

Pacific Journal of Mathematics

A CHARACTERIZATION OF WEAK* CONVERGENCE

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1. Introduction. Let X be a locally compact, Hausdorff space and $\{\mu_i; i \in D\}$ be a net of Radon measures on X (in the sense of Caratheodory). The weak* or vague limit of this net is the Radon measure ν such that

$$\lim_i \int f d\mu_i = \int f d\nu$$

for every continuous function f vanishing outside some compact set. In this paper, we construct in § 3 a Radon measure φ^* from a given base \mathcal{B} for the topology of X and $\liminf_i \mu_i$ and then, in § 4, we give necessary and sufficient conditions for φ^* to be the weak* limit of the μ_i . In particular, if the latter exists then it is the φ^* generated when \mathcal{B} is the family of all open sets.

The measure φ^* is obtained from another measure φ by a standard regularizing process. The definition of φ easily extends to abstract spaces but that of φ^* makes essential use of the topology. Thus, it is of some importance to know when $\varphi = \varphi^*$, that is, when a measure constructed through an abstract process from the μ_i turns out to be, in the topological situation, the weak* limit of the μ_i . In Theorem 3.3 we give a condition for $\varphi = \varphi^*$ and in § 5 we give an example to show that the condition cannot be eliminated.

We refer to standard texts such as Halmos [1], Kelley [2], and Munroe [3] for the elementary properties and concepts of topology and measure theory used in this paper.

2. Notation.

- 2.1 ω denotes the set of natural numbers.
- 2.2 0 denotes both the empty set and the smallest number in ω .
- 2.3 μ is a Caratheodory (outer) measure on X if and only if μ is a function on the family of all subsets of X such that $\mu 0 = 0$ and

$$0 \leq \mu A \leq \sum_{n \in \omega} \mu B_n \leq \infty \quad \text{whenever } A \subset \bigcup_{n \in \omega} B_n \subset X.$$

- 2.4 For μ a Caratheodory measure on X , A is μ -measurable if and only if $A \subset X$ and for every $T \subset X$

$$\mu T = \mu(T \cap A) + \mu(T - A).$$

- 2.5 For X a topological space, μ is a Radon measure on X if and

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only if μ is a Caratheodory measure on X such that:

- (i) open sets are μ -measurable,
- (ii) if C is compact then $\mu C < \infty$,
- (iii) if α is open then $\mu\alpha = \sup \{\mu C; C \text{ compact, } C \subset \alpha\}$,
- (iv) if $A \subset X$ then $\mu A = \inf \{\mu\alpha; \alpha \text{ open, } A \subset \alpha\}$.

2.6 For X a topological space, $C_0(X)$ is the family of all real-valued continuous functions on X vanishing outside some compact set.

2.7 $(D, <)$ is a directed set if and only if $D \neq 0$, D is partially ordered by $<$ so that for any $i, j \in D$ there exists $k \in D$ with $i < k$ and $j < k$.

2.8 A net is a function on a directed set.

2.9 \bar{A} denotes the closure of A .

3. **The lim inf measure.** Let X be a regular topological space; \mathcal{B} be a base for the topology of X , closed under finite unions and intersections; $(D, <)$ be a directed set and, for each $i \in D$, μ_i be a Radon measure on X .

For each $a \in \mathcal{B}$, let

$$g\alpha = \lim_{i \in D} \mu_i \alpha (= \sup_{j \in D} \inf_{\substack{i \in D \\ j < i}} \mu_i \alpha) < \infty$$

and let φ be the Caratheodory measure on X generated by g and \mathcal{B} (see method I of Munroe [3]), i.e. for each $A \subset X$,

$$\varphi A = \inf \left\{ \sum_{\alpha \in H} g\alpha; H \text{ countable, } H \subset \mathcal{B}, A \subset \bigcup_{\alpha \in H} \alpha \right\}.$$

As we show in § 5, φ need not be a Radon measure even when X is compact and Hausdorff. For this reason, for any $A \subset X$ let

$$\varphi^* A = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi C.$$

We then have the following:

3.1 **THEOREM.** φ is a Caratheodory measure on X such that:

- (i) if A and B are disjoint, closed, compact sets then $\varphi(A \cup B) = \varphi A + \varphi B$.
- (ii) if $A \subset X$ then $\varphi A = \inf \{\varphi\alpha; \alpha \text{ open, } A \subset \alpha\}$.
- (iii) if C is compact and for every $\alpha \in \mathcal{B}$, $g\alpha = \lim_i \mu_i \alpha$ then

$$\varphi C = \inf \{g\alpha; \alpha \in \mathcal{B}, C \subset \alpha\}.$$

3.2 **THEOREM.** φ^* is a Radon measure on X such that:

- (i) $\varphi^* \leq \varphi$.
- (ii) if C is compact then $\varphi^* C = \varphi C$.

