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The lattice ordered rings known as f-rings, introduced by Birkhoff and Pierce in [1], have been studied very intensively in the last few years. In particular Pierce has shown in [4] that the f-rings without nonzero nilpotents are precisely the (isomorphic images of) lattice ordered subdirect unions of totally ordered rings with integrity, and Johnson in [2] has gone on to prove that any Archimedean f-ring with no nonzero nilpotents can be represented as a lattice ordered ring of continuous extended realvalued functions on a locally compact Hausdorff space.

Since many commonly occurring examples of partially ordered rings are not lattice ordered it is natural to ask whether these two results can be generalised so as to be independent of the lattice structure. Such a generalisation is given here when multiplication is assumed commutative.

Theorem 1 characterises the subdirect unions of totally ordered commutative rings with integrity; Theorem 2 sharpens this result and Theorem 3 completes the programme by extending Johnson's representation.

The plan of the paper is as follows:

Section 1 is an introduction to the subject matter and methods of the paper; the succeeding three sections contain proofs of Theorems 1, 2 and 3 respectively and § 5 shows that for f-rings the representations given preserve the lattice structure.

1. Introduction. Throughout this paper "ring" will be an abbreviation for "commutative associative ring".

A *partially ordered* (or *po-*) *ring* is a ring whose elements are partially ordered in such a way that if $a \geqq b$ then $a + c \geqq b + c$ for all c and $ac \geqq bc$ for all $c \geqq 0$. Among the *po-rings* those with integrity (i.e. without divisors of zero) and a total ordering (the *toi-rings*) are particularly simple and it is our first aim to find out when a *po*-ring can suitably be built up from *toi*-rings. To make this more precise:

If $\{R_i\}_{i \in I}$ is a nonempty family of *toi*-rings their *direct union*, $\sum R_i$, is formed by taking the class of all functions $a: I \rightarrow \bigcup R_i$ with $a(i) \in R_i$ for all i, and defining addition by $(a + b)(i) = a(i) + b(i)$ for all i; multiplication by $(ab)(i) = a(i)b(i)$ for all i, and order by $a \geqq b$

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when $a(i) \geqq b(i)$ for all i. $\sum R_i$ is then a *po*-ring (in fact it is an *f*-ring). A *subdirect union* of the family $\{R_i\}_{i \in I}$ is a subring, R, of $\sum R_i$ satisfying $R(i) = R_i$ for all i, together with the partial ordering induced on it by the partial ordering of $\sum R_i$. If in addition, whenever R contains a it contains a^+, defined by $a^+(i) = a(i) \vee 0$ for all i, it is called a *lattice ordered* subdirect union of $\{R_i\}_{i \in I}$ (and is an *f*-ring).

A mapping, h, from one *po*-ring to another is called a *homomorphism* if it is a ring homomorphism such that $h(a) \geqq h(b)$ when $a \geqq b$: it is called an *isomorphism* if it is a ring isomorphism with $h(a) \geqq h(b)$ if and only if $a \geqq b$.

Suppose R is a *po*-ring and $\mathfrak{H}$ is a nonempty class of homomorphisms, h, of R onto *toi*-rings R_h respectively. Suppose further that if $a \in R$ and $a \ngeqq 0$ then there is an $h \in \mathfrak{H}$ with $h(a) < 0$. For any $a \in R$ let $\tilde{a}$ be the function on $\mathfrak{H}$ defined by $\tilde{a}(h) = h(a)$ for all $h \in \mathfrak{H}$. Then $\tilde{R} = \{\tilde{a}: a \in R\}$, with the natural induced structure, is a subdirect union of $\sum R_h$, and the map $a \rightarrow \tilde{a}$ is an isomorphism of R onto $\tilde{R}$.

To generate the homomorphisms needed we look at the *semirings* in R (i.e. the nonempty subsets, S, of R with $SS \cup (S + S) \subset S$). Under conditions stated in the next section, if $a_0 \ngeqq 0$ then maximalisation by Zorn's Lemma yields a semiring P, with $a_0 \notin P$ and $P'P' \subset -P'$,[1] which contains all $a \geqq 0$ and all squares in R. From this a homomorphism onto a *toi*-ring arises as follows:

$I = P \cap -P$ is a prime ring ideal in R. For,

(i) if $a, b \in I$ then clearly $a - b \in I$;

(ii) if $a \in I$ and $c \in R$ then $c \in P$ or $c \in -P$ (otherwise $-((-c)c) = c^2 \in P'$) and in either case $ac \in I$;

(iii) if $a \in I'$ and $b \in I'$ then $a \in P'$ or $-a \in P'$ and $b \in P'$ or $-b \in P'$; whence $ab \in P'$ or $-ab \in P'$ and certainly $ab \in I'$. Let h be the canonical homomorphism of R onto R/I, which is a ring with integrity. A simple calculation shows that $h(P)$ is a semiring, $h^{-1}(h(P)) = P$, $h(P) \cup -h(P) = h(R)$ and $h(P) \cap -h(P) = \{0\}$. So if we define $h(a) \geqq h(b)$ to mean $h(a) - h(b) \in h(P)$, (i.e. $a - b \in P$) then this is a total ordering making R/I into a *toi*-ring which is called the *quotient ring* of R by P and is denoted by R/P. Since P contains all $a \geqq 0$, $a_0 \in P'$ and $h^{-1}(hP) = P$, h is a homomorphism of R onto R/P and $h(a_0) < 0$.

