefficients. We set

$$q(z) = \exp[t(f(z)) + it^*(f(z))] ;$$

this is single-valued in a neighborhood of r.

If $f(r) = \sigma$, the expansion in that neighborhood is

$$q(z) = \frac{1}{c_N} z^{-N} + - \sum_{N+1}^{\infty} l_i z^i .$$

Similarly, there is a neighborhood of r in which

$$r(z) = \exp[p(z) + ip^*(z)]$$

can be expanded as

$$r(z) = z + \sum_{1}^{\infty} m_i z^i .$$

Hence

$$\frac{1}{c_N} = \lim_{z \to r} q(z)\,(r(z))^N$$

which yields the first conclusion. The second is proved in the same way. This concludes the proof.

If we let λ equal N or $-N$ according as $f(r)$ is σ or α, then the function

$$t(f(z)) + \lambda p(z) - \sum_i{}' \mu_i g(s_i, z) + \sum_j{}' \nu_j g(a_i, z)$$

is harmonic on Ω, when the summations are over the inverse images in $\Omega \backslash r$. On applying Jensen's formula (4) and substituting from (5) or (6) we obtain the alternative expression

$$\log \left| \frac{1}{c_N} \right| = \frac{1}{2\pi} \int_{\partial\Omega} t(f(z)) d^* p(z) + k\lambda + \sum_i{}' \mu_i g(s_i, r) - \sum_j{}' \nu_j g(a_i, r) .$$

We shall need the following property of subharmonic functions:

LEMMA 2.4. *Let u be an u.s.c. function on a region W.*

(i) *If u is subharmonic on W, then for every regular Ω whose closure is in W, and every $z \in \Omega$,*

$$u(z) \leqq \frac{1}{2\pi} \int_{\partial\Omega} u(w) d^* p_\Omega(w, z) . \tag{7}$$

(ii) *If for every $z \in W$, there is a regular Ω such that*

$$u(z) \leqq \frac{1}{2\pi} \int_{\partial h} u(w) d^* p_\Omega(w, z)$$

over every level line ∂h of $p_\Omega(w, z)$, then u is subharmonic on W.

Proof. To prove (i) we take an arbitrary Ω and $z \in \Omega$, and let $\{v_n\}$ be a descending sequence of continuous functions on $\partial\Omega$ tending to u. For any $w \in \partial\Omega$, we have by (2)

$$\overline{\lim_{z \to w}}\, u(z) \leqq \lim_{z \to w} \frac{1}{2\pi} \int_{\partial\Omega} v_n(w) d^* p(w, z)$$

for all v_n. By applying the monotone convergence theorem and the maximum principle we obtain the desired result.

For (ii) we let z_0 be an arbitrary point in the region, and choose a parametric disk about z_0. In terms of the associated unit disk the

hypothesis yields

$$u(z_0) \leqq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

for $0 < r \leqq 1$. The subharmonicity of u follows from the theory of functions on the plane.

We immediately obtain

COROLLARY 2.5. *If $u(z)$ is subharmonic on $\bar{\Omega}_2$ and $z \in \Omega_1 \subseteqq \Omega_2$, then*

$$\int_{\partial\Omega_1} u(w) d^* p_1(w, z) \leqq \int_{\partial\Omega_2} u(w) d^* p_2(w, z)\ , \tag{8}$$

where $p_i(w, z)$, $i = 1, 2$, is the capacity function on Ω_i.

3. **Argument principle.** Using the same notation as before we let Ω be a relatively compact regularly imbedded open set in the surface R, and $n(\Omega, \alpha)$, $n(\Omega, \delta)$ the number of inverse images (with multiplicities) in Ω of points α and δ in S that are not on the image of $\partial\Omega$. We have

$$\frac{1}{2\pi}\int_{\partial\Omega} d^* t(f(z), \alpha, \delta) = n(\Omega, \delta) - n(\Omega, \alpha) \tag{9}$$

where $\partial\Omega$ is oriented counter-clockwise, and t belongs to any principal family.

This statement follows from removing small disks at each of the inverse images of α and δ, applying Green's formula to $t(f(z), \alpha, \delta)$ over the remainder of Ω, and taking the limit as the disks shrink to points.

We choose $r \in \Omega$. If $p(z, r)$ is the capacity function on Ω, and k is its value on $\partial\Omega$, we let $\Omega_h = \{z \in \Omega \mid p(z, r) < h\}$ and ∂h be the boundary of Ω_h.

THEOREM 3.1. *If α and δ are not in the image of ∂h, and if α, δ and $f(r)$ are distinct, then*

$$\frac{1}{2\pi}\frac{d}{dh}\int_{\partial h} t(f(z), \alpha, \delta) d^* p(z, r) = n(h, \delta) - n(h, \alpha)\ , \tag{10}$$

where $n(h, \delta)$ and $n(h, \alpha)$ are the number of inverse images (with multiplicities) of δ and α in Ω_h, and t belongs to any principal family.

Proof. We let $t(f(z))$ and $p(z)$ represent $t(f(z), \alpha, \delta)$ and $p(z, r)$, $\{a_i\}$ and $\{d_j\}$ be the finite number of inverse images of α and δ in Ω_h.