3.3 THEOREM. *If every open set in X is the countable union of compacta then $\varphi^* = \varphi$.*

Proofs

Proof of 3.1

(i) Let A, B be closed, compact and $A \cap B = 0$. Since X is regular and \mathcal{B} is closed to finite unions, there exist $\alpha, \beta \in \mathcal{B}$ such that $A \subset \alpha, B \subset \beta$ and $\alpha \cap \beta = 0$. Given $\varepsilon > 0$, choose $\gamma_n \in \mathcal{B}$ for $n \in \omega$ so that $A \cup B \subset \bigcup_{n \in \omega} \gamma_n$ and

$$\sum_{n \in \omega} g\gamma_n \leq \varphi(A \cup B) + \varepsilon.$$

Let $\gamma'_n = \gamma_n \cap \alpha$ and $\gamma''_n = \gamma_n \cap \beta$. Then $\gamma'_n, \gamma''_n \in \mathcal{B}, A \subset \bigcup_{n \in \omega} \gamma'_n, B \subset \bigcup_{n \in \omega} \gamma''_n$ and hence

$$\varphi A + \varphi B \leq \sum_{n \in \omega} (g\gamma'_n + g\gamma''_n) \leq \sum_{n \in \omega} g\gamma_n \leq \varphi(A \cup B) + \varepsilon.$$

Since ε is arbitrary and φ is a Caratheodory measure we have $\varphi(A \cup B) = \varphi A + \varphi B$.

(ii) Let $A \subset X$. If $\varphi A = \infty$ then the conclusion is trivial. So, let $\varphi A < \infty$ and $\varepsilon > 0$. Then there exists a countable $H \subset \mathcal{B}$ such that $A \subset \bigcup_{\alpha \in H} \alpha$ and

$$\sum_{\alpha \in H} g\alpha \leq \varphi A + \varepsilon$$

and therefore

$$\varphi\left(\bigcup_{\alpha \in H} \alpha\right) \leq \sum_{\alpha \in H} \varphi\alpha \leq \sum_{\alpha \in H} g\alpha \leq \varphi A + \varepsilon.$$

(iii) Suppose for every $\alpha \in \mathcal{B}, g\alpha = \lim_i \mu_i \alpha$. Then for $\alpha_0, \dots, \alpha_n$ in \mathcal{B} we have

$$\begin{aligned} \sum_{k=0}^n g\alpha_k &= \lim_i \sum_{k=0}^n \mu_i \alpha_k \\ &= \lim_i \mu_i \left(\bigcup_{k=0}^n \alpha_k \right) \\ &= g\left(\bigcup_{k=0}^n \alpha_k \right). \end{aligned}$$

Hence for any compact C ,

$$\varphi C = \inf \{g\alpha ; \alpha \in \mathcal{B}, C \subset \alpha\}.$$

Proof of 3.2

(i) Clearly, for any compact $C, \varphi C < \infty$ and, for any open α ,

$$\varphi^* \alpha = \sup \{ \varphi C ; C \text{ compact, } C \subset \alpha \} \leq \varphi \alpha .$$

Thus, for any $A \subset X$, using 3.1 (ii) we have

$$\begin{aligned} \varphi^* A &= \inf \{ \varphi^* \alpha ; \alpha \text{ open, } A \subset \alpha \} \\ &\leq \inf \{ \varphi \alpha ; \alpha \text{ open, } A \subset \alpha \} \\ &= \varphi A . \end{aligned}$$

(ii) For any compact C and open $\alpha \supset C$, we have $\varphi C \leq \varphi^* \alpha$, hence $\varphi C \leq \varphi^* C$. By (i) then $\varphi^* C = \varphi C$.