It is convenient to write $a(P)$ for $h(a)$ and to use abbreviations similar to writing $a \geqq b, (P)$ for $a(P) \geqq b(P)$.

The representation of a *po*-ring as a ring of real valued functions on some set would be very useful. Unfortunately it seems difficult to find a simple general condition permitting this, which does not make all the functions used bounded. Nevertheless, a *po*-ring of the type here considered which is also Archimedean (that is $na \leqq b$, $n =$

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[1] $P' = R \backslash P$.

1, 2, $\cdots$ implies $a \leqq 0$) can be represented using functions with values in the extended real numbers. The possibility of this is suggested by the observation that in a *toi*-ring R if $ab \geqq 0$ and $a > 0$ then $cb \geqq 0$ for all $c \geqq 0$ so that if

$$\begin{aligned}\bar{a} = \inf \{m/n\colon\ & m \text{ and } n \text{ are integers, } n > 0, \text{ and} \\ & mb \geqq nab \text{ for all } b > 0\} \\ (= \sup \{m/n\colon\ & m \text{ and } n \text{ are integers, } n > 0, \text{ and} \\ & mb \leqq nab \text{ for all } b > 0\})\end{aligned}$$

it follows by routine calculations that $\bar{a} \geqq 0$ when $a \geqq 0$, $\overline{ab} = \bar{a}\bar{b}$ unless $\bar{a} = 0$ and $\bar{b} = \pm\infty$ or vice versa, and $\overline{a + b} = \bar{a} + \bar{b}$ unless $\bar{a}$ and $\bar{b}$ are infinite and of opposite sign. Here the infimum is taken in the extended reals and the infimum of the empty set is $+\infty$. The main problem is to guarantee that the substitution of $\bar{a}$ for a, which is usually far from being $(1 - 1)$, still leaves enough information for reconstruction of the original *po*-ring; it is here that the assumption that the ring is Archimedean is required.

The following notation will be standard for the rest of the paper:

If R is a *po*-ring then $R^+ = \{x\colon x \geqq 0\}$ is the class of *quasi positive* elements of R and $R^{++} = \{x\colon x > 0\}$ is the class of *positive* elements of R.

$\boldsymbol{Z}$ is the *po*-ring of integers.

$\boldsymbol{R}$ is the *po*-ring of real numbers and $\bar{\boldsymbol{R}}$ the "quasi *po*-ring" of the extended real numbers with the usual topology of the two point compactification.

If a set X is fixed in some context and $Y \subset X$ then Y' will denote $X \backslash Y$. The empty set is denoted by ϕ. The set with x as its only element will sometimes be denoted simply by x.

If A and B are subsets of a partially ordered set then $A \leqq B$ means that every element of A is less than or equal to every element in B.

2. f^*-rings.

Lemma 1 below, on the semirings in a ring, is the key to the rest of the paper. It is used in this section to produce a characterisation of the isomorphic images of subdirect sums of *toi*-rings (Theorem 1).

A semiring S in a ring R is said to be *normal* with respect to a nonempty subset H of R if no expression of the form

$$\sum_{i=1}^{N} (-1)^{n_i+1} s_i a_{i,1} a_{i,2} \cdots a_{i,n_i} - a_1 a_2 \cdots a_{2q} - s \tag{1}$$

is zero, where each a is in H, each s is in S, each n is in $\boldsymbol{Z}^{++}$, q is in $\boldsymbol{Z}^{++}$ and N is in $\boldsymbol{Z}^{+}$.

If S contains all squares in R and $H = \{a\}$ then S is normal with respect to H if and only if $sa - a^{2n} \in S'$ for all $s \in S$ and all $n \in \boldsymbol{Z}^{++}$.

Normality of S with respect to H implies $H \subset S'$. For if $a \in H \cap S$ then $(-1)^{1+1}aa + (-1)^{1+1}aa - aa - aa = 0$.

A semiring P in a ring R is called *prime* if $P'P' \subset -P'$.

The usefulness of normality is due to the following result:

LEMMA. *If S is a semiring containing all squares in a ring R, and H is a nonempty subset of R then there is a prime semiring P in R with $P \supset S$ and $P' \supset H$ if and only if S is normal with respect to H.*

Proof. (i) If such a P exists then for any $a_1, a_2, \cdots, a_n \in P'$ and any $s \in S$, $(-1)^{n+1}sa_1a_2 \cdots a_n \leqq 0, (P)$ (see § 1 for this notation); and if n is even $-a_1a_2 \cdots a_n < 0, (P)$. So any expression of the form (1) is $<0, (P)$ and cannot be equal to zero.

