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**INVERSION AND REPRESENTATION THEOREMS FOR A
GENERALIZED LAPLACE TRANSFORM**

J. M. C. JOSHI

INVERSION AND REPRESENTATION THEOREMS FOR A GENERALIZED LAPLACE TRANSFORM

J. M. C. JOSHI

1. Introduction. In a series of recent papers I have discussed various properties and inversion theorems etc. for the transform

$$(1.1) \quad F(x) = \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \int_0^\infty (xy)^\beta {}_1F_1(\beta + \eta + 1; \alpha + \beta + \eta + 1; -xy) f(y) dy .$$

where $f(y) \in L(0, \infty)$, $\beta \geq 0$, $\eta > 0$.

$$= A \int_0^\infty (xy)^\beta \psi(x, y) f(y) dy$$

where for convenience we denote $\Gamma(\beta + \eta + 1)/\Gamma(\alpha + \beta + \eta + 1)$ by A and ${}_1F_1(a; b; -xy)$ by $\psi(xy)$; a and b standing respectively for $\beta + \eta + 1$ and $\alpha + \beta + \eta + 1$. For $\alpha = \beta = 0$ (1.1) reduces to the wellknown Laplace transform

$$(1.2) \quad F(x) = \int_0^\infty e^{-xy} f(y) dy .$$

The transform (1.1), which may be called a generalization of the Laplace transform, arises if we apply Kober's operators of fractional integration [2] to the function $x^\beta e^{-x}$ [1].

The object of the present paper is to obtain an inversion and a representation theorem for the transform (1.1) by using properties of Kober's operators defined below.

2. Definition of operations. The operators given by Kober are defined as follows.

$$I_{\eta, \alpha}^+ [f(x)] = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha} \int_0^x (x-u)^{\alpha-1} u^\eta f(u) du$$

$$K_{\zeta, \alpha}^- [f(x)] = \frac{1}{\Gamma(\alpha)} x^\zeta \int_x^\infty (u-x)^{\alpha-1} u^{-\zeta-\alpha} f(u) du$$

where $f(x) \in L_p(0, \infty)$, $1/p + 1/q = 1$, if $1 < p < \infty$ and $1/p$ or $1/q = 0$ if p or $q = 1$, $\alpha > 0$, $\zeta > -(1/p)$, $\eta > -(1/q)$.

The Mellin transform $Mf(x)$ of a function $f(x) \in L_p(0, \infty)$ is defined as

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$$\bar{M}f(x) = \int_0^\infty f(x)x^{it}du \tag{p = 1}$$

and

$$= \lim_{x \rightarrow \infty}^{\text{index } \mathcal{V}} \int_{1/x}^x f(x)x^{it-1/q}d\eta \tag{p > 1} .$$

The inverse Mellin transform $M^{-1}\phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

$$(2.1) \quad M^{-1}\phi(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \phi(t)x^{-it}dt \tag{q = 1}$$

and

$$= \frac{1}{2\pi} \lim_{T \rightarrow \infty}^{\text{index } p} \int_{-T}^T \phi(t)x^{-it-1/p}dt \tag{q > 1} .$$

If Mellin transform is applied to Kober's operators and the orders of integrations are interchanged we obtain, under certain conditions

$$\bar{M}\{I_{\eta, \alpha}^+ f(x)\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)}{\Gamma\left(\alpha + \left\{\eta + \frac{1}{q} - it\right\}\right)} \bar{M}f(x)$$

and

$$\bar{M}\{K_{\xi, \alpha}^- f(x)\} = \frac{\Gamma\left(\xi + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\xi + \frac{1}{p} + it\right)\right]} \bar{M}f(x) .$$

But

$$\bar{M}(e^{-x} \cdot x^\beta) = \int_0^\infty e^{-x} x^{\beta+it-1/q} dx = \Gamma\left(\beta + it + \frac{1}{p}\right), \text{ if } \text{Re}\left(\beta + \frac{1}{p}\right) > 0 .$$

