HOMOMORPHISMS OF $d$-SIMPLE INVERSE SEMIGROUPS WITH IDENTITY

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Munn determined all homomorphisms of a regular Rees matrix semigroup $S$ into a Rees matrix semigroup $S^*$ [3, 2]. This generalized an earlier theorem due to Rees [7, 2].

We consider the homomorphism problem for an important class of $d$-simple semigroups.

Let $S$ be a $d$-simple inverse semigroup with identity. Such semigroups are characterized by the following conditions [1, 4, 2].

A1: $S$ is $d$-simple.
A2: $S$ has an identity element.
A3: Any two idempotents of $S$ commute.

It is shown by Clifford [1] that the structure of $S$ is determined by that of its right unit semigroup $P$ and that $P$ has the following properties:

B1: The right cancellation law hold in $P$.
B2: $P$ has an identity element.
B3: The intersection of two principal left ideals of $P$ is a principal left ideal of $P$.

Two elements of $P$ are $L$-equivalent if and only if they generate the same principal left ideal.

Since any homomorphic image of a $d$-simple inverse semigroup with identity is a $d$-simple inverse semigroup with identity [5], we may limit our discussion to homomorphisms of $S$ into $S^*$ where $S^*$, as well as $S$, is of this type.

In §1, we consider two such semigroups $S$ and $S^*$ with right unit semigroups $P$ and $P^*$ respectively. We determine the homomorphisms of $S$ into $S^*$ in terms of certain homomorphism of $P$ into $P^*$, and we show that $S$ is isomorphic to $S^*$ if and only if $P$ is isomorphic to $P^*$.

In §2, we show that if $P$ is a semigroup satisfying B1 and B2 on which $L$ is a congruence relation then $P$ is a Schreier extension of its group of units $U$ by $P/L$ and that $P/L$ satisfies B1, B2, and has a trivial group of units. $P$ satisfies B3 if and only if $P/L$ satisfies B3. The converse of this theorem is also given. In this case, we determine the homomorphisms of $P$ into $P^*$ in terms of the homomor-

Received June 25, 1963.
phisms of \( U \) into \( U^* \) and those of \( P/L \) into \( P^*/L^* \) and give the corresponding isomorphism theorem. In §3, some examples are given.

It is a pleasure to acknowledge several helpful conversations with Professor A. H. Clifford.

**Section 1.** The correspondence between the homomorphism of \( S \) and those of \( P \).

We first summarize the construction of Clifford referred to in the introduction.

Let \( S \) be any semigroup with identity element. We say that the two elements are \( ^- \)-equivalent if they generate the same principal right ideal: \( aS = bS \). \( L \)-equivalent elements are defined analogously. Two elements \( a \) and \( b \) are called \( d \)-equivalent if there exists an element of \( S \) which is \( L \)-equivalent to \( a \) and \( R \)-equivalent to \( b \) (This implies the existence of an element of \( S \) which is \( R \)-equivalent to \( a \) and \( L \)-equivalent to \( b \)). We shall say that \( S \) is \( d \)-simple if it consists of a single class of \( d \)-equivalent elements.

Now let \( P \) be any semigroup satisfying \( B1, B2 \) and \( B3 \). From each class of \( L \)-equivalent elements of \( P \), let us pick a fixed representative. \( B3 \) states that if \( a \) and \( b \) are elements of \( P \), there exists \( c \) in \( P \) such that \( Pa \cap Pb = Pc \). \( c \) is determined by \( a \) and \( b \) to within \( L \)-equivalence. We define \( avb \) to be the representative of the class to which \( c \) belongs. We observe also that

\[(1.1) \quad avb = bva\, .\]

We define a binary operation \( x \) by

\[(1.2) \quad (axb)b = avb\]

for each pair of elements \( a, b \) of \( P \).

Now let \( P^{-1}oP \) denote the set of ordered pairs \((a, b)\) of elements of \( P \) with equality defined by

\[(1.3) \quad (a, b) = (a', b') \text{ if } a' = \rho a \text{ and } b' = \rho b \text{ where } \rho \text{ is a unit in } P \text{ (\( \rho \) has a two sided inverse with respect to 1, the identity of } P)\].

We define product in \( P^{-1}oP \) by

\[(1.4) \quad (a, b)(c, d) = ((cx)b)a, (bxc)d\].