(ii) Conversely, if S is normal with respect to H then Zorn's lemma shows that there is a maximal semiring, P, among the semirings containing S which are normal with respect to H. It will be proved that P is as required.

Since P contains all squares in R, if $x \in R$ then the semiring, P_x, generated by $P \cup \{x\}$ is $\boldsymbol{Z}^{+}x + xP + P$. So if $x \in P'$ and $y \in P'$, since neither P_x nor P_y is normal with respect to H, there are identities of the form

$$\sum_{i=1}^{N} (-1)^{n_i+1}(s_i' + s_i)a_{i,1} \cdots a_{i,n_i} - a_1a_2 \cdots a_{2q} - (s' + s) = 0$$

and

$$\sum_{j=1}^{M} (-1)^{m_j+1}(t_j' + t_j)b_{j,1} \cdots b_{j,m_j} - b_1b_2 \cdots b_{2r} - (t' + t) = 0 ,$$

where every a and b is in H, every n and m is in $\boldsymbol{Z}^{++}$, q and r are in $\boldsymbol{Z}^{++}$, M and N are in $\boldsymbol{Z}^{+}$, every s and t is in P, every s' is in $\boldsymbol{Z}^{+}x + xP$ and every t' is in $\boldsymbol{Z}^{+}y + yP$.

Collection of the terms involving x, y respectively to one side of the equations (taking the rest to the other side) followed by multiplication of the new equalities yields, after rearrangement, the following,

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{M} (-1)^{n_i+m_j+2} s_i' t_j' a_{i,1} \cdots a_{i,n_i} b_{j,1} \cdots b_{j,m_j} + s't' \\
&\quad + \sum_{i=1}^{N} (-1)^{n_i+2} s_i' t' a_{i,1} \cdots a_{i,n_i} + \sum_{j=1}^{M} (-1)^{m_j+2} t_j' s' b_{j,1} \cdots b_{j,m_j} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{M} (-1)^{n_i+m_j+1} s_i t_j a_{i,1} \cdots a_{i,n_i} b_{j,1} \cdots b_{j,m_j} \\
&\quad + \sum_{i=1}^{N} (-1)^{n_i+1} s_i t a_{i,1} a_{i,2} \cdots a_{i,n_i} \\
&\quad + \sum_{i=1}^{N} (-1)^{n_i+2r+1} s_i a_{i,1} \cdots a_{i,n} b_1 \cdots b_{2r} \\
&\quad + \sum_{j=1}^{M} (-1)^{m_j+1} t_j s b_{j,1} \cdots b_{j,m_j} \\
&\quad + \sum_{j=1}^{M} (-1)^{m_j+2q+1} t_j b_{j,1} \cdots b_{j,m_j} a_1 \cdots a_{2q} + (-1)^{2r+1} s b_1 \cdots b_{2r} \\
&\quad + (-1)^{2q+1} t a_1 \cdots a_{2q} - st - a_1 a_2 \cdots a_{2q} b_1 b_2 \cdots b_{2r} = 0 .
\end{aligned}
$$

If $xy \in -P$ this would contradict the hypothesis that P is normal with respect to H.

It is clear that $P \supset S$ and $P' \supset H$, so the proof is complete.

COROLLARY. *If H has only one element, a, then there is a P as required if and only if $sa - a^{2n} \in S'$ for all $s \in S$ and all $n \in \boldsymbol{Z}^{++}$.*

The full force of Lemma 1 is not required until § 4; up to that point the corollary will be sufficient.

From now on A will always denote a *po*-ring, $\mathscr{S}$ the class of all semirings in A which contain A^+ and $\mathscr{P}$ the class of prime semirings in A which contain A^+. If $\mathscr{D}$ is a subset of $\mathscr{P}$ such that for any $a \notin A^+$ there is a $D \in \mathscr{D}$ with $a(D) < 0$ then $\mathscr{D}$ will be said to be *distinguishing*.

A is called an *f^*-ring* if A^+ contains all squares in A and is normal with respect to every single point set $\{a\}$ with $a \notin A^+$.

We have:

THEOREM 1. *A is isomorphic to a subdirect union of toi-rings if and only if it is an f^*-ring.*

Proof. (i) If A is an f^*-ring then the Corollary to Lemma 1 shows that $\mathscr{P}$ is distinguishing, so that from the discussion in the previous section, A is isomorphic to a subdirect union of *toi*-rings $\{A/P\}_{P \in \mathscr{P}}$.

(ii) If A can be identified with a subdirect union R of *toi*-rings $\{R_i\}_{i \in I}$ then $a \in A \backslash A^+$ implies $a(i) < 0$ for some $i \in I$, say $a(i_0) < 0$. Consequently, if $s \in A^+$ and $n \in \boldsymbol{Z}^{++}$, $(sa - a^{2n})(i_0) < 0$ and $sa - a^{2n} \notin A^+$.