Therefore

$$\bar{M}\{I_{\eta, \alpha}^+(x^\beta e^{-x})\} = \frac{\Gamma\left[\left(\eta + \frac{1}{q} - it\right)\right] \Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left\{\eta + \frac{1}{q} - it\right\}\right]}$$

and

$$\bar{M}\{K_{\xi,\alpha}^-(x^\beta e^{-x})\} = \frac{\Gamma\left(\beta + it + \frac{1}{p}\right) P\left(\zeta + it + \frac{1}{p}\right)}{\Gamma\left[\alpha + \left\{\zeta + \frac{1}{p} + it\right\}\right]} .$$

By (2.1) we then have

$$(2.2) \quad I_{\eta,\alpha}^+(x^\beta e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\left(\eta + \frac{1}{q} - it\right) \Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} x^{-it-1/p} dt$$

and

$$K_{\xi,\alpha}^-(x^\beta e^{-x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right) \Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} x^{-it-1/p} dt ,$$

provided that $1/p > 0$, $\eta + 1/q > 0$ and $\zeta + 1/p > 0$.

3. Inversion theorem. We now define an inversion operator which will serve to invert (1.1).

An operator is defined for integral values of n by the relations

$$\begin{aligned} W_0[G(x)] &= G(x) , \\ W_n[G(x)] &= (-)^n n^{\beta+n+1} \left(\frac{d}{dx}\right)^n [x^{-\beta} G(x)], \quad (n = 1, 2, \dots) \\ Q_{n,t}[G(x)] &= \frac{1}{\Gamma(n + 1 + \beta - \alpha)} [W_n[G(x)]]_{n=t} \quad (n = 1, 2, \dots) . \end{aligned}$$

THEOREM 3.1. *If $f(t)$ is bounded in $(0 < t < \infty)$ then, provided that the integral (1.1) converges, $\eta > 0$, $\beta \geq 0$*

$$f(t) = \lim_{n \rightarrow \infty} Q_{n,t}[F(x)]$$

for almost all positive t .

Proof. Let x be any number greater than zero. Then, since the integral (1.1) converges, we can differentiate under the integral sign. Also (2.2) gives

$$(3.1) \quad \left(\frac{d}{dx}\right)[x^{-\beta} I_{n,\alpha}(x^\beta e^{-x})] = -x^{-\beta} I_{\eta+1,\alpha}[x^\beta e^{-x}] .$$

Using this relation we get

$$\begin{aligned} W_n[F(n)] &= (-)^n n^{\beta+n+1} \int_0^\infty x^{-\beta} y^n I_{\eta+n,\alpha}\{(xy)^\beta e^{-xy}\} f(y) dy \\ &= \frac{\Gamma(\beta + \eta + n + 1)}{\Gamma(\alpha + \beta + \eta + n + 1)} \int_0^\infty y^{\beta+n} {}_1F_1(\beta + \eta + n + 1; \\ &\qquad\qquad\qquad \alpha + \beta + \eta + n + 1 - xy) f(y) dy . \end{aligned}$$

Therefore

$$\begin{aligned} Q_{n,t}\{F(x)\} &= \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \left(\frac{n}{t}\right)^{\beta+n+1} \frac{1}{\Gamma(n + \beta + 1 - \alpha)} \\ &\quad \times \int_0^\infty y^{\beta+n} {}_1F_1(n + \beta + \eta + 1; \alpha + \beta + \eta + 1 + n; -xy) f(y) dy \\ &= \frac{1}{\Gamma(n + \beta + 1 - \alpha)} \left(\frac{n}{t}\right)^{\beta+n+1} \frac{\Gamma(a)}{\Gamma(b)} \\ &\quad \times \int_0^\infty y^{\beta+n} {}_1F_1(a + n; b + n; -xy) f(y) dy \end{aligned}$$

in the notation of § 1.

$$\begin{aligned} &= \frac{\Gamma(a + n)}{\Gamma(b + n)\Gamma(n + \beta + 1 - \alpha)} \left(\frac{n}{t}\right)^{n+\beta+1} \\ &\quad \times \int_0^\infty (tv)^{n+\beta} {}_1F_1(a + n; b + n; -nv) f(tv) dt \\ &= \frac{\Gamma(a + n)}{\Gamma(b + n)\Gamma(n + \beta + 1 - \alpha)} \left(\frac{n}{t}\right)^{n+\beta+1} \\ &\quad \times \int_0^\infty v^{n+\beta} {}_1F_1(\beta + \eta + n + 1; \alpha + \beta + \eta + n + 1; -nv) f(tv) dt \end{aligned}$$

by a simple change of variable. Now by using a result of Slater [4] we have

$$\frac{\Gamma(a + n)}{\Gamma(b + n)} {}_1F_1(a + n; b + n; -v) \sim (nv)^{a-b} e^{-nv} \quad (n \rightarrow \infty).$$