There is a finite h' such that $\Omega_{h'}$ does not contain any of these inverse images. We remove small disks about the a_i's and d_j's and apply Green's formula to $t(f(z))$ and $p(z)$ over the remainder of $\Omega_h \backslash \bar{\Omega}_{h'}$. After evaluating and letting the disks shrink to points we obtain

$$\int_{\partial h - \partial h'} t(f(z)) d^* p(z) + 2\pi \left[\sum_i \mu_i p(a_i) - \sum_j \nu_j p(d_j) \right] = h \int_{\partial h} d^* t(f(z)) - h' \int_{\partial h'} d^* t(f(z)) ,$$

since $p(z)$ is the capacity function on both Ω_h and $\Omega_{h'}$. In this relationship μ_i and ν_j are the multiplicities of the corresponding inverse images. The differentiation of this equation yields

$$\frac{d}{dh} \int_{\partial h} t(f(z)) d^* p(z) = \frac{d}{dh} \left[h \int_{\partial h} d^* t(f(z)) \right] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\varepsilon \int_{\partial h} d^* t(f(z)) - (h - \varepsilon) \int_{\partial (h-\varepsilon) - \partial h} d^* t(f(z)) \right].$$

Since the last term vanishes for sufficiently small ε, we substitute from (9) and obtain the required relationship.

We note that (10) is an invariant property of principal families.

4. **Logarithmic capacities.** A logarithmic capacity of a compact set E properly contained in an arbitrary surface S can be defined in relation to any principal family $\mathscr{T}$ if $\sigma \in S \backslash E$. We let μ be a regular positive unit mass distribution on E. Since $t(\zeta, \eta, \sigma)$ is l.s.c. on E, we define the logarithmic potential of μ relative to $\mathscr{T}$ as

$$p_\mu(\eta) = \int_E t(\zeta, \eta, \sigma) \, d\mu(\zeta)$$

on $S \backslash \sigma$. The following proposition carries over from the plane:

LEMMA 4.1. *The logarithmic potential $p_\mu(\eta)$ is harmonic on $S \backslash (E \cup \sigma)$ and superharmonic on $S \backslash \sigma$. In the neighborhood of σ, $p_\mu(\eta) - \log |\eta - \sigma|$ is bounded.*

Proof. We let

$$t_n(\zeta, \eta, \sigma) = \min \{n, t(\zeta, \eta, \sigma)\} ,$$

and

$$p_{\mu n}(\eta) = \int_E t_n(\zeta, \eta, \sigma) \, d\mu(\zeta) .$$

By Lemmas 1.5 and 1.6, $t_n(\zeta, \eta, \sigma)$ is continuous in (ζ, η), $\zeta \in E$, $\eta \in S \backslash \sigma$,

and as E is compact there is for any arbitrary point $\eta_0 \in S\backslash\sigma$ and $\varepsilon > 0$, a neighborhood Δ of η_0 such that

$$| t_n(\zeta, \eta, \sigma) - t_n(\zeta, \eta_0, \sigma) | < \varepsilon, \zeta \in E, \eta \in \Delta ,$$

It follows that $p_{\mu n}(\eta)$ is continuous and $p_\mu(\eta)$ l.s.c. on $S\backslash\sigma$.

Let Γ be a disk about η_0 such that $\partial\Gamma$ is a level line of $t(\zeta, \eta_0, \sigma)$. We orient $\partial\Gamma$ clockwise about η_0. Since $t(\zeta, \eta, \sigma)$ is bounded below for all $\zeta \in E, \eta \in \partial\Gamma$, and $-t(\zeta, \eta_0, \sigma)$ is the capacity function on Γ, we have by Corollary 1.2 and (2),

$$\frac{1}{2\pi} \int_{\partial\Gamma} d^*t(\eta, \eta_0, \sigma) \int_E t_n(\zeta, \eta, \sigma)\, d\mu(\zeta) = \int_E t'_n(\zeta, \eta_0, \sigma)\, d\mu(\zeta)$$

where for each ζ, $t'_n(\zeta, \eta, \sigma)$ is the harmonic function in η on Γ with boundary values $t_n(\zeta, \eta, \sigma)$. Further, by superharmonicity, $t'_n(\zeta, \eta_0, \sigma) \leqq t(\zeta, \eta_0, \sigma)$ for each ζ and for all n. We substitute in the above equation and apply the monotonic convergence theorem as $n \to \infty$. We obtain

$$\frac{1}{2\pi} \int_{\partial\Gamma} d^*t(\eta, \eta_0, \sigma) \int_E t(\zeta, \eta, \sigma)\, d\mu(\zeta) \leqq \int_E t(\zeta, \eta_0, \sigma)\, d\mu(\zeta)$$

and $p_\mu(\eta)$ is superharmonic by (7).

If $\eta_0 \notin E \cup \sigma$, Γ can be chosen such that $\bar{\Gamma} \subseteqq S\backslash(E \cup \sigma)$. Since $t(\zeta, \eta, \sigma)$ is harmonic on $\bar{\Gamma}$, the same method establishes the harmonicity of $p_\mu(\eta)$ on $S\backslash(E \cup \sigma)$ by (2) and the maximum principle.

To establish the final part of the lemma we need only note that by Lemmas 1.5 and 1.6 there is a neighborhood Δ of σ such that

$$t(\zeta, \eta, \sigma) = t(\eta, \zeta, \sigma) = \log | \eta - \sigma | + h(\eta, \zeta), \zeta \in E, \eta \in \Delta\backslash\sigma ,$$

where $h(\eta, \zeta)$ is bounded.

We deduce the following proposition:

COROLLARY 4.2. *If μ is as above and $f: R \to S$ is analytic, then, for a regular $\Omega \subset R$,*

$$\int_E d\mu(\zeta) \int_{\partial\Omega} t(f(z), \zeta, \sigma)\, d^*p(z, r) = \int_{\partial\Omega} d^*p(z, r) \int_E t(f(z), \zeta, \sigma)\, d\mu(\zeta) . \tag{11}$$

where t belongs to any principal family $\mathcal{T}$ and p is the capacity function on Ω. The iterated integral is either finite or $+\infty$.