Thus A is normal with respect to $\{a\}$. Also, for any $a \in A$, $(a^2)(i) = a(i)^2 \geqq 0$ for all $i \in I$, so $a^2 \in A^+$. Thus A is an f^*-ring.

3. Ring Archimedean f^*-rings.

In this section a class of f^*-rings is introduced which includes the Archimedean f^*-rings and for which a considerable sharpening of Theorem 1 is possible (see Theorem 2 below).

A *po*-ring R is called *ring* (or *r*-) *Archimedean* if $Z^+a + R^+a \leqq b$ implies $a \leqq 0$. An Archimedean *po*-ring is necessarily r-Archimedean, but the converse is not true, since every totally ordered field is r-Archimedean.

The following two measures of size will be used.

In any *toi*-ring R an element, a, is called a *ring* (r—) *order unit* if $Z^+a + R^+a - R^+ = R$, and is called *ring* (r-) *infinitesimal* if $Z^+a^2 + R^+a^2 \leqq |a|$. Notice that if for some $q > 0$, $(Z^+|a| + R^+|a|)q \leqq q$ then a is r-infinitesimal and $(Z^+|a| + R^+|a|)p \leqq p$ for all $p \geqq 0$. A *toi*-ring is r-Archimedean if and only if every positive element is an r-order unit.

The main result to be proved is:

THEOREM 2. *A necessary and sufficient condition that A be an r-Archimedean f^*-ring is that it be isomorphic to a subdirect union of r-Archimedean toi-rings with no nonzero r-infinitesimal elements.*

It will be convenient to divide up the proof into a number of lemmas.

LEMMA 2. *Let A be an r-Archimedean f^*-ring and $\mathscr{D}$ a distinguishing subclass of $\mathscr{P}$. If a (D) is r-infinitesimal in A/D for all $D \in \mathscr{D}$ such that $a \notin D$ then $a \geqq 0$.*

Proof. For each $D \in \mathscr{D}$ either (i) $a \geqq 0$ or (ii) $a < 0, (D)$ and $[Z^+(-a) + A^+(-a)](-a) \leqq (-a), (D)$. In either case $[Z^+(-a) + A^+(-a)]a^2 \leqq a^2(D)$. Therefore, since $\mathscr{D}$ is distinguishing, $[Z^+(-a) + A^+(-a)]a^2 \leqq a^2$; whence, A being r-Archimedean, $(-a)^3 \leqq 0$, and in an f^*-ring this implies $-a \leqq 0$, i.e. $a \geqq 0$.

LEMMA 3. *In any toi-ring R if a is not r-infinitesimal then $|a|$ is an r-order unit.*

Proof. If $Z^+|a| + R^+|a| \leqq b$ while $(n_0|a| + p_0|a|)|a| > |a|$ with $n_0 \in Z^+$ and $p_0 \in R^+$, then $b > 0$ and $(n_0|a| + p_0|a|)b > b \geqq (n_0b + p_0b)|a| = (n_0|a| + p_0|a|)b$, which is impossible.

Let $\mathscr{M}$ be the class of maximal elements in $\mathscr{P}$ (under set inclusion).

LEMMA 4. *If* $P \in \mathscr{P}$, $a \in P'$ *and* $|a(P)|$ *is an r-order unit in* A/P *then no* $Q \in \mathscr{P}$ *can contain* $P \cup \{a\}$, *therefore there is an* $M \in \mathscr{M}$ *with* $a \notin M \supset P$.

Proof. Suppose such a Q does exist and take $q \in Q'$. Since $-a(P)$ is an r-order unit in A/P there are $n \in \boldsymbol{Z}^+$ and $p \in P$ such that $n[(-a) + p(-a)] \geqq q(P)$. So $n(-a) + p(-a) - q \in P$ and $q \in P + na + pa \subset Q$, contrary to the hypothesis that $q \in Q'$.

The three previous lemmas show that $\mathscr{M}$ is distinguishing for r-Archimedean f^*-rings. However, a stronger result is needed to prove the Theorem.

LEMMA 5. *In any toi-ring R the class, I, of r-infinitesimal elements is a prime ring ideal such that if* $|c| \leqq |a|$ *and* $a \in I$ *then* $c \in I$.

Proof. If $a \in I$ and $|c| < |a|$, then for any $n \in Z^+$ and $p, q \in R^+$, $(n|c| + p|c|)q \leqq (n|a| + p|a|)q \leqq q$, so $c \in I$.

If $a, b \in I$, $n \in \boldsymbol{Z}^+$ and $p, q \in R^+$, $(2n|a-b| + 2p|a-b|)q \leqq (2n|a|+2p|a|)q+(2n|b|+2p|b|)q \leqq 2q$, whence $(n|a-b|+p|a-b|)q \leqq q$ and $a - b \in I$.

If $a \in I$ and $e \in R$ then $ae \in I$, for if not then, by Lemma 3, there are $n \in \boldsymbol{Z}^+$ and $p \in R^+$ such that $n|ae| + p|ae| > 2|e|$. But, since $a \in I$, $|e| \geqq n|ae| + p|ae|$, and these two inequalities together yield the contradiction, $0 > |e|$.