Therefore

$$\lim_{n \rightarrow \infty} Q_{n,t}\{F(n)\} = \lim_{n \rightarrow \infty} \frac{n^{\beta+n+1-\alpha}}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty v^{n+\beta-\alpha} e^{-nv} f(tv) dv .$$

But [3] we have for almost all positive t

$$\lim_{n \rightarrow \infty} \frac{n^{\beta+n+1-\alpha}}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty y^{n+\beta-\alpha} e^{-ny} \{f(ty) - f(t)\} dy = 0$$

and so we have our theorem.

5. Representation theorem. In this section we propose to give a set of necessary and sufficient conditions for the representation of a function as an integral of the form (1.1). We shall need a lemma which we now prove.

LEMMA 4.1. *If n is a positive integer and x and t are positive variables then*

$$\left(\frac{\partial}{\partial t}\right)^n \left[t^{\beta+n-1} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right] = \frac{n^n}{t^{n+1-\beta}} I_{\eta+n,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\}.$$

Proof. It is plain that

$$\left(\frac{t}{x}\right)^{\beta+n-1} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\}$$

is a homogeneous function of zero order. Therefore applying Euler's theorem we get

$$t \left(\frac{\partial}{\partial t}\right) \left[\left(\frac{t}{x}\right)^{\beta+n-1} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right] + n \left(\frac{\partial}{\partial x}\right) \left[\left(\frac{t}{x}\right)^{\beta+n-1} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right] = 0$$

or

$$\left(\frac{\partial}{\partial t}\right) \left[\frac{t^{\beta+n-1}}{x^{\beta+n}} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right] = - \left(\frac{\partial}{\partial x}\right) \left[\frac{t^{\beta+n-2}}{x^{\beta+n-1}} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right]$$

or

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \left[\frac{t^{\beta+n-1}}{x^{\beta+n}} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right] &= - \frac{\partial^2}{\partial t \partial x} \left[\frac{t^{\beta+n-2}}{x^{\beta+n-1}} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right] \\ &= - \left(\frac{\partial}{\partial x}\right) \left[\frac{\partial}{\partial t} \left\{ \frac{t^{\beta+n-2}}{x^{\beta+n-1}} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right\} \right] \\ &= (-)^2 \frac{\partial^2}{\partial x^2} \left[\frac{t^{\beta+n-3}}{x^{\beta+n-2}} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right]. \end{aligned}$$

Proceeding in the same manner we have

$$\frac{\partial^n}{\partial t^n} \left[\frac{t^{\beta+n-1}}{x^{\beta+n}} I_{\eta,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\} \right] = \frac{t^{\beta-n-1}}{x^\beta} I_{\eta+n,\alpha} \left\{ \left(\frac{x}{t}\right)^\beta e^{-x/t} \right\}$$

using (3.1).

THEOREM 4.1. *The necessary and sufficient conditions that a given function $F(x)$ may have the representation (1.1) with $f(y)$ bounded and $\text{Re } \eta > 0$ $\text{Re } \beta \geq 0$ are that*

(i) $F(x)$ has derivatives of all orders in $0 < x < \infty$.

- (ii) $F(x)$ tends to zero as x tends to infinity and
- (iii) $|Q_{n,t}\{F(x)\}| < M$ for all integral n ($0 < t < \infty$).