Clifford's main theorem states: Starting with a semigroup \( P \) satisfying \( B1, B2, B3 \) equations (1.2), (1.3), and (1.4) define a semigroup \( P^{-1}oP \) satisfying \( A1, 2, 3 \). \( P \) is isomorphic with the right unit subsemigroup of \( P^{-1}oP \) (the right unit subsemigroup of \( P^{-1}oP \) is the set of elements
of $P^{-o}oP$ having a right inverse with respect to 1. This set is easily shown to be a semigroup. Conversely, if $S$ is a semigroup satisfying $A1, 2, 3$ its right unit subsemigroup $P$ satisfies $B1, 2, 3$ and $S$ is isomorphic with $P^{-o}oP$.

The following results are also obtained:

The elements $(1, a)$ of $P^{-o}oP$ constitute a subsemigroup thereof isomorphic to $P$. We have

$$ (1, a)(1, b) = (1, ab) $$

for $a, b$ in $P$.

The ordered pair $(1, 1)$ is the identity of $P^{-o}oP$, i.e.

$$ (a, b)(1, 1) = (1, 1)(a, b) = (a, b) $$

for $a, b$ in $P$.

The right inverse of $(1, a)$ is $(a, 1)$, i.e.

$$ (1, a)(a, 1) = (1, 1) $$

for $a$ in $P$.

$$ (a, c) = (a, 1)(1, c) $$

for all $a$ and $c$ in $P$.

We identify $S$ with $P^{-o}oP$ and $P$ with $\{(1, a) : a \in P\}$.

$$(avb)c = \rho(acvbc)$$

where $a, b,$ and $c$ are in $P$ and $\rho$ is a unit in $P$.

The idempotent elements of $P^{-o}oP$ are just those elements of the form $(a, a)$ where $a$ in $P$.

$$(a, a)(b, b) = (avb, avb)$$

for all $a, b$ in $P$.

$$(alb)(a, b \text{ in } P)$$

if and only if $a = \rho b$ where $\rho$ is a unit in $P$.

Let $P$ and $P^*$ be semigroups satisfying $B1$, and $B2$ and $B3$. Let $v$ and $u$ be the ‘join’ operations on $P$ and $P^*$ respectively defined on page 2. Let $N$ be a homomorphism of $P$ into $P^*$. $N$ is called a semilattice homomorphism (or sl-homomorphism) if

$$ P^*((avb)N) = P^*(aN) \cap P^*(bN) $$

i.e. $(avb)N$ $LaNubN$ in $P^*$.

It is easily seen that we always have $P^*((avb)N) \subseteq P^*(aN) \cap P^*(bN)$. However, the reverse inclusion is not generally valid. For example, we might have $P = G^+$, $P^* = G^*$, where $G$ and $G^*$ are lattice-ordered groups. An order-preserving homomorphism of $G$ into $G^*$ need not preserve the lattice operations.

**Theorem 1.1.** Let $S$ and $S^*$ be semigroups satisfying $A1, A2,$ and
A3, and let \( P \) and \( P^* \) be their right unit subsemigroups, Let \( N, \) be a sl-homomorphism of \( P \) into \( P^* \), and let \( k \) be an element of \( P^* \).

For each element \((a, b)\) of \( S \), define

\[
(a, b)M = [(aN)k, (bN)k]
\]

the square brackets indicating an element of \( S^* \). Then \( M \) is a homomorphism of \( S \) into \( S^* \). Conversely, every homomorphism of \( S \) into \( S^* \) is obtained in this fashion.

**PROOF.** To show that \( M \) is single valued, let \((a, b) = (a', b')\). Then, \( a' = \rho a \) and \( b' = \rho b \) where \( \rho \) is a unit in \( P \) by (1.3). Thus, \( a'N = \rho NaN \) and \( b'N = \rho NbN \). Thus, since \( \rho N \) is a unit of \( P^* \), \((a, b)M = (a', b')M \) by (1.3). To show that \( M \) is a homomorphism let \( \times \) and \( \otimes \) be the operations defined on \( P \) and \( P^* \) respectively by (1.2). Thus, using (1.2), (1.9), (1.13), and (1.12) obtain

\[
((rN)k \otimes (nN)k)(nN)k = (rN)k \ u(nN)k = w(rNunN)k = w\rho^* \ ((rvn)N)k = w\rho^* \ ((r \times n)n)N)k = w\rho^* ((r \times n)(nN)k \text{ where } w \text{ and } \rho^* \text{ are units in } P^*.
\]

Thus, from B1,

\[
(rN)k \otimes (nN)k = w\rho^* ((r \times n)N).
\]