I has now been proved to be an ideal: it remains to prove that it is prime.

If $a, b \in I'$ there are $m, n \in \boldsymbol{Z}^+$ and $p, q \in R^+$ such that for any $s > 0$, $(m|a| + p|a|)s > s$ and $(n|b| + q|b|)s > s$, whence, by multiplication $(mn|ab| + (mp + nq + pq)|ab|)s^2 > s^2 > 0$, and so $ab \in I'$.

Let $\mathscr{M}^* = \{M \in \mathscr{M} : A/M$ contains no nonzero r-infinitesimal elements$\}$.

Then we have:

LEMMA 6. *If* $M \in \mathscr{M} \backslash \mathscr{M}^*$ *then every element of* A/M *is r-infinitesimal.*

Proof. Let $I_M = \{x \in A : x(M)$ is r-infinitesimal$\}$ and let $P = I_M + M$. Lemma 5 shows immediately that P is a semiring containing A. Furthermore if $a, b \in P'$ then $-a(M)$ and $-b(M)$ are positive and non-r-infinitesimal in A/M. So $a(M)b(M)$ is positive and non-r-infinitesimal in A/M, and $-ab \in P'$.

The maximality of M and the supposition that $M \notin \mathscr{M}^*$ imply therefore that $P = A$. So if $a \in A$ there is a $b \in I_M$ with $|b(M)| \geqq |a(M)|$, whence $a(M)$ is r-infinitesimal.

The following simple result proves to be important.

LEMMA 7. *If a is a non-r-infinitesimal positive element of a toi-ring R then there is a $b \in R^+$ such that $b^2 > a$*

Proof. If $a^2 \geqq a$ there is nothing to prove. If $a^2 < a$ then, since a is not r-infinitesimal, there are $n \in \boldsymbol{Z}^+$ and $p \in R^+$ with $(na + pa)a > a$; whence $(na + pa)^2a^2 > a^2 > a^3$, $(na + pa)^2a^2 > a^3$ and $(na + pa)^2 > a$. So $na + pa$ may be taken for b.

Proof of Theorem 2.

(i) *Necessity.* $\mathscr{M}^*$ is a distinguishing subset of $\mathscr{P}$; for if $a \ngeqq 0$ Lemma 2 shows that there is a $P \in \mathscr{P}$ with $a \in P'$ and $a(P)$ not r-infinitesimal and by Lemma 4 there is an $M \in \mathscr{M}$ containing A^+ with $a \notin M$, so $\mathscr{M}$ is distinguishing. Lemma 6 and a second application of Lemma 4 show that $\mathscr{M}^*$ is distinguishing.

Reference to the introduction completes the proof.

(ii) *Sufficiency.* Suppose A is identified with a subdirect union of a family $\{R_i\}_{i \in I}$ of *toi*-rings without nonzero r-infinitesimal elements. If $a \in A$ satisfies $\boldsymbol{Z}^+a + aA^+ \leqq b$ and $a(i) > 0$ for some $i \in I$ then $\boldsymbol{Z}^+a(i) + p^2(i)a(i) \leqq b(i)$ for all $p \in A^+$; and by Lemma 7, $\boldsymbol{Z}^+a(i) + R_i^+a(i) \leqq b(i)$. So, since R_i is r-Archimedean, $a(i) \leqq 0$, contrary to hypothesis. Thus $a \leqq 0$ and A is r-Archimedean.

4. Archimedean f^*-rings.

A ring of $\bar{\boldsymbol{R}}$-valued functions on a nonempty set X is a nonempty class, R, of $\bar{\boldsymbol{R}}$-valued functions on X such that

(i) If $\{f_i\}_{i \in I}$ is any finite subclass of R there is at least one point x in X where every $f_i(x)$ is finite.

(ii) If f, g and h are in R and $f(x) \geqq g(x)$ for all x where $h(x)$ is finite then $f(x) \geqq g(x)$ for all x in X.

(iii) If f and g are in R then there are functions s, p and n in R such that $s(x) = f(x) + g(x)$ whenever $f(x)$ and $g(x)$ are not infinite and of opposite sign, $p(x) = f(x)g(x)$ unless $f(x) = 0$ and $g(x) = \pm\infty$ or vice versa, and $n(x) = -f(x)$ for all x in X.

Condition (ii) shows that such s, p and n are unique, so they may be denoted by $f + g$, fg and $-f$ respectively.

Subsets of X of the form $\{x : f(x) = \pm\infty\}$ are called *nul-sets* (a name suggested by integration theory and Condition (ii)).

