INVERSION AND REPRESENTATION THEOREMS FOR A GENERALIZED LAPLACE TRANSFORM

J. M. C. Joshi
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1. Introduction. In a series of recent papers I have discussed various properties and inversion theorems etc. for the transform

\[ F(x) = \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \int_0^\infty (xy)^{\beta} \psi(xy) \, f(y) \, dy, \]

(1.1)

where \( f(y) \in L^0, \infty \), \( \beta \geq 0 \), \( \eta > 0 \).

\[ = A \int_0^\infty (xy)^{\beta} \psi(x, y) \, f(y) \, dy \]

where for convenience we denote \( \Gamma(\beta + \eta + 1)/\Gamma(\alpha + \beta + \eta + 1) \) by \( A \) and \( \psi(xy) \) by \( \psi(xy) \); \( a \) and \( b \) standing respectively for \( \beta + \eta + 1 \) and \( a + \alpha \). For \( \alpha = \beta = 0 \) (1.1) reduces to the wellknown Laplace transform

\[ F(x) = \int_0^\infty e^{-xy} f(y) \, dy. \]

(1.2)

The transform (1.1), which may be called a generalization of the Laplace transform, arises if we apply Kober's operators of fractional integration [2] to the function \( x^\beta e^{-x} \) [1].

The object of the present paper is to obtain an inversion and a representation theorem for the transform (1.1) by using properties of Kober's operators defined below.

2. Definition of operations. The operators given by Kober are defined as follows.

\[ I_{\eta,a}^+[f(x)] = \frac{1}{\Gamma(\alpha)} x^{-\eta-a} \int_0^x (x-u)^{a-1} u^\eta f(u) \, du \]

\[ K_{\xi,a}^-[f(x)] = \frac{1}{\Gamma(\alpha)} x^\xi \int_0^\infty (u-x)^{a-1} u^{-\xi-a} f(u) \, du \]

where \( f(x) \in L_p(0, \infty) \), \( 1/p + 1/q = 1 \), if \( 1 < p < \infty \) and \( 1/p \) or \( 1/q \) 0 if \( p \) or \( q=1 \), \( \alpha > 0 \), \( \xi > -(1/p) \), \( \eta > -(1/q) \).

The Mellin transform \( \tilde{M}f(x) \) of a function \( f(x) \in L_p(0, \infty) \) is defined as

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The inverse Mellin transform $M^{-1} \phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

$$(2.1) \quad M^{-1} \phi(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-T}^{T} \phi(t)x^{-it}dt \quad (q = 1)$$

and

$$= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-r}^{r} \phi(t)x^{-it-1/q}dt \quad (q > 1).$$

If Mellin transform is applied to Kober's operators and the orders of integrations are interchanged we obtain, under certain conditions

$$\tilde{M}\{I_{\eta x}^+ f(x)\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)}{\Gamma\left(\alpha + \left\{\eta + \frac{1}{q} - it\right\}\right)} \tilde{M}f(x)$$

and

$$\tilde{M}\{K_{\xi x}^- f(x)\} = \frac{\Gamma\left(\xi + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\xi + \frac{1}{p} + it\right)\right]} \tilde{M}f(x).$$

But

$$\tilde{M}(e^{-\alpha} \cdot x^\beta) = \int_0^\infty e^{-\alpha}x^{\beta+it-1/q}dx = \Gamma\left(\beta + it + \frac{1}{p}\right), \text{ if } Re\left(\beta + \frac{1}{p}\right)>0.$$
\[
\bar{M}\{K_{\xi,\alpha}(\omega^\beta e^{-\tau})\} = \frac{\Gamma(\beta + it + \frac{1}{p})P(\zeta + it + \frac{1}{p})}{\Gamma(\alpha + \frac{1}{p} + it)}.
\]

By (2.1) we then have

\[
(2.2) \quad I_{\eta,\alpha}(\omega^\beta e^{-\tau}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\eta + \frac{1}{q} - it)\Gamma(\beta + \frac{1}{p} + it)}{\Gamma(\alpha + \eta + \frac{1}{q} - it)} \omega^{-it-1/p} dt
\]

and

\[
K_{\xi,\alpha}(\omega^\beta e^{-\tau}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\zeta + \frac{1}{p} + it)\Gamma(\beta + \frac{1}{p} + it)}{\Gamma(\alpha + \zeta + \frac{1}{p} + it)} \omega^{-it-1/p} dt,
\]

provided that \(1/p > 0, \eta + 1/q > 0\) and \(\zeta + 1/p > 0\).