Proof. First let us suppose that $F(x)$ has the representation (1.1). Under the conditions of the theorem it is obvious that all the derivatives of $F(x)$ exist. Also

$$\begin{aligned} F(x) &\leq M' \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \\ &\quad \times \int_0^\infty (xy)^\beta {}_1F_1(\beta + \eta + 1; \alpha + \beta + \eta + 1; -xy) dy \\ &= \frac{M' \Gamma(\eta) \Gamma(\beta + 1)}{x \Gamma(\alpha + \eta)} \end{aligned}$$

since $f(y)$ is bounded. So $F(x)$ tends to zero as x tends to infinity. To prove the necessity of (iii) we see, as in Theorem 3.1, that

$$|Q_{n,t}\{F(x)\}| \leq \left\{ \frac{n^{\beta+n+1-\alpha}}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty v^{n+\beta-\alpha} e^{-nv} dv \right\} \left\{ \text{lub}_{0 \leq t < \infty} |f(tv)| \right\} = M.$$

To prove the sufficiency let us suppose that the conditions are satisfied. If we now set

$$J_n = \int_0^\infty I_{\eta,\alpha}\{(xy)^\beta e^{-xy}\} Q_{n,y}\{F(x)\} dy$$

we have

$$\begin{aligned} J_n &= \frac{1}{\Gamma(n + 1 + \beta - \alpha)} \int_0^\infty \frac{n}{t^2} I_{\eta,\alpha}\left\{\left(\frac{nx}{t}\right)^\beta e^{-nx/t}\right\} W_n\{F(x)\} dn \\ &= (-)^n \int_0^\infty nt^{n+\beta-1} I_{\eta,\alpha}\left\{\left(\frac{nx}{t}\right)^\beta e^{-nx/t}\right\} \left(\frac{d}{dt}\right)^n \{t^{-\beta} F(t)\} dt. \end{aligned}$$

It will be seen in the course of the argument that this integral exists. Integrating by parts we have

$$\begin{aligned} J_n &= \frac{(-)^n n}{\Gamma(n + \beta + 1 - \alpha)} \left[t^{n+\beta-1} I_{\eta,\alpha}\left\{\left(\frac{nx}{t}\right)^\beta e^{-nx/t}\right\} \left(\frac{d}{dt}\right)^{n-1} \{t^{-\beta} F(t)\} \right]_0^\infty \\ &\quad + \frac{(-)^{n-1} n}{\Gamma(n + 1 + \beta - \alpha)} \int_0^\infty \left(\frac{d}{dt}\right)^{n-1} \{t^{-\beta} F(t)\} \left(\frac{\partial}{\partial t}\right) \{t^{n+\beta-1} I_{\eta,\alpha} \phi\} dt \end{aligned}$$

where

$$\phi \equiv \left(\frac{nx}{t}\right)^\beta e^{-nx/t}.$$

Now

$$\begin{aligned} I_{\eta\alpha}\phi &= O(t^{\eta+1}) \quad (t \rightarrow 0) \\ &= O(1) \quad \beta = O(t \rightarrow \infty) \\ &= O(1) \quad \beta > 0(t \rightarrow \infty) \end{aligned}$$

for [1]

$$I_{\eta,\alpha}(\phi) = \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \left(\frac{nx}{t}\right)^\beta {}_1F_1\left(\beta + \eta + 1; \alpha + \beta + \eta + 1; -\frac{nx}{t}\right).$$

Also the hypotheses of the theorem by implications mean that

$$F(x) = O(x^{-1})$$

and in general

$$F^{(n)}(x) = O(x^{-n-1})$$

and

$$\begin{aligned} &\left(\frac{d}{dt}\right)^{n-1} [t^{-\beta}F(t)] \\ &= \{(-)^{n-1}\beta(\beta + 1) \cdots (\beta + n - 2)t^{-\beta-n+1}F(t) + \cdots t^{-\beta}F^{(n-1)}(t)\}. \end{aligned}$$

Therefore the integrated part

$$= O[t^{\eta+1}\{A_1F(t) + \cdots t^{n-1}F^{(n-1)}(t)\}] \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Also it is

$$= O[A_1F(t) + \cdots tF^{(n-1)}(t)] \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Therefore the integrated part is zero and integrating by parts again

$$\begin{aligned} J_n &= \frac{(-)^{n-1}n}{\Gamma(n + \beta + 1 - \alpha)} \left[\frac{\partial}{\partial t} (t^{n+\beta-1}I_{\eta\alpha}\phi) \left(\frac{d}{dt}\right)^{n-2} \{t^{-\beta}F(t)\} \right]_0^\infty \\ &+ \frac{(-)^{n-2}n}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty \left(\frac{d}{dt}\right)^{n-2} \{t^{-\beta}F(t)\} \frac{\partial^2}{\partial t^2} (t^{n+\beta-1}I_{\eta,\alpha}\phi) dt. \end{aligned}$$