Proof. There exists a closed disk $D \subseteqq S$ about σ such that

(a) $D \cap E$ is void,

(b) $t(\alpha, \zeta, \sigma) < 0, \alpha \in D, \zeta \in E$ by 1.6,

(c) the intersection of D and the image of $\partial\Omega$ consists of a finite number (possibly zero) of Jordan arcs β_i, $i = 1, \cdots, n$, each of which

passes through σ.

We divide $\partial\Omega$ into the inverse images γ_i, $i = 1, \cdots, n$, of β_i, and the remainder γ.

On γ the function $t(f(z), \zeta, \sigma)$ is uniformly bounded below for $\zeta \in E$ and we may apply Fubini's theorem to

$$\int_E d\mu(\zeta) \int_\gamma t(f(z), \zeta, \sigma)\, d^*p(z, r) .$$

The integral is either finite or $+\infty$.

For each i, we exhaust $\beta_i \backslash \sigma$ by a sequence of compact sets F_j. By (c) the restriction of $d^*p(z, r)$ to $f^{-1}(F_j) \cap \gamma_i$ induces a positive mass function on $F_j \subseteqq D \backslash \sigma$. Its logarithmic potential

$$p_{\mu ij}(\zeta) = \int_{f^{-1}(F_j)\cap\gamma_i} t(f(z), \zeta, \sigma)\, d^*p(z, r)$$

is harmonic on $S \backslash D$ by 4.1. By (b) the functions $p_{\mu ij}$ form a decreasing sequence; by Harnack's principle its limit

$$p_{\mu i}(\zeta) = \int_{\gamma_i} t(f(z), \zeta, \sigma)\, d^*p(z, r)$$

is either $-\infty$, or harmonic on $S \backslash D$.

We may assume that $p(z, r)$ is zero on $\partial\Omega$; then $\exp\{p(z, r) + ip^*(z, r)\}$ (choosing any branch of p^*) maps γ_i onto an arc of the unit circle. For any $\zeta_0 \in E$ we have

$$p_{\mu i}(\zeta_0) = \int_\alpha^\beta \log |e^{i\theta} - 1|\, d\theta + c$$

where c is some finite constant. Since this integral is bounded with respect to α and β, $p_{\mu i}(\zeta_0)$ is finite, and $p_{\mu i}(\zeta)$ is bounded on E.

Consequently, by (b), we may apply Fubini's theorem to

$$\int_E d\mu(\zeta) \int_{\gamma_i} t(f(z), \zeta, \sigma)\, d^*p(z, r)$$

for each i, and the integral is finite. Summing over γ and γ_i, we obtain the required relation.

5. **First main theorem.** To develop a first main theorem for analytic mappings $f: R \to S$ where R and S are arbitrary Riemann surfaces, we fix a point $\sigma \in S$ and define a principal family $\mathscr{T}$; we then select points $\tau \in S$ and $r \in R$ such that σ, τ and $f(r)$ are distinct. A parametric disk is selected at r.

Let $\overset{+}{t} = \max\{t, 0\}$. For a regular region $\Omega \Subset R$ such that $r \in \Omega$, the proximity function $m(\Omega, f)$, the counting function $N(\Omega, f)$ and the characteristic function $T(\Omega, f)$ of f on Ω are defined as

$$m(\Omega, f) = \frac{1}{2\pi} \int_{\partial\Omega} \overset{+}{t}(f(z), \sigma, \tau)\, d^*p(z, r) ,$$

$$N(\Omega, f) = \sum_i g(s_i, r) , \qquad \{s_i\} = f^{-1}(\sigma) \cap \Omega ,$$

$$T(\Omega, f) = m(\Omega, f) + N(\Omega, f) ,$$

where p and g are the capacity and Green's functions on Ω, and s_i is repeated in accordance with its multiplicity.

The proximity $m(\Omega, \alpha)$ and the counting functions $N(\Omega, \alpha)$ at the point α are defined as $m(\Omega, f)$ and $N(\Omega, f)$ when $\alpha = \sigma$; otherwise we define

$$m(\Omega, \alpha) = \frac{1}{2\pi} \int_{\partial\Omega} \overset{+}{t}(f(z), \alpha, \sigma)\, d^*p(z, r) ,$$

$$N(\Omega, \alpha) = \sum g(a_i, r) , \qquad \{a_i\} = f^{-1}(\alpha) \cap \Omega .$$

where a_i is repeated in accordance with its multiplicity.

The first main theorem reads:

THEOREM 5.1. *For every* $\alpha \in S \backslash f(r)$,

$$m(\Omega, \alpha) + N(\Omega, \alpha) = T(\Omega, f) + 0(1) ,$$

where $O(1)$ *is a bounded function with respect to* Ω.