3. **Inversion theorem.** We now define an inversion operator which will serve to invert (1.1).

An operator is defined for integral values of \(n\) by the relations

\[
W_0[G(x)] = G(x),
\]

\[
W_n[G(x)] = (-)^n \omega^{\beta+n+1} \left(\frac{d}{dx}\right)^n [\omega^{-\beta} G(x)], (n = 1, 2, \cdots)
\]

\[
Q_{n,t}[G(x)] = \frac{1}{\Gamma(n + 1 + \beta - \alpha)} \left[ W_n[G(x)] \right]_{n=n/\alpha} (n = 1, 2, \cdots).
\]

**Theorem 3.1.** If \(f(t)\) is bounded in \((0 < t < \infty)\) then, provided that the integral (1.1) converges, \(\eta > 0, \beta \geq 0\)

\[
f(t) = \lim_{n \to \infty} Q_{n,t}[F(x)]
\]

for almost all positive \(t\).

**Proof.** Let \(x\) be any number greater than zero. Then, since the integral (1.1) converges, we can differentiate under the integral sign. Also (2.2) gives

\[
(3.1) \quad \left(\frac{d}{dx}\right) [\omega^{-\beta} I_{\eta,\alpha}(\omega^\beta e^{-\tau})] = -\omega^{-\beta} I_{\eta+1,\alpha}[\omega^\beta e^{-\tau}].
\]

Using this relation we get
\[ W_n[F(n)] = (-)^n n^{\beta+n+1} \int_0^\infty x^{-\beta} y^n I_{\eta+n,a}(xy) e^{-xy} f(y) dy \]
\[ = \frac{\Gamma(\beta + \eta + n + 1)}{\Gamma(\alpha + \beta + \eta + n + 1)} \int_0^\infty y^{\beta+n} F_1(\beta + \eta + n + 1; \alpha + \beta + \eta + n + 1; -xy) f(y) dy . \]

Therefore

\[ Q_{n,t} F(x) \]
\[ = \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \left( \frac{n}{t} \right)^{\beta+n+1} \frac{1}{\Gamma(n + \beta + 1 - \alpha)} \times \int_0^\infty y^{\beta+n} F_1(a + n + b + n; -xy) f(y) dy \]
\[ = \frac{\Gamma(\alpha + n)}{\Gamma(b + n) \Gamma(n + \beta + 1 - \alpha)} \left( \frac{n}{t} \right)^{n+\beta+1} \times \int_0^\infty (tv)^{\nu+n+1} F_1(a + n; b + n; -nv) f(tv) dt \]
\[ = \frac{\Gamma(\alpha + n)}{\Gamma(b + n) \Gamma(n + \beta + 1 - \alpha)} \left( \frac{n}{t} \right)^{n+\beta+1} \times \int_0^\infty \nu^{n+\beta+1} F_1(\beta + \eta + n + 1; \alpha + \beta + \eta + n + 1; -nv) f(tv) dt \]

by a simple change of variable. Now by using a result of Slater \[4\] we have

\[ \frac{\Gamma(a + n)}{\Gamma(b + n)} F_1(a + n; b + n; -v) \sim (nv)^{\alpha-b} e^{-nv} \quad (n \to \infty). \]

Therefore

\[ \lim_{n \to \infty} Q_{n,t} F(n) = \lim_{n \to \infty} \frac{n^{\beta+n+1-\omega}}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty \nu^{n+\beta-\omega} e^{-\nu y} f(tv) dv . \]

But \[3\] we have for almost all positive \( t \)

\[ \lim_{n \to \infty} \frac{n^{\beta+n+1-\omega}}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty \nu^{n+\beta-\omega} e^{-\nu y} (f(y) - f(t)) dy = 0 \]

and so we have our theorem.
5. Representation theorem. In this section we propose to give a set of necessary and sufficient conditions for the representation of a function as an integral of the form (1.1). We shall need a lemma which we now prove.