(iii) To see that φ^* is a Radon measure, we now only need to check that open sets are φ^* -measurable. Let α be open, $T \subset X$ and $\varepsilon > 0$. Let T' be open, $T \subset T'$ and $\varphi^* T' < \varphi^* T + \varepsilon$. Note that if C is compact, β is open and $C \subset \beta$ then, by regularity, $\bar{C} \subset \beta$. Thus, since $T' \cap \alpha$ is open, there exists a closed, compact $C_1 \subset T' \cap \alpha$ with $\varphi^*(T' \cap \alpha) \leq \varphi C_1 + \varepsilon$. Also, since $T' - C_1$ is open, there exists a closed compact $C_2 \subset T' - C_1$ with $\varphi^*(T' - C_1) \leq \varphi C_2 + \varepsilon$. Then

$$\begin{aligned} \varphi^*(T \cap \alpha) + \varphi^*(T - \alpha) &\leq \varphi^*(T' \cap \alpha) + \varphi^*(T' - C_1) \\ &\leq \varphi C_1 + \varphi C_2 + 2\varepsilon \\ &= \varphi(C_1 \cup C_2) + 2\varepsilon \quad (\text{by 3.1 (i)}) \\ &\leq \varphi^* T' + 2\varepsilon \\ &\leq \varphi^* T + 3\varepsilon . \end{aligned}$$

Proof of 3.3. We need only show that $\varphi^* A = \varphi A$ for open A . Given such A , by assumption, $A = \bigcup_{n \in \omega} C_n$ where the C_n are compact and $C_n \subset C_{n+1}$. Because of regularity, we may assume that the C_n are closed compact. We shall show that $\varphi A = \lim_n \varphi C_n$. To this end, let $\varepsilon > 0$ and define α_n and C'_n by recursion as follows: let $C' = C_0$ and, for any $n \in \omega$, let α_n be open, $C'_n \subset \alpha_n$, $\varphi \alpha_n \leq \varphi C'_n + \varepsilon/2^{n+1}$ and

$$C'_{n+1} = C_{n+1} - \bigcup_{j=0}^n \alpha_j .$$

Then the C'_n are closed compact, mutually disjoint and $A \subset \bigcup_{n \in \omega} \alpha_n$. Thus,

$$\begin{aligned} \varphi A &\leq \sum_{n \in \omega} \varphi \alpha_n \leq \sum_{n \in \omega} \varphi C'_n + \varepsilon \\ &= \lim_N \sum_{n=0}^N \varphi C'_n + \varepsilon = \lim_N \varphi \left(\bigcup_{n=0}^N C'_n \right) + \varepsilon \\ &\leq \lim_N \varphi C_N + \varepsilon . \end{aligned}$$

4. **Weak* convergence.** Let X be a locally compact, Hausdorff

space, \mathcal{M} be the family of Radon measures on X , μ be a net in \mathcal{M} . It is well known that \mathcal{M} can be identified with the set of positive linear functionals on $C_0(X)$ so that the weak* or vague limit of the μ_i is defined by

4.1. DEFINITION. $(W^*)\text{-}\lim_i \mu_i = \nu$ if and only if $\nu \in \mathcal{M}$ and, for every $f \in C_0(X)$,

$$\lim_i \int f d\mu_i = \int f d\nu .$$

On the other hand, for any base \mathcal{B} for the topology of X , let

4.2. DEFINITION. $\mathcal{B}\text{-}\underline{\text{Lim}}_i \mu_i$ be the measure φ^* defined in § 3. If \mathcal{B} is the family of all open sets then we simply write $\underline{\text{Lim}}_i \mu_i$ instead of $\mathcal{B}\text{-}\underline{\text{Lim}}_i \mu_i$.

We then have the following:

4.3. THEOREM. $(W^*)\text{-}\lim_i \mu_i$ exists if and only if there exists a base \mathcal{B} for the topology of X , closed under finite unions and intersections, such that, for every $\alpha \in \mathcal{B}$, $\lim_i \mu_i \alpha < \infty$, in which case,

$$(W^*)\text{-}\lim_i \mu_i = \mathcal{B}\text{-}\underline{\text{Lim}}_i \mu_i = \underline{\text{Lim}}_i \mu_i .$$

The proof of this theorem is given in Lemmas A, B, C, D, E below. A restricted version of Lemma B was proved by Wulfsohn [4].