It is easily seen that any ring of $\bar{\boldsymbol{R}}$-valued functions on a set X

is an Archimedean f^*-ring. Conversely, if A is an Archimedean f^*-ring, and for each $a \in A$ $\bar{a}$ denotes the function $P \rightarrow \overline{a(P)}$ defined on $\mathscr{P}(\overline{a(P)})$ was defined in the Introduction), then Lemma 8 below and the remarks in the Introduction show that for any distinguishing subset $\mathscr{D}$ of $\mathscr{P}$ $\bar{A} \mid \mathscr{D} = \{\bar{a} \mid \mathscr{D} : a \in A\}$ is a ring of $\bar{R}$-valued functions on $\mathscr{D}$, and the map $a \rightarrow \bar{a} \mid \mathscr{D}$ is an isomorphism of A onto $\bar{A} \mid \mathscr{D}$.

If $\mathscr{D}$ is any subset of $\mathscr{P}$, $a, b \in A$ and $\lambda \in \bar{\mathbf{R}}$ it is convenient to adopt conventions similar to $\mathscr{D}(\bar{a} \geqq \lambda)$ for $\{D \in \mathscr{D} : \overline{a(D)} \geqq \lambda\}$ and $\mathscr{D}(a \geqq b)$ for $\{D \in \mathscr{D} : a(D) \geqq b(D)\}$.

Lemma 8. *If A is an Archimedean f^*-ring and $\mathscr{D}$ is a distinguishing subset of $\mathscr{P}$ and if $\mathscr{D}(\bar{a} < \bar{b})$ is a nul-set then $a \geqq b$.*

Proof. There is a $c \in A^+$ with $\mathscr{D}(\bar{c} = \infty) \supset \mathscr{D}(\bar{a} < \bar{b})$; so $e \overset{\text{def}}{=} c + a^2 + b^2$ satisfies $\mathscr{D}(a \neq 0) \cup \mathscr{D}(b \neq 0) \subset \mathscr{D}(e \neq 0)$ and $\mathscr{D}(\bar{a} < \bar{b}) \cup \mathscr{D}(\bar{a} = \pm\infty) \cup \mathscr{D}(\bar{b} = \pm\infty) \subset \mathscr{D}(\bar{e} = \infty)$.

Consider the following three situations which may occur for a $D \in \mathscr{D}$:

(i) $b > a, (D)$ and $\bar{e}(D) = \infty$; whence $\mathbf{Z}^+(b-a) \leqq e(b-a), (D)$ and so $\mathbf{Z}^+(b-a)2e \leqq e^4 + (b-a)^2, (D)$.

(ii) $b > a, (D)$, and $\bar{e}(D) < \infty$; whence $\bar{a}(D)$ and $\bar{b}(D)$ are finite, $\overline{(b-a)}(D) = 0$, and so $\mathbf{Z}^+(b-a)2e \leqq 2e, (D)$.

(iii) $b \leqq a, (D)$.

In all cases $\mathbf{Z}^+(b-a) \leqq e^4 + (b-a)^2 + 2e, (D)$. So $\mathbf{Z}^+(b-a)e \leqq e^4 + (b-a)^2 + 2e$ and, A being Archimedean, $(b-a)e \leqq 0$. This, in an f^*-ring with e as here defined, implies $b - a \leqq 0$, that is $a \geqq b$.

Corollary. *No nul-set can contain a nonempty set of the form $\mathscr{D}(\bar{a} > 0)$.*

Let $\mathscr{M}^{**} = \{M \in \mathscr{M}^* : \exists a \in A$ with $\bar{a}(M)$ nonzero$\}$.

Lemma 8 shows that $\mathscr{M}^{**}$ is distinguishing and so the mapping $a \rightarrow \bar{a} \mid \mathscr{M}^{**}$ is an isomorphism of A onto $\bar{A} \mid \mathscr{M}^{**}$.

Two natural topologies for $\mathscr{M}^{**}$, $\mathscr{T}_1$ with the sets of the form $\mathscr{M}^{**}(a > 0)$ as a subbase, and $\mathscr{T}_2$ with the sets of the form $\mathscr{M}^{**}(\bar{a} > 0)$ as a subbase, turn out to be the same.

Lemma 9. *$\mathscr{T}_1 = \mathscr{T}_2 (= \mathscr{T}$ say). $\mathscr{T}$ is Hausdorff and is the weak topology induced on $\mathscr{M}^{**}$ by $\bar{A}$.*

Proof. $\mathscr{T}_2 \supset \mathscr{T}_1$, for if $M \in \mathscr{M}^{**}(a > 0)$ there is a $b \in A$ with $\bar{b}(M) > 0$, and since $a(M)$ is an r-order unit, there are, using Lemma 7, $n \in \mathbf{Z}^+$ and $e \in A^+$ such that $na + e^2a > b, (M)$. So $M \in \mathscr{M}^{**}(\overline{na + e^2a} > 0) \subset \mathscr{M}^{**}(na + e^2a > 0) \subset \mathscr{M}^{**}(a > 0)$. Conversely, $\mathscr{T}_1 \supset \mathscr{T}_2$, for if $M \in \mathscr{M}^{**}(\bar{a} > 0)$ then for some $n \in \mathbf{Z}^{++}$,

$M \in \mathscr{M}^{**}(\bar{a} > 1/n)$; so $na^3 > a^2, (M)$ and $M \in \mathscr{M}^{**}(na^3 - a^2 > 0) \subset \mathscr{M}^{**}(\bar{a} \geqq 1/n) \subset \mathscr{M}^{**}(\bar{a} > 0)$. $\mathscr{T}$ is Hausdorff. If $M_1, M_2 \in \mathscr{M}^{**}$ and $M_1 \neq M_2$ there are $a_1 \in M_1 \backslash M_2$ and $a_2 \in M_2 \backslash M_1$. Whence $a = a_1 - a_2 \in (-M_1') \cap M_2'$, that is $M_2 \in \mathscr{M}^{**}(a < 0)$ and $M_1 \in \mathscr{M}^{**}(a > 0)$.