**Lemma 4.1.** If $n$ is a positive integer and $x$ and $t$ are positive variables then

$$
\left( \frac{\partial}{\partial t} \right)^n \left[ t^{\beta + n - 1} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right] = n^n t^{n+1-\beta} I_{\eta + n, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\}.
$$

**Proof.** It is plain that

$$
\left( \frac{t}{x} \right)^{\beta + n - 1} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\}
$$
is a homogeneous function of zero order. Therefore applying Euler's theorem we get

$$
t \left( \frac{\partial}{\partial t} \right) \left[ \left( \frac{t}{x} \right)^{\beta + n - 1} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right] = n \left( \frac{\partial}{\partial x} \right) \left[ \left( \frac{t}{x} \right)^{\beta + n - 1} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right] = 0,
$$
or

$$
\left( \frac{\partial}{\partial t} \right) \left[ \frac{t^{\beta + n - 1}}{x^{\beta + n}} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right] = - \left( \frac{\partial}{\partial x} \right) \left[ \frac{t^{\beta + n - 2}}{x^{\beta + n - 1}} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right].
$$
or

$$
\frac{\partial^2}{\partial t^2} \left[ \frac{t^{\beta + n - 1}}{x^{\beta + n}} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right] = - \frac{\partial^2}{\partial t \partial x} \left[ \frac{t^{\beta + n - 2}}{x^{\beta + n - 1}} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right] = - \left( \frac{\partial}{\partial x} \right) \left[ \frac{t^{\beta + n - 2}}{x^{\beta + n - 1}} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right].
$$

Proceeding in the same manner we have

$$
\frac{\partial^n}{\partial t^n} \left[ \frac{t^{\beta + n - 1}}{x^{\beta + n}} I_{\eta, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\} \right] = \frac{t^{\beta - n - 1}}{x^\beta} I_{\eta + n, \alpha} \left\{ \left( \frac{x}{t} \right)^{\beta} e^{-x/t} \right\},
$$

using (3.1).

**Theorem 4.1.** The necessary and sufficient conditions that a given function $F(x)$ may have the representation (1.1) with $f(y)$ bounded and $\text{Re} \eta > 0 \text{ Re} \beta \geq 0$ are that

(i) $F(x)$ has derivatives of all orders in $0 < x < \infty$. 
(ii) $F(x)$ tends to zero as $x$ tends to infinity and
(iii) $|Q_{n,t}\{F(x)\}| < M$ for all integral $n$ ($0 < t < \infty$).

**Proof.** First let us suppose that $F(x)$ has the representation (1.1). Under the conditions of the theorem it is obvious that all the derivatives of $F(x)$ exist. Also

$$F(x) \leq \frac{M'}{\Gamma(\alpha + \beta + \eta + 1)} \int_{0}^{\infty} \frac{(xy)^{\beta} F_{n}(\beta + \eta + 1; \alpha + \beta + \eta + 1; -xy)dy}{x^{\eta + 1}}$$

since $f(y)$ is bounded. So $F(x)$ tends to zero as $x$ tends to infinity.