LEMMA A. Let $\nu \in \mathcal{M}$ and

$$\mathcal{B} = \{ \alpha : \alpha \text{ is open, } \bar{\alpha} \text{ is compact and } \nu(\text{boundary } \alpha) = 0 \} .$$

Then \mathcal{B} is a base for the topology of X and is closed under finite unions and intersections.

Proof. Let A be open and $a \in A$. Then there exists $f \in C_0(X)$ such that: $0 \leq f(x) \leq 1$ for $x \in X$, $f(a) = 1$ and $f(x) = 0$ for $x \notin A$. Since $\int f d\nu < \infty$, there exists $0 < t < 1$ such that $\nu(f^{-1}\{t\}) = 0$. Let $\alpha = \{x : f(x) > t\}$. Then α is open, $a \in \alpha \subset A$ and boundary $\alpha = f^{-1}\{t\}$ so that $\alpha \in \mathcal{B}$. Thus, \mathcal{B} is a base. It is closed to finite unions and intersections since boundary $(\alpha \cup \beta) \cup \text{boundary } (\alpha \cap \beta) \subset \text{boundary } \alpha \cup \text{boundary } \beta$ for any open α, β .

LEMMA B. $(W^*)\text{-}\lim_i \mu_i = \nu$ if and only if $\nu \in \mathcal{M}$ and $\lim_i \mu_i \alpha = \nu \alpha$ for every open α with $\bar{\alpha}$ compact and $\nu(\text{boundary } \alpha) = 0$.

Proof. Let $(W^*)\text{-}\lim_i \mu_i = \nu$, α be open, $\bar{\alpha}$ compact, $\nu(\text{boundary } \alpha)$

$\alpha) = 0$. For any compact $C \subset \alpha$, let $f \in C_0(X)$, $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ for $x \in C$, $f(x) = 0$ for $x \notin \alpha$. Then

$$\nu C \leq \int f d\nu = \lim_i \int f d\mu_i \leq \varliminf_i \mu_i \alpha.$$

Hence

$$\nu \alpha \leq \varliminf_i \mu_i \alpha.$$

Now, since ν (boundary α) = 0, given $\varepsilon > 0$, let β be open, $\bar{\alpha} \subset \beta$ and $\nu\beta \leq \nu\bar{\alpha} + \varepsilon = \nu\alpha + \varepsilon$. Let $f \in C_0(X)$, $0 \leq f(x) \leq 1$ for $x \in X$, $f(x) = 1$ for $x \in \bar{\alpha}$, $f(x) = 0$ for $x \notin \beta$. Then

$$\varliminf_i \mu_i \alpha \leq \lim_i \int f d\mu_i = \int f d\nu \leq \nu\beta \leq \nu\alpha + \varepsilon.$$

Thus,

$$\nu \alpha = \lim_i \mu_i \alpha.$$

Conversely, suppose $\nu \in \mathcal{M}$ and $\lim_i \mu_i \alpha = \nu \alpha$ for every open α with $\bar{\alpha}$ compact and ν (boundary α) = 0. Let $f \in C_0(X)$, $\varepsilon > 0$. Then there exist $t_k \neq 0$ for $k = 0, \dots, n$ such that $t_k < t_{k+1}$, $t_0 \leq f(x) \leq t_n$ for $x \in X$, $\nu(f^{-1}\{t_k\}) = 0$ and

$$\sum_{k=0}^{n-1} t_{k+1} \nu \alpha_k - \varepsilon \leq \int f d\nu \leq \sum_{k=0}^{n-1} t_k \nu \alpha_k + \varepsilon$$

where

$$\alpha_k = \{x : t_k < f(x) < t_{k+1}\}$$

so that α_k is open, $\bar{\alpha}_k$ is compact and ν (boundary α_k) = 0. Then $\lim_i \mu_i \alpha_k = \nu \alpha_k$ and

$$\begin{aligned} \int f d\nu &\leq \lim_i \sum_{k=1}^{n-1} t_k \mu_i \alpha_k + \varepsilon \\ &\leq \varliminf_i \int f d\mu_i + \varepsilon. \end{aligned}$$