Finally, $\mathscr{T}$ is the weak topology induced by $\bar{A}$ on $\mathscr{M}^{**}$. For, by definition, $\mathscr{T}$ is coarser than this weak topology. Conversely, if $\lambda > -\infty \, \mathscr{M}^{**}(\bar{a} \geqq \lambda) = \cap \{\mathscr{M}^{**}(\overline{sae^2} \geqq r\bar{e}^2)\colon r/s < \lambda,\ s > 0 \text{ and } e \in A\}$, and so is closed with respect to $\mathscr{T}$.

Next it is shown that $\mathscr{M}^{**}(\bar{a} \geqq \varepsilon)$ is compact for all $\varepsilon > 0$ and all $a \in A$.

It is sufficient to prove the following result.

LEMMA 10. *If $a \in A$ then $\mathscr{M}^{**}(\bar{a} \leqq -1)$ is compact.*

Proof. Alexander's Theorem ([3] p. 139) shows that it is sufficient to prove that any cover of $\mathscr{M}^{***}(\bar{a} \leqq -1)$ by sets of the form $\mathscr{M}^{**}(c < 0)$, $c \in A$, has a finite subcover.

Accordingly, suppose C is a subset of A such that $\{\mathscr{M}^{**}(c < 0)\colon c \in C\}$ covers $\mathscr{M}^{**}(\bar{a} \leqq -1)$ and contains no finite subcover. A contradiction will be derived from this.

Consider any $M \in \mathscr{M}^{**}(\bar{a} \leqq -1)$ and any rational number m/n with $n>0$, $m>2$ and $2/3<m/n<1$. Since $\bar{a}(M) \leqq -1$, $naa^4 < -ma^4, (M)$ so $naa^4 + (m-2)a^4 < -2a^4 < -a^2, (M)$, that is $[na \cdot a^2 + (m-2)a^2]a^2 + a^2 < 0, (M)$. Thus $[na \cdot a^2 + (m-2)a^2]a^2 + a^2 \in N = \cap\{M'\colon \bar{a}(M) \leqq -1\}$.

Let $K = \{na \cdot a^2 + (m-2)a^2\colon m \geqq 2,\ n > 0 \text{ and } 2/3 < m/n < 1\}$.

If $\{c_i\}_{i=1}^r \subset C$ there is an $M \in \mathscr{M}^{**}(\bar{a} \leqq -1)$ with $\{c_i\}_{i=1}^r \subset M$. So the semiring, S, generated by $A^+ \cup C$ is normal with respect to N and there is a $P \in \mathscr{P}$ with $P \supset S$ and $P \cap N = \phi$. For any $k \in K$, $ka^2 + a^2 < 0, (P)$, so $k(P)$ is not r-infinitesimal in A/P. There is therefore an $M_0 \in \mathscr{M}$ with $M_0 \cap K = \phi$. Now for any element $na \cdot a^2 + (m-2)a^2$ of K $na \cdot a^2 + (m-2)a^2 < 0, (M_0)$; whence $\bar{a}(M_0) \leqq -(m-2)/n$. Consequently $\bar{a}(M_0) \leqq -1$, so $M_0 \in \mathscr{M}^{**}$, while $M_0 \supset C$, which is contrary to the hypothesis on C.

$\mathscr{M}^{**}$ may include semirings M such that $\bar{A}(M) \subset \{0, \pm\infty\}$. Lemma 8 shows that these are not algebraically significant (i.e. $\mathscr{M}^{***} \overset{\text{def}}{=} \{M \in \mathscr{M}^{**}\colon \exists a \in A \text{ with } \bar{a}(M) \notin \{0, \pm\infty\}\}$ is distinguishing). Considered as a subspace of the topological space $\{\mathscr{M}^{**}, \mathscr{T}\}$, $\mathscr{M}^{***}$ is a Hausdorff space. Further, since for all $a \in A$ and all $\lambda, \varepsilon \in R^+$, $\mathscr{M}^{**}(\lambda \geqq \bar{a} \geqq \varepsilon)$ is a closed, and therefore compact, subset of $\{\mathscr{M}^{**}, \mathscr{T}\}$ which is contained in $\mathscr{M}^{***}$. So $\mathscr{M}^{***}$ is a locally compact Hausdorff space; for if $D \in \mathscr{M}^{***}$ and $D \in \mathscr{M}^{**}(\bar{a}>0)$ there is a $b \in A$ with $\infty > \bar{b}(D) > 0$, so $\mathscr{M}^{**}(\bar{a} \geqq 1/2\, \bar{a}(D) \wedge 1) \cap \mathscr{M}^{**}(2\bar{b}(D) \geqq \bar{b} \geqq 1/2\, \bar{b}(D))$ is a compact neighbourhood of D in $\mathscr{M}^{***}$.