Now

$$\left(\frac{\partial}{\partial t}\right) \{t^{\beta+n-1}I_{\eta,\alpha}\phi\} = [(n - 1)t^{\beta+n-2}I_{\eta,\alpha}\phi + \cdots + nnt^{\beta+n-3}I_{\eta+1,\alpha}(\phi)]$$

and

$$\begin{aligned} &\left(\frac{d}{dt}\right)^{n-2} \{t^{-\beta}F(t)\} \\ &= \{(-)^{n-2}\beta(\beta + 1) \cdots (\beta + n - 3)t^{-\beta-n+2}F(t) + \cdots t^{-\beta}F^{(n-2)}(t)\}. \end{aligned}$$

Therefore as before the integrated part again approaches zero when t tends to zero and t tends to infinity. Proceeding in the same manner we obtain

$$\begin{aligned}
 J_n &= \frac{n}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty t^{-\beta} F(t) \frac{\partial^n}{\partial t^n} \{t^{\beta+n-1} I_{n,\alpha} \phi\} dt \\
 &= \frac{n}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty t^{-\beta} F(t) \frac{(nx)^n}{t^{n+1}} t^\beta I_{n,\alpha}(\phi) dt
 \end{aligned}$$

by the Lemma 4.1. Hence

$$J_n = \frac{n^{\beta+n+1} n^{\alpha+\beta} \Gamma(a)}{\Gamma(n + \beta + 1 - \alpha) \Gamma(b)} \int_0^\infty t^{-\beta-n-1} {}_1F_1\left(a; b; -\frac{nx}{t}\right) F(t) dt .$$

It is clear that this integral exists under the hypotheses of the theorem and therefore all the previous integrals exist. By a simple substitution this gives on using the asymptotic expansion of ${}_1F_1(a; b; x)$ [4]

$$J_n \sim \frac{n^{\beta+n+1} n^{\alpha+\beta}}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty u^{\beta+n-1} e^{-nxu} F\left(\frac{1}{u}\right) du .$$

Let

$$(1/u)F\left(\frac{1}{u}\right) \equiv \psi(u) .$$

Now

$$(1/u)F(1/u) = o(1) \quad (u \rightarrow \infty) \quad \text{and} \quad F\left(\frac{1}{u}\right) = o(1) \quad (u \rightarrow 0) .$$

Hence it is easily seen

- (i) $\psi(u) \in L$ ($1/R \leq t < R$) for every $R > 1$.
- (ii) $\int_0^\infty \psi(u) e^{-cu} du$ converges for any fixed $c > 0$, and
- (iii) $\int_0^1 u \psi(u) du$ also converges. Therefore [3]

$$\lim_{n \rightarrow \infty} J_n = \frac{1}{u} \psi\left(\frac{1}{u}\right) = F(u) .$$

Now if

$$\chi(x, y) = \frac{\Gamma(a)}{\Gamma(b)} (xy)^\beta {}_1F_1(a; b; -xy) .$$

Then $\chi(xy) \in L$ in $0 \leq y < \infty$ under the conditions assumed for the convergence of (1.1). Therefore by a theorem on weak compactness of a set of functions [5] the inequalities in the hypothesis (iii) of the theorem imply the existence of a subset $\{n_i\}$ of the positive integers

and a bounded function $f(y)$ such that

$$\lim_{i \rightarrow \infty} \int_0^{\infty} [Q_{n_i, y}\{F(x)\}]\chi(x, y) = \int_0^{\infty} \chi(x, y)f(y)dy .$$

Hence

$$F(x) = \int_0^{\infty} \chi(x, y)f(y)dy$$

and the theorem is established.

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REFERENCES

1. A. Erdelyi, *On some functional transformations*, Rend. del Semin. Matematico. **10** (1950-51), 217-234.
2. H. Kober, *On fractional integrals and derivatives*, Quart. Jour. Math. **11** (1940), 193-211.
3. J. M. C. Joshi, *A singular integral and a real inversion theorem for a generalized Laplace transform*, to appear.
4. L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, 1960.
5. D. V. Widder, *The Laplace Transform*, Princeton University Press 1946.

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