Now, let β_k be open, $\bar{\beta}_k$ be compact, ν (boundary β_k) = 0, $\bar{\alpha}_k \subset \beta_k$ and $\nu\beta_k \leq \nu\alpha_k + \varepsilon/(n |t_{k+1}|)$. Then $\lim_i \mu_i \beta_k = \nu\beta_k$ and

$$\begin{aligned} \varliminf_i \int f d\mu_i &\leq \lim_i \sum_{k=0}^{n-1} t_{k+1} \mu_i \beta_k \\ &= \sum_{k=0}^{n-1} t_{k+1} \nu\beta_k \\ &\leq \sum_{k=0}^{n-1} t_{k+1} \nu\alpha_k + \varepsilon \\ &\leq \int f d\nu + 2\varepsilon. \end{aligned}$$

LEMMA C. If $(W^*)\text{-lim}_i \mu_i = \nu$ and

$$\mathcal{B} = \{\alpha : \alpha \text{ is open, } \bar{\alpha} \text{ is compact, } \nu(\text{boundary } \alpha) = 0\}$$

then

$$\nu = \mathcal{B}\text{-}\underline{\text{Lim}}_i \mu_i .$$

Proof. Let $g\alpha = \underline{\text{lim}}_i \mu_i \alpha$ for any $\alpha \in \mathcal{B}$, φ be the measure generated by g and \mathcal{B} (see § 3). Then, in view of Lemma B and 3.1 (iii), for any compact $C \subset X$,

$$\varphi C = \inf \{g\beta ; \beta \in \mathcal{B} ; C \subset \beta\} .$$

Now, for any open $\alpha \supset C$ there exists, by Lemma A, $\beta \in \mathcal{B}$ with $C \subset \beta \subset \alpha$. Therefore, using Lemma B, and the outer regularity of ν , we have

$$\begin{aligned} \nu C &= \inf \{\nu\alpha ; \alpha \text{ open, } C \subset \alpha\} \\ &= \inf \{\nu\beta ; \beta \in \mathcal{B}, C \subset \beta\} \\ &= \inf \{g\beta ; \beta \in \mathcal{B}, C \subset \beta\} \\ &= \varphi C . \end{aligned}$$

Hence, for any $A \subset X$,

$$\begin{aligned} \nu A &= \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \nu C \\ &= \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi C = \mathcal{B}\text{-}\underline{\text{Lim}}_i \mu_i A . \end{aligned}$$

LEMMA D. Let \mathcal{B} be a base for the topology of X , closed under finite unions and intersections, such that for any $\alpha \in \mathcal{B}$, $\lim_i \mu_i \alpha < \infty$. Then

$$\mathcal{B}\text{-}\underline{\text{Lim}}_i \mu_i = (W^*)\text{-}\lim_i \mu_i .$$

Proof. For $\alpha \in \mathcal{B}$, let $g\alpha = \underline{\text{lim}}_i \mu_i \alpha = \lim_i \mu_i \alpha$, φ be the measure generated by g and \mathcal{B} and $\varphi^* = \mathcal{B}\text{-}\underline{\text{Lim}}_i \mu_i$ (see § 3). Then, by Theorem 3.2, $\varphi^* \in \mathcal{M}$. Let α be open, $\bar{\alpha}$ compact, $\varphi^*(\text{boundary } \alpha) = 0$. By 3.2 (ii), we have

$$\varphi^* \alpha = \varphi^* \bar{\alpha} = \varphi \bar{\alpha}$$

and by 3.1 (iii),

$$\varphi \bar{\alpha} = \inf \{g\beta ; \beta \in \mathcal{B}, \bar{\alpha} \subset \beta\} .$$

Given $\varepsilon > 0$, let $\beta \in \mathcal{B}$, $\bar{\alpha} \subset \beta$ and $g\beta \leq \varphi^* \alpha + \varepsilon$. Then

$$\overline{\lim}_i \mu_i \alpha \leq \lim_i \mu_i \beta = g\beta \leq \varphi^* \alpha + \varepsilon .$$

On the other hand, let C be compact, $C \subset \alpha$ and $\varphi^* \alpha < \varphi^* C + \varepsilon = \varphi C + \varepsilon$. Then there exists $\gamma \in \mathcal{B}$ such that $C \subset \gamma \subset \alpha$ and therefore

$$\varphi C \leq g\gamma = \lim_i \mu_i \gamma \leq \underline{\lim}_i \mu_i \alpha .$$

Thus,

$$\overline{\lim}_i \mu_i \alpha \leq \varphi^* \alpha \leq \underline{\lim}_i \mu_i \alpha$$

so that $\lim_i \mu_i \alpha = \varphi^* \alpha$. By Lemma B then $\varphi^* = (W^*)\text{-}\lim_i \mu_i$.