Proof. When $\alpha = \sigma$ it is trivial; when $\alpha \neq \sigma$, Jensen's formula (4) is

$$t(f(r), \alpha, \sigma) = \frac{1}{2\pi} \int_{\partial\Omega} \overset{+}{t}(f(z), \alpha, \sigma)\, d^*p(z, r) + \sum_i g(a_i, r) - \frac{1}{2\pi} \int_{\partial\Omega} \overset{+}{t}(f(z), \sigma, \alpha)\, d^*p(z, r) - \sum_i g(s_i, r) ,$$

which is

$$m(\Omega, \alpha) + N(\Omega, \alpha) = \frac{1}{2\pi} \int_{\partial\Omega} \overset{+}{t}(f(z), \sigma, \alpha)\, d^*p(z, r) + N(\Omega, f) + O(1) . \tag{12}$$

For $\zeta \in S$ we define

$$q(\zeta) = \overset{+}{t}(\zeta, \sigma, \alpha) - \overset{+}{t}(\zeta, \sigma, \tau) .$$

There is a neighborhood Δ of σ in which both $t(\zeta, \sigma, \alpha)$ and $t(\zeta, \sigma, \tau)$ are positive. Hence in Δ,

$$q(\zeta) = t(\zeta, \sigma, \alpha) - t(\zeta, \sigma, \tau) = t(\zeta, \tau, \alpha) ,$$

which is bounded. Outside Δ, $q(\zeta)$ is obviously bounded. It follows that

$$\overset{+}{t}(f(z), \sigma, \alpha) = \overset{+}{t}(f(z), \sigma, \tau) + O(1) .$$

We conclude the proof by substituting this in (12).

We note that if $\mathscr{T}'$ and $\mathscr{T}''$ are principal families defined with respect to the same point σ, then the functions $t'(\zeta, \sigma, \tau)$ and $t''(\zeta, \sigma, \tau)$ belonging to these families differ by a bounded harmonic function. Consequently the corresponding characteristic functtions T_1 and T_2 are related by $T_1(\Omega, f) = T_2(\Omega, f) + O(1)$ where $O(1)$ is bounded with respect to Ω.

Before defining functions of bounded characteristic we shall develop an alternative representation of the characteristic function. For this purpose we prove the following lemma.

LEMMA 5.2. *$N(\Omega, \zeta)$ is continuous on $S\backslash f(r)$.*

Proof. Let α be an arbitrary point in $S\backslash f(r)$, and let $a_1, \cdots, a_q$ with multiplicities $\nu_1, \cdots, \nu_p$ be the inverse images of α in Ω.

We can construct open connected neighborhoods D', D of α in $S\backslash f(r)$, and E_j', E_j of a_j in $\Omega\backslash r$ for every j, such that the following properties hold:

(a) Each neighborhood lies in a parametric disk about its associated point.

(b) Every inverse image of $\zeta \in D'\backslash\alpha$ is simple and ζ has ν_j inverse images in E_j'.

(c) Every $z \in E_j'\backslash a_j$ is simple.

(d) $\bar{E}_j \subseteqq E_j'$.

(e) Every $\zeta \in D\backslash\alpha$ has ν_j roots in E_j, and $\bar{D} \subseteqq D'$.

$(E_j'\backslash a_j, f)$ is a smooth covering surface of S. If $\gamma(t)$ is an arc in D from an arbitrary $\delta \in D$ to α, its path of determination $\gamma'(t)$ from an inverse image of δ in E_j cannot intersect $E_j'\backslash E_j$ and must tend to a_j. Similarly if the inverse image is not in an E_j', $\gamma'(t)$ must tend to $\partial\Omega$. Hence every component of the inverse image of D that intersects Ω is either a neighborhood of some a_j or intersects $\partial\Omega$.

Let $D_0 = \{\xi \,|\, |\zeta - \alpha| < \rho_0\}$ be a disk in D in terms of the local coordinates. Let F_{j0} be the component of the inverse image of D_0 that contains a_j, let $G_{j0}, j = 1, \cdots, n$ be the components that intersect the inverse images $b_j, j = 1, \cdots, n$, of α on $\partial\Omega$, and let $H_{j0}, j = 1, \cdots, m$, be the other components that intersect Ω. The number of components is finite since $\partial\Omega$ and ∂D_0 are analytic curves.

We define a real-valued function $h_j(z)$ on H_{j0} by $h_j(z) = |f(z) - \alpha|$. For each H_{j0} there exists $r_j > 0$ such that $h_j(z) > r_j$ for $z \in H_{j0} \cap \Omega$,

and there exists a positive $r_0 < r_j$ for all j. Let $D_0' = \{\zeta \mid |\zeta - \alpha| < r_0\}$.

Let M be a uniform bound of the number of inverse images in Ω of $\zeta \in S$. For $\varepsilon > 0$ and every j, there exist neighborhoods $\Delta(b_j) \subseteqq G_{j0}$ of b_j such that

$$|g(z, r) - g(a_j, r)| < \frac{\varepsilon}{M}, \qquad z \in \Delta(a_j),$$

and

$$|g(z, r)| < \frac{\varepsilon}{M}, \qquad z \in \Delta(b_j)$$

where $g(z, r)$ is the Green's function on Ω and vanishes outside Ω. Then $|N(\Omega, \zeta) - N(\Omega, \alpha)| < \varepsilon$ in the intersection of D_0' and the images of $\Delta(a_j)$ and $\Delta(b_j)$ for all j. This completes the proof.

LEMMA 5.3. *If μ is a regular positive unit measure on a compact set $E \subseteqq S \backslash \sigma$, and if $p_\mu(\eta) = \int_E t(\eta, \zeta, \sigma)\, d\mu(\zeta)$ is the logarithmic potential with respect to any family $\mathscr{T}$, then*

$$-p_\mu(f(r)) = -\frac{1}{2\pi}\int_{\partial\Omega} p_\mu(f(z))\, d^*p(z, r) + N(\Omega, f) - \int_E N(\Omega, \zeta)\, d\mu(\zeta). \tag{13}$$

Proof. By Lemmas 1.5 and 5.2 we may integrate Jensen's formula (4) over E and obtain

$$\int_E t(f(r), \sigma, \zeta)\, d\mu(\zeta) = \frac{1}{2\pi}\int_E d\mu(\zeta) \int_{\partial\Omega} t(f(z), \sigma, \zeta)\, d^*p(z, r) + N(\Omega, f) - \int_E N(\Omega, \zeta)\, d\mu(\zeta).$$

We apply (11) and obtain the required result, which is the natural generalization of Frostman's formula.