To prove the necessity of (iii) we see, as in Theorem 3.1, that

$$|Q_{n,t}\{F(x)\}| \leq \left\{ \frac{\Gamma(n + \beta + 1 - \alpha)}{\Gamma(n + \beta + 1 - \alpha)} \int_{0}^{\infty} \frac{g^{n+\beta-\alpha} e^{-\eta y}dv}{y} \right\} \left\{ \text{lub } |f(y)| \right\} = M.$$ 

To prove the sufficiency let us suppose that the conditions are satisfied. If we now set

$$J_{n} = \int_{0}^{\infty} I_{\eta,\alpha}(xy)e^{-xy}Q_{n,y}\{F(x)\}dy$$

we have

$$J_{n} = \frac{1}{\Gamma(n + 1 + \beta - \alpha)} \int_{0}^{\infty} \frac{n^{n+\beta-1} I_{\eta,\alpha}\left\{ \left( \frac{nx}{t} \right)^{\beta} e^{-\eta z/t} \right\}}{t^{n+\beta-1}} W_{n}(F(x))dn$$

$$= \left\{ \frac{(-)^{n}n}{\Gamma(n + 1 + \beta - \alpha)} \int_{0}^{\infty} \frac{\left( \frac{nx}{t} \right)^{\beta} e^{-\eta z/t}}{t^{n+\beta-1}} \left\{ \frac{d}{dt} \right\}^{n-1} [t^{-\beta} F(t)]dt \right\}.$$ 

It will be seen in the course of the argument that this integral exists. Integrating by parts we have

$$J_{n} = \frac{(-)^{n}n}{\Gamma(n + 1 + \beta - \alpha)} \left[ t^{n+\beta-1} I_{\eta,\alpha}\left\{ \left( \frac{nn}{t} \right)^{\beta} e^{-\eta n/t} \right\} \right]^{\infty}_{0} \left\{ \frac{d}{dt} \right\}^{n-1} [t^{-\beta} F(t)] + \frac{(-)^{n-1}n}{\Gamma(n + 1 + \beta - \alpha)} \int_{0}^{\infty} \left\{ \frac{d}{dt} \right\}^{n-1} [t^{-\beta} F(t)] \left( \frac{\partial}{\partial t} \right) [t^{n+\beta-1} I_{\eta,\alpha} \phi] dt$$

where

$$\phi \equiv \left( \frac{nx}{t} \right)^{\beta} e^{-\eta z/t}.$$
Now
\[ I_{\eta,\alpha}^\beta = 0(t^{\eta+1}) \quad (t \to 0) \]
\[ = 0(1) \quad \beta = 0(t \to \infty) \]
\[ = 0(1) \quad \beta > 0(t \to \infty) \]
for [1]
\[ I_{\eta,\alpha}(\phi) = \frac{\Gamma(\beta + \gamma + 1)}{\Gamma(\alpha + \beta + \gamma + 1)} \left( \frac{n^x}{t} \right)^\beta F_1(\beta + \gamma + 1; \alpha + \beta + \gamma + 1; -\frac{n^x}{t}) . \]

Also the hypotheses of the theorem by implications mean that
\[ F(x) = 0(x^{-1}) \]
and in general
\[ F^{(n)}(x) = 0(x^{-n-1}) \]

and
\[ \left( \frac{d}{dt} \right)^{n-1} [t^{-\beta} F(t)] \]
\[ = \{(-)^{n-1}\beta(\beta + 1) \cdots (\beta + n - 2)t^{-\beta - n+1} F(t) + \cdots t^{-\beta} F^{(n-1)}(t) \} . \]

Therefore the integrated part
\[ = 0[t^{\eta+1} \{A_1 F(t) + \cdots t^{n-1} F^{(n-1)}(t) \}] \to 0 \quad \text{as} \quad t \to 0 . \]

Also it is
\[ = 0[A_1 F(t) + \cdots t F^{(n-1)}(t)] \to 0 \quad \text{as} \quad t \to \infty . \]

Therefore the integrated part is zero and integrating by parts again
\[ J_n = \frac{(-)^{n-1}n}{\Gamma(n + \beta + 1 - \alpha)} \left[ \frac{\partial}{\partial t} (t^{n+\beta-1} I_{\eta,\alpha} \phi) \right]_0^{\infty} \left( \frac{d}{dt} \right)^{n-2} [t^{-\beta} F(t)] \]
\[ + \frac{(-)^{n-2}n}{\Gamma(n + \beta + 1 - \alpha)} \int_0^{\infty} \left( \frac{d}{dt} \right)^{n-2} [t^{-\beta} F(t)] \frac{\partial^2}{\partial t^2} (t^{n+\beta-1} I_{\eta,\alpha} \phi) dt . \]