The following analogue of [2] Theorem 4.1 has now been proved.

THEOREM 3. *If A is an Archimedean f^*-ring the mapping $a \longrightarrow \bar{a} \mid \mathscr{M}^{***}$ is an isomorphism of A onto a ring $\bar{A} \mid \mathscr{M}^{***}$ of extended real valued functions on $\mathscr{M}^{***}$. The weak topology induced on $\mathscr{M}^{***}$ by $\bar{A} \mid \mathscr{M}^{***}$ is Hausdorff and locally compact and relative to it each set $\mathscr{M}^{***}(\lambda \geqq \bar{a} \geqq \varepsilon)$ with $a \in A$ and $\lambda, \varepsilon \in \boldsymbol{R}^{++}$ is compact. No function is infinite at every point of a nonempty set of the form $\mathscr{M}^{***}(\bar{a} > 0)$.*

The rest of Johnson's theorem seems to require that A be an f-ring.

5. *f*-**rings.** A commutative f-ring is a *po*-ring A which is lattice ordered in such a way that if $a \wedge b = 0$ then $ac \wedge b = 0$ for all $c \in A^+$.

An f-ring without nonzero nilpotents is an f^*-ring. For if $b, c \in A$ and $b \wedge c = 0$ then $bc \wedge bc = 0$, that is $bc = 0$. So for any $a \notin A^+$, $s \in A^+$ and $n \in \boldsymbol{Z}^{++}$, $sa - a^{2n} = sa^+ - sa^- - (a^+)^{2n} - (a^-)^{2n} \leqq sa^+ - (a^-)^{2n}$. And the latter expression is not in A^+ since $a^+ \wedge a^- = 0$ yields $sa^+ \wedge (a^-)^{2n} = 0$; whence $(sa^+ - (a^-)^{2n})^- = (a^-)^{2n} \neq 0$. Furthermore if A is an f^*-ring which is lattice ordered and such that $a \wedge b = 0$ implies $ab = 0$ then for any $P \in \mathscr{P}$, $(a \wedge b)(P) = a(P) \wedge b(P)$. For if $a \wedge b = c$ then $(a - c) \wedge (b - c) = 0$, so $(a - c)(b - c) = 0$; whence $(a - c)(P)(b - c)(P) = 0$. But A/P is a ring with integrity, so $(a - c)(P) = 0$ or $(b - c)(P) = 0$. Therefore, since $(a - c) \geqq 0$ and $(b - c) \geqq 0$, $(a - c)(P) \wedge (b - c)(P) = 0$ and $a(P) \wedge b(P) = c(P) = (a \wedge b)(P)$. Consequently the isomorphisms set up in Theorems 1 and 2 are isomorphisms onto a lattice ordered subdirect union of *toi*-rings which preserve lattice relations.

As for Theorem 3, it follows that for any $a, b \in A$ and any $M \in \mathscr{M}^{***}$, $\bar{a}(M) \wedge \bar{b}(M) = \bar{a} \wedge \bar{b}(M)$. Whence the sets $\{\mathscr{M}^{***}(a > 0)\}_{a \in A}$ form a basis for $\mathscr{T}$ and so does the class of sets $\{\mathscr{M}^{***}(\bar{a} > 0)\}_{a \in A}$. So each function a is finite on a dense subset of $\mathscr{M}^{***}$ (i.e. it is an *extended function* in the sense of [2]). Finally, Lemma 2.6 (ii) of [2] may be used to prove that the topology of $\mathscr{M}^{***}$ is precisely the weak topology induced by the bounded functions in $\bar{A} \mid \mathscr{M}^{***}$.

Note added in proof. Lemma 3, together with the remark at the end of the fourth paragraph of § 3, shows that for any toi ring, R, the following three properties are equivalent:

(i) R is r-Archimedean,

(ii) R has no nonzero r-infinitesimal elements,

(iii) Every element of R^{++} is an r-order unit.

So Theorem 2 can be sharpened. For example, we may omit "with no nonzero r-infinitesimal elements".

REFERENCES

1. G. Birkhoff, and R. S. Pierce, *Lattice ordered rings*, An. Acad. Brasil Ci., **28** (1956), 41-69.

2. D. J. Johnson, *A representation theory for a class of Archimedean lattice ordered rings*, Proc. London Math. Soc., **12** (1962), 207-225.

3. J. L. Kelley, *General Topology*, Van Nostrand. (New York 1955).

4. R. S. Pierce, *Radicals in function rings*, Duke Math. J., **23** (1956), 253-261.

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