The characterization of $T(\Omega, f)$ that we need is a consequence of the next theorem.

For a fixed $\sigma, \tau \in S$ and $r \in R$ such that σ, τ and $f(r)$ are distinct, we shall write $t(\zeta)$ for $t(\zeta, \sigma, \tau)$, $t_m(\zeta)$ for $\max\{m, t(\zeta)\}$ and $p(z)$ for $p(z, r)$.

THEOREM 5.4. *If $E_m = \{\zeta \mid t(\zeta) = m\}$ where m is finite and t belongs to the principal family $\mathscr{T}_1(I)$ with respect to the identity partition, then*

$$t_m(f(r)) = \frac{1}{2\pi}\int_{\partial\Omega} t_m(f(z))\, d^*p(z) + N(\Omega, f) - \frac{1}{2\pi}\int_{E_m} N(\Omega, \zeta)\, d^*t(\zeta)\,. \tag{14}$$

Proof. We first prove this theorem for the case in which some extra hypotheses hold, and then remove the restrictions.

We assume that either S is closed or that S is a regular region containing the image of $\bar{\Omega}$ and that $m \neq \lim t(\zeta)$ as $\zeta \to \xi \in \partial S$. We choose a unit mass distribution on the compact set E_m (oriented clockwise about σ) such that $d\mu = 1/2\pi\, d^*t(\zeta)$. Its logarithmic potential is

$$p_\mu(\eta) = \frac{1}{2\pi}\int_{E_m} t(\zeta, \eta, \sigma)\, d^*t(\zeta) \tag{15}$$

E_m divides S into two components, one containing σ and the other τ; we shall call them the σ_m- and τ_m-components. If S is a regular region one of these components is a neighborhood of the ideal boundary; we suppose that it is the σ_m-component.

If $m < t(\eta) < \infty$, then the flux of $t(\zeta, \eta, \sigma)$ is zero over the boundary, $E_m \cup \partial S$, of the σ_m-component and is also zero over the boundary, ∂S, of S; since $E_m \cap \partial S$ is void, it follows that the flux over E_m is zero. As $t(\zeta)$ is constant on E_m, it follows from (15) that

$$-p_\mu(\eta) = -\frac{1}{2\pi}\int_{E_m} [t(\zeta, \eta, \sigma)\, d^*t(\zeta) - t(\zeta)\, d^*t(\zeta, \eta, \sigma)]\,.$$

The application of Green's formula to $t(\zeta, \eta, \sigma)$ and $t(\zeta)$ over the τ_m-component proves that the right-hand side equals $-t(\tau, \eta, \sigma) = t_m(\eta)$.

If $-\infty < t(\eta) < m$, we write $t(\zeta, \eta, \sigma) = t(\zeta, \tau, \sigma) + t(\zeta, \eta, \tau)$ in (15). The flux of $t(\zeta, \tau, \sigma)$ is 2π and the flux of $t(\zeta, \eta, \tau)$ is zero over the boundary, E_m, of the τ_m-component. The first integral equals $-m$. We add a zero term and obtain

$$-p_\mu(\eta) = m - \frac{1}{2\pi}\int_{E_m} [t(\zeta, \eta, \tau)\, d^*t(\zeta) - t(\zeta)\, d^*t(\zeta, \eta, \tau)]$$

from (15). We apply Green's formula to $t(\zeta, \eta, \tau)$ and $t(\zeta)$ over the σ_m-component, and it follows that $-p_\mu(\eta) = m - t(\sigma, \eta, \tau) = m = t_m(\eta)$ by (1).

We obtain the same results if we suppose that the τ_m-component is a neighborhood of the ideal boundary.

Since the application of Lemma 4.1 to (15) shows that $p_\mu(\eta)$ is continuous at τ, we conclude that $-p_\mu(\tau) = t_m(\tau)$.

If $\eta \in E_m$ we note that $t(\zeta, \eta, \sigma)$ is superharmonic in the neighborhood of E_m. We consider the level lines $E_{m-\varepsilon}$ and $E_{m+\varepsilon}$, $\varepsilon > 0$. For sufficiently

small ε, either $t(\zeta)$ is the capacity function on the $\tau_{m-\varepsilon}$-, τ_m- and $\tau_{m+\varepsilon}$-components or $-t(\zeta)$ is the capacity function on the corresponding σ-components. In either case we apply (8) to (15) and obtain

$$t_{m-\varepsilon}(\eta) \leqq -p_\mu(\eta) \leqq t_{m+\varepsilon}(\eta) ,$$

which yields $-p_\mu(\eta) = t_m(\eta)$.

We substitute in (13) and obtain (14).

To remove the restrictions we shall denote the intersection of E_k and the image of $\bar{\Omega}$ by E_k'. Then E_k' is compact and

$$\int_{E_k'} N(\Omega, \zeta)\, d^*t(\zeta) = \int_{E_k} N(\Omega, \zeta)\, d^*t(\zeta) .$$

If S is a regular region and $m = \lim t(\zeta)$ as $\zeta \to \xi \in \partial S$, we take $\varepsilon_0 > 0$ sufficiently small that

$$\{\zeta \mid \operatorname{grad} t(\zeta) = 0 \text{ and } m + \varepsilon \leqq t(\zeta) \leqq m\} \subseteq E_m .$$

For $\varepsilon_0 > \varepsilon > 0$ we map $E'_{m+\varepsilon}$ into E_m along the level lines of $t^*(\zeta)$. These are well defined as the different branches of $t^*(\zeta)$ differ by an additive constant. The mapping is one-to-one except that onto each of the finite number of zeros of grad $t(\zeta)$ on E_m is mapped a finite number (one more than the order of the zero) of points on $E'_{m+\varepsilon}$.