Now, from (1.2), (1.1), and (1.15), we have

\[
((nN)k \otimes (rN)k)(rN)k = (nN)k \ u(rN)k = (rN)k \ u(nN)k = w\rho^* \ ((rvn)N)k = w\rho^* \ ((r \times n)n)N)k = w\rho^* \ ((n \times r)r)N)k = w\rho^* \ ((n \times r)N) \ (rN) \ k.
\]

Therefore, by B1,

\[
(nN)k \otimes (rN)k = w\rho^* ((n \times r)N).
\]

Thus, by (1.14), (1.4), (1.15), (1.16), and (1.3), \((m, n)M(r, s)M = [(mN)k, (nN)k] \ [r(rN)k, (sN)k] = [(rN)k \otimes (nN)k] (mN)k, \ ([nN)k \otimes (rN)k]) (sN)k = [w\rho^* ((r \times n)N) \ (mN)k, \ w\rho^* ((n \times r)N) \ (sN)k] = [(r \times n)m, (n \times r)s]M = ((m, n) \ (r, s))M.
\]

Conversely, let \( M \) be a homomorphism of \( S \) into \( S^* \). Then, by (1.6) and (1.10),

\[
(1, 1)M = [k, k]
\]

for some \( k \) in \( P^* \). Now suppose that \((1, n)M = [a, b] \) and \((n, 1)M = [c, d] \) for \( n \) in \( P \). It thus follows from (1.7) and (1.6) that \([a, b] \ [c, d] [a, b] = [a, b] \) and \([c, d] [a, b] = [c, d] \). From (1.8) and (1.7), it easily follows that \([b, a] \ [a, b] = [a, b] \) and \([b, a] [a, b] = [b, a] \). Hence, \([b, a] \) and \([c, d] \) are inverses of \([a, b] \) (2, p. 27). Therefore, it follows from a theorem of Munn and Penrose (4; 2, p. 28, Theorem 1.17) that \([b, a] = [c, d] \). Thus

\[
(1, n)M = [a, b]
\]
\[ (n, 1)M = [b, a] \]
Now, from (1.7), (1.17), and (1.18), \([a, b][b, a] = [k, k]\). Thus, from (1.8) and (1.7), we have \([a, a] = [k, k]\). Hence, by (1.3), \(a = \rho k\) where \(\rho\) is a unit of \(P^*\). Therefore, by (1.18) and (1.3),

\[
\begin{align*}
(1, n)M &= [\rho k, b] = [k, \rho^{-1}b] = [k, c] \\
(n, 1)M &= [b, \rho k] = [\rho^{-1}b, k] = [c, k]
\end{align*}
\]

where \(c = \rho^{-1}b\). Now, again using (1.8) and (1.7), \([c, k][k, c] = [c, c]\). Thus, by (1.11), \([k, k][c, c] = [kuc, kuc] = [c, c]\). Therefore, by (1.3) (1.12), \(P^*(kuc) = P^*c\). Hence, by the definition of \(u\), \(P^*k \cap P^*c \subseteq P^*k\). Thus, we may write \(c = B_nk\) where \(B_n\) in \(P^*\). Thus, from (1.19), we have

\[
\begin{align*}
(1, n)M &= [k, B_nk] \\
(n, 1)M &= [B_nk, k].
\end{align*}
\]

It follows easily from (1.8), (1.20) and (1.7) that

\[
(1.21) (m, n)M = [B_mk, B_nk].
\]

Thus, to complete the proof, we must show that \(n \rightarrow B_n\) is a homomorphism of \(P\) into \(P^*\) and that \(P^* (B_m \cup B_n) \subseteq P^*B_{mn}\). It follows from (1.20), (1.3), and (B1) that \(n \rightarrow B_n\) is single valued. To show that \(n \rightarrow B_n\) is a homomorphism we first note that from (1.5) and (1.20), \([k, B_mk][k, B_nk] = [k, B_mk].\) Thus, by (1.4)

\[
(1.22) [(k \otimes B_mk)k, (B_mk \otimes k)B_nk] = [k, B_{mn}k].
\]

From (1.2), the definition of \(u\), and (1.12)

\[
(1.23) (k \otimes B_mk)B_mk = ku (B_mk) = wB_mk
\]

where \(w\) is a unit of \(P^*\). Thus, by (B1)

\[
(1.24) k \otimes (B_mk) = w.
\]

By virtue of (1.2), (1.1), and (1.23), \(((B_