Now
\[ \left( \frac{\partial}{\partial t} \right) [t^{\beta+n-1} I_{\eta,\alpha} \phi] = [(n - 1)t^{\beta+n-2} I_{\eta,\alpha} \phi + \cdots + nnt^{\beta+n-3} I_{\eta+1,\alpha}(\phi)] \]
and
\[ \left( \frac{d}{dt} \right)^{n-2} [t^{-\beta} F(t)] \]
\[ = \{(-)^{n-2}\beta(\beta + 1) \cdots (\beta + n - 3)t^{-\beta - n+2} F(t) + \cdots t^{-\beta} F^{(n-2)}(t) \} . \]
Therefore as before the integrated part again approaches zero when $t$ tends to zero and $t$ tends to infinity. Proceeding in the same manner we obtain

\[
J_n = \frac{n}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty t^{-\beta} F(t) \frac{\partial^n}{\partial t^n} \{t^{\beta+n-1} I_{n+\alpha} \phi \} \, dt
\]

\[
= \frac{n}{\Gamma(n + \beta + 1 - \alpha)} \int_0^\infty t^{-\beta} F(t) \left( \frac{n x}{t^{n+1}} \right)^{\beta} I_{n+\alpha} \phi \, dt
\]

by the Lemma 4.1. Hence

\[
J_n = \frac{n^{n+\beta+1} \Gamma(n+\beta)}{\Gamma(n + \beta + 1 - \alpha) \Gamma(b)} \int_0^\infty u^{-\beta-n-1} F_1(a; b; -n x/t) F(t) \, dt.
\]

It is clear that this integral exists under the hypotheses of the theorem and therefore all the previous integrals exist. By a simple substitution this gives on using the asymptotic expansion of $F_1(a; b; x)$ \[4\]

\[
J_n \sim \frac{n^{\beta+n+1} \Gamma(n+\beta)}{\Gamma(n + \beta + 1 - \alpha) \Gamma(b)} \int_0^\infty u^{\beta+n-1} e^{-n x u} F\left( \frac{1}{u} \right) \, du.
\]

Let

\[
(1/u) F\left( \frac{1}{u} \right) = \psi(u).
\]

Now

\[
(1/u) F(1/u) = 0(1) \quad (u \to \infty) \quad \text{and} \quad F\left( \frac{1}{u} \right) = 0(1) \quad (u \to 0).
\]

Hence it is easily seen

(i) $\psi(u) \in L (1/R \leq t < R)$ for every $R > 1$.

(ii) $\int_1^\infty \psi(u) e^{-c u} \, du$ converges for any fixed $c > 0$, and

(iii) $\int_1^1 u \psi(u) \, du$ also converges. Therefore \[3\]

\[
\lim_{n \to \infty} J_n = \frac{1}{u} \psi\left( \frac{1}{u} \right) = F(u).
\]

Now if

\[
\chi(x, y) = \frac{\Gamma(a)}{\Gamma(b)} (xy)\beta F_1(a; b; -xy).
\]

Then $\chi(xy) \in L$ in $0 \leq y < \infty$ under the conditions assumed for the convergence of (1.1). Therefore by a theorem on weak compactness of a set of functions \[5\] the inequalities in the hypothesis (iii) of the theorem imply the existence of a subset $\{n_i\}$ of the positive integers
and a bounded function \( f(y) \) such that

\[
\lim_{t \to \infty} \int_0^\infty [Q_{\alpha, \beta} \{F(x)\}] \chi(x, y) f(y) dy = \int_0^\infty \chi(x, y) f(y) dy.
\]

Hence

\[
F(x) = \int_0^\infty \chi(x, y) f(y) dy
\]

and the theorem is established.

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REFERENCES

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