On the image of $\bar{\Omega}$ we set the measures $d\mu_\varepsilon = 1/2\pi\, d^*t(\zeta)$ on $E'_{m+\varepsilon}$, $0 < \varepsilon < \varepsilon_0$. By Helly's theorem there exists a limiting measure that is obviously on E'_m. By the continuity of the normal derivative of $t(\zeta)$ it is, under the above mapping, $d^*t(\zeta)$, a.e. Hence, if $N_q = \min(N, q)$, we obtain

$$\lim_{\varepsilon \to 0} \int_{E'_{m+\varepsilon}} N_k(\Omega, \zeta)\, d^*t(\zeta) \leqq \int_{E'_m} N(\Omega, \zeta)\, d^*t(\zeta) .$$

The opposite inequality is obtained by Fatou's lemma. Consequently,

$$\lim_{\varepsilon \to 0} \int_{R_{m+\varepsilon}} N(\Omega, \zeta)\, d^*t(\zeta) = \int_{E_m} N(\Omega, \zeta)\, d^*t(\zeta) .$$

We now establish (14) for m by applying it to $m + \varepsilon$, which is permissible, and letting $\varepsilon \to 0$.

If S is arbitrary we consider an exhaustion of S by regular regions W such that W contains σ, τ and the image of $\bar{\Omega}$. We denote by $t_W(\zeta) = t_W(\zeta, \sigma, \tau)$ the function in the $\mathscr{T}_1(I)$ family defined with respect to W, and we set $E_{Wm} = \{\zeta \in W \mid t_W(\zeta) = m\}$.

Let W_0 be a regular region containing the image of $\bar{\Omega}$. We first consider m such that $E_m \cap \bar{W}_0$ contains no zeros of grad $t(\zeta)$, and cover it with a finite number of parametric disks. We select $\varepsilon_0 > 0$, such that the set

$$F = \{\zeta \mid \varepsilon_0 \leqq t(\zeta) \leqq m + \varepsilon_0\} \cap \bar{W}_0$$

is contained in these disks and does not contain any zeros of grad $t(\zeta)$. On each disk we use $t(\zeta)$ and any branch of $t^*(\zeta)$ as local variables.

Since E_m and ∂W_0 are analytic manifolds, their intersection consists of a finite number of components. Consequently there exists a compact $F' \subseteqq F$ such that the intersection of E_m and the image of $\bar{\Omega}$ is contained in the interior of F', and that $\partial F'$ intersects E_m at a finite number of points, each of which has a neighborhood in which $\partial F'$ lies on a level line of $t^*(\zeta)$. We set $E_m^* = E_m \cap F'$

Since $t_W(\zeta)$ and its normal derivative tend uniformly on compact sets to $t(\zeta)$ and its normal derivative, there is for any $\varepsilon > 0$, a W_ε such that

$$E_{Wm} \subseteqq \{\zeta \mid m - \varepsilon < t(\zeta) < m + \varepsilon\} \cap F' ,$$

and that the maximum angle between E'_{Wm} and E_m is less than $\pi/2$, for $W \supseteqq W_\varepsilon$. For sufficiently small ε we can map E^*_{Wm} univalently onto E_m^* along the level lines of $t^*(\zeta)$.

We have set up the set we need for the proof. We apply (14) to the region W and let $W \to S$. It is only necessary to examine the convergence of the last term. On $\bar{W}_0$ we choose a set of measures $d\mu_W = 1/2\pi\, d^*t_W(\zeta)$ on E^*_{Wm}. For sufficiently large W,

$$\int_{E^*_{Wm}} N(\Omega, \zeta)\, d^*t_W(\zeta) = \int_{E_{Wm}} N(\Omega, \zeta)\, d^*t_W(\zeta) .$$

We apply Helly's theorem as before and obtain the necessary convergence. Consequently, the theorem holds for open S if there is no zero of grad $t(\zeta)$ on E_m.

If grad $t(\zeta)$ has a zero on E_m, we apply (14) to $E_{m+\varepsilon}$ and take the limit as $\varepsilon \to 0$. To obtain the convergence of the last term, we choose the set of measures $d\mu_\varepsilon = 1/2\pi\, d^*t(\zeta)$ on $E^*_{m+\varepsilon}$ and apply Helly's theorem. This completes the proof.

By taking $m = 0$ in (14) we immediately obtain a generalization of Cartan's formula:

COROLLARY 5.5. *If the characteristic $T(\Omega, f)$ is defined in terms of a principal family $\mathscr{T}_1(I)$, then*

$$T(\Omega, f) = \overset{+}{t}(f(r)) + \frac{1}{2\pi} \int_{E_0} N(\Omega, \zeta)\, d^*t(\zeta) . \tag{16}$$

As a side issue we shall strengthen Lemma 5.2.

LEMMA 5.6. *If $f(\partial\Omega)$ is the image of $\partial\Omega$, then $N(\Omega, \zeta)$ is LP on $S \backslash f(\partial\Omega)$.*

Proof. Let $\alpha \in S\backslash(f(\partial\Omega) \cup f(r))$. We take σ at α, a parametric disk Δ at α, and an arbitrary τ. Let $t(\zeta) = t(\zeta, \sigma, \tau) \in \mathscr{T}_1(I)$. There exists m_0 such that $\{\zeta \mid t(\zeta) \leqq m_0\} \subseteqq \Delta\backslash(f(\partial\Omega) \cup f(r))$.

For $m \geqq m_0$, (14) yields

$$m = \frac{1}{2\pi}\int_{\partial\Omega} m d^*p(z) + N(\Omega, \alpha) - \frac{1}{2\pi}\int_{E_m} N(\Omega, \zeta)\, d^*t(\zeta)\,,$$

which is

$$N(\Omega, \alpha) = \frac{1}{2\pi}\int_{E_m} N(\Omega, \zeta)\, d^*t(\zeta)\,.$$

Since $-t(\zeta)$ is the capacity function on the neighborhood of α bounded by $m \geqq m_0$, the function $N(\Omega, \zeta)$ is harmonic on $S\backslash(f(\partial\Omega) \cup f(r)$.

Let the multiplicity of r be k. By the construction used in Lemma 5.2 there is for any n, a neighborhood of $f(r)$ such that each ζ therein has k inverse images in

$$\{z \mid g(z, r) > n\}\,,$$

and a uniformly bounded number of other inverse images, for all of which $g(z, r)$ is uniformly bounded above. Hence $N(\Omega, \zeta)$ has a logarithmic singularity with coefficient k. This completes the proof.

6. **Functions of bounded characteristic.** The remark after Theorem 5.1 shows that if the characteristic function $T(\Omega, f)$ is bounded with respect to Ω when it is defined in terms of one principal family $\mathscr{T}$, then it is also bounded when defined in terms of another family. We shall show that this property is also independent of the points σ, τ, r, provided that σ, τ and $f(r)$ are distinct.

For a fixed family $\mathscr{T}_1(I)$ and a fixed τ, we define

$$x(\Omega, q) = \frac{1}{2\pi}\int_{\partial\Omega} \overset{+}{t}(f(z), \sigma, \tau)\, d^*p_\Omega(z, q)\,,$$

$$y(\Omega, q) = \sum_i g_\Omega(s_i, q)\,, \qquad \{s_i\} = f^{-1}(\sigma) \cap \Omega\,,$$

$$x'(\Omega, q) = \frac{1}{2\pi}\int_{\partial\Omega} \overset{+}{t}(f(z), \tau, \sigma)\, d^*p_\Omega(z, q)\,,$$

$$y'(\Omega, q) = \sum_i g_\Omega(t_i, q)\,, \qquad \{t_i\} = f^{-1}(\tau) \cap \Omega\,,$$

and $u = x + y$, $u' = x' + y'$, where Ω is a regular region in R, and s_i, t_i are repeated in accordance with their multiplicities.

LEMMA 6.1. *If Ω exhausts R, then the limits of $y(\Omega, q)$ and*

$u(\Omega, q)$ are either LP (harmonic with positive logarithmic singularities) functions or $+\infty$; if the limit of $u(\Omega, q)$ is LP, then the limit of $x(\Omega, q)$ is harmonic.

Proof. The classical method is employed. We first prove that $\Omega_1 \subseteqq \Omega_2$ implies $u(\Omega_1, q) \leqq u(\Omega_2, q)$. Let $z \in \bar{\Omega}_1$. We write $t(\zeta)$ for $t(\zeta, \sigma, \tau)$. If $\overset{+}{t}(f(z)) > 0$, then $\overset{+}{t}(f(z)) = t(f(z)) = u(\Omega_2, z) - u'(\Omega_2, z)$ by Jensen's formula (14). Hence $\overset{+}{t}(f(z)) \leqq u(\Omega_2, z)$ for all $z \in \bar{\Omega}_1$. Consequently,

$$\frac{1}{2\pi}\int_{\partial\Omega_1} \overset{+}{t}(f(z))\, d^*p_1(z, q) \leqq \frac{1}{2\pi}\int_{\partial\Omega_1}\Big[u(\Omega_2, z) - \sum_i g_1(s_i, z)\Big]\, d^*p(z, q)\ ,$$

which is, by transposition,

$$u(\Omega_1, q) \leqq u(\Omega_2, q)\ .$$

For any fixed Ω_0 we exhaust R by $\Omega \supseteqq \Omega_0$. By the application of Harnack's principle to $u(\Omega, q) - u(\Omega_0, q)$ over Ω_0, we find that the limit of $u(\Omega, q)$ is LP or $+\infty$ over Ω_0 and hence over R.

By the maximum principle, $y(\Omega_1, q) \leqq y(\Omega_2, q)$ when $\Omega_1 \subseteqq \Omega_2$, and the same proof carries through.

If the limit of $u(\Omega, q)$ is LP, so is that of $y(\Omega, q)$: further, both functions have the same singularities. By taking the limit of $x(\Omega, q) = u(\Omega, q) - y(\Omega, q)$, we obtain the harmonicity of the limit of $x(\Omega, q)$. This completes the proof.

THEOREM 6.2. *If $T(\Omega, f)$ is bounded with respect to Ω, then it is bounded for any choice of r, τ, σ if $f(r)$, σ and τ are distinct.*

Proof. A subscript indicates functions defined in terms of the new parameters.

(a) If r is changed to r_1 such that $f(r_1) \neq \sigma$, then $T_1(\Omega, f) = u(\Omega, r_1)$ is bounded since the limit of $u(\Omega, q)$ is LP.

(b) If τ is changed to τ_1, we have

$$|\, T_1(\Omega, f) - T(\Omega, f)\,| \leqq \frac{1}{2\pi}\int_{\partial\Omega}\Big|\overset{+}{t}_1(f(z)) - \overset{+}{t}(f(z))\Big|\, d^*p(z, r)\ .$$

The integrand is bounded since the function $q(\zeta)$ in the proof of Theorem 5.1 is bounded.

(c) If σ is changed to σ_1, we may by (b) take σ_1 as τ in defining $T(\Omega, f)$. From the definitions of the terms

$$T(\Omega, f) - (m_1(\Omega, \sigma) + N_1(\Omega, \sigma))$$

is a constant function. It follows from Theorem 5.1 that $T(\Omega, f) - T_1(\Omega, f) = O(1)$.

We have established the fact that the following class of functions is well-defined.

DEFINITION. *An analytic function $f: R \to S$, where R is an arbitrary open surface and S an arbitrary surface, is of bounded characteristic, $f \in MS$, if $T(\Omega, f)$ is bounded with respect to $\Omega \subseteqq R$.*

It follows from Lemma 6.1 that if $f \in MB$, then the limit of $y(\Omega, q)$ is a positive superharmonic function on R. Consequently R is hyperbolic and we set

$$N(R, \zeta, r) = \lim_{\Omega \to R} N(\Omega, \zeta) = \lim_{\Omega \to R} \sum_i g_\Omega(z_i, r) , \qquad \{z_i\} = f^{-1}(\zeta) \cap \Omega .$$

Since ([3] p. 429) $N(R, \zeta, r) = \sum_i g(z_i, r)$, $\{z_i\} = f^{-1}(\zeta)$, where g is Green's function on R, it follows ([3] p. 418) that the class MB is identical with the Lindelöfian maps. We are able to obtain a characterization in terms of $N(R, \zeta, q)$.

THEOREM 6.3. *If $f: R \to S$ is analytic, the following statements are equivalent:*

(a) $f \in MB$

(b) *there exists $s \in R$ and open $U \subseteqq S$ such that $N(R, \zeta, s) < \infty$ for $\zeta \in U$,*

(c) $N(R, \zeta, s) < \infty$, $s \in R$, $\zeta \in S \backslash f(r)$.

Proof. To prove that (b) implies (a) we select $\alpha \in U \backslash f(s)$ and a parametric disk Δ at α such that $\bar{\Delta} \subseteqq U \backslash f(s)$.

Set $\Delta_n = \{\zeta \in \bar{\Delta}_n \mid N(R, \zeta, s) \leqq n\}$ then by Lemma 5.2 $N(R, \zeta, s)$ is lower semi-continuous and Δ_n is closed. Also $\bar{\Delta} = \cup \Delta_n$. By Baire's category theorem there exists M such that Δ_M has an interior point.

Let $Q \subseteqq \Delta_M$ be an open region, and $\tau \in Q \backslash (\sigma \cup f(s))$. We define a family $\mathscr{T}_1(I)$ at σ; $t(\zeta, \sigma, \tau)$ has a level line E in Q. There is a principal family $\mathscr{T}_1'(I)$ such that $E = \{\zeta \mid t'(\zeta) = 0\}$, $t' \in \mathscr{T}_1'(I)$. Substitution in (16) yields

$$T(\Omega, f) \leqq \overset{+}{t}'(f(s)) + \int_E M d^* t'(\zeta) < \infty$$

for all Ω. Hence $f \in MB$.

(c) implies (b) trivially. To show that (a) implies (c) we note that by Lemma 6.1, $N(\Omega, \zeta)$ is bounded above for $s \in R$, $\zeta \in S \backslash f(s)$, whenever $f \in MB$. This completes the proof.

An extremal decomposition characterization of MB functions is given by the following:

THEOREM 6.4. *An analytic $f: R \to S$ is of class MB if and only*

if $t(f(z))$ is the difference between two LP functions, where t may be from any principal family $\mathscr{T}$.

Proof. From the proof of Lemma 6.1,

$$t(f(z)) = u(\Omega, z) - u'(\Omega, z)$$

for all Ω. If $f \in MB$, then the limits of u and u' are LP functions. This proves the necessity.

For the sufficiency we assume $t(f(z)) = v(z) - w(z)$, $v, w \in LP$.

The singularities of $u(\Omega, z)$ are positive singularities of $t(f(z))$, and so among the singularities of $v(z)$. Hence $v(z) - u(\Omega, z)$ is superharmonic on Ω and attains its minimum on $\partial\Omega$.

Let $w \in \partial\Omega$. By (2), $x(\Omega, z)$ is the harmonic function on Ω with boundary values $t(f(w))$ and

$$\lim_{z \to w} x(\Omega, z) = \overset{+}{t}(f(w))$$

for any approach to w; also $y(\Omega, z) \to 0$ as $z \to w$, and $v(w) \geqq t(f(w))$. Consequently $v(z) - u(\Omega, z) \geqq 0$ on Ω.

Since $v(z) \in LP$, there exists $r \in \Omega$ such that $v(r) < \infty$. Hence $u(\Omega, r)$ is bounded for all Ω, and $f \in MB$. This concludes the proof.

The integrand of the proximity function used by Sario [8] is

$$\begin{aligned} s(\zeta, \alpha) &= t(\zeta, \alpha, \sigma) + \log(1 + e^{2t(\zeta)})(1 + e^{2t(\alpha)}), \quad \alpha \neq \sigma \\ &= \log(1 + e^{2t(\zeta)}), \qquad \alpha = \sigma . \end{aligned}$$

where $t \in \mathscr{T}_0$ and $t(\zeta) = t(\zeta, \sigma, \tau)$. A comparison of the characteristic functions, evaluated at σ, shows that the functions of bounded characteristic with respect to Sario's characteristic function are the same as those treated above.

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