LEMMA E. *Let \mathcal{B} be a base for the topology of X , closed under finite unions and for every $\alpha \in \mathcal{B}$, $\lim_i \mu_i \alpha < \infty$. Then*

$$\mathcal{B}\text{-}\underline{\lim}_i \mu_i = \underline{\lim}_i \mu_i .$$

Proof. For any open α , let $g\alpha = \underline{\lim}_i \mu_i \alpha$, φ_1 be the measure generated by g and \mathcal{B} and φ_2 be the measure generated by g and the family of all open sets. We have to show that for any compact C , $\varphi_1 C = \varphi_2 C$. Now, clearly $\varphi_2 C \leq \varphi_1 C$. Suppose $\varphi_2 C < \infty$ and $\varepsilon > 0$. Let α_i be open for $i = 0, \dots, n$, $C \subset \bigcup_{i=0}^n \alpha_i$ and

$$\sum_{i=1}^n g\alpha_i \leq \varphi_2 C + \varepsilon .$$

For each $x \in C$ there exists $\beta \in \mathcal{B}$ such that $x \in \beta \subset \alpha_i$ for some $i = 0, \dots, n$. Since C is compact, there is a finite family $H \subset \mathcal{B}$ which covers C and is a refinement of $\{\alpha_0, \dots, \alpha_n\}$. For each i , let β_i be the union of all those elements in H which are contained in α_i . Then $\beta_i \in \mathcal{B}$, $\beta_i \subset \alpha_i$ and $C \subset \bigcup_{i=0}^n \beta_i$. Thus,

$$\varphi_1 C \leq \sum_{i=0}^n g\beta_i \leq \sum_{i=0}^n g\alpha_i \leq \varphi_2 C + \varepsilon .$$

5. Remarks. Let \mathcal{B} , g , φ be as in § 3. The following example shows that φ need not be a Radon measure.

Let X be the set of all ordinals up to and including the first uncountable ordinal Ω . Then, in the order-topology, X is compact Hausdorff. For each $i < \Omega$, let μ_i be the point mass at i , that is, $\mu_i \alpha = 1$ if $i \in \alpha$ and $\mu_i \alpha = 0$ if $i \notin \alpha$. Let

$$\mathcal{B} = \{\alpha ; \alpha \text{ is open and } \Omega \notin (\bar{\alpha} - \alpha)\} .$$

For any $\alpha \in \mathcal{B}$, if $\Omega \notin \alpha$ then α is countable and hence $g\alpha = \underline{\lim}_i \mu_i \alpha = 0$; if $\Omega \in \alpha$ then $g\alpha = 1$. Let $A = X - \{\Omega\}$. Then A is open and, being

uncountable, for any countable family $H \subset \mathcal{B}$ which covers A there exists $\alpha \in H$ with $g\alpha = 1$. Thus, $\varphi A = 1$. On the other hand, if C is compact $C \subset A$ then C is countable and hence $\varphi C = 0$. Thus,

$$\varphi A \neq \sup \{ \varphi C; C \text{ compact, } C \subset A \}.$$

Note, however, that if, instead of taking \mathcal{B} as above, we let \mathcal{B} be the family of all open sets in X then there exist uncountable, disjoint $\alpha, \beta \in \mathcal{B}$ with $A = \alpha \cup \beta$. Then $g\alpha = g\beta = 0$ so that $\varphi A = 0$. In this case, φ is the point mass at Ω and $\varphi = \varphi^*$.

We are unable to determine if this holds true in general for compact or locally compact Hausdorff spaces, i.e. if $\varphi = \varphi^*$ whenever \mathcal{B} is the family of all open sets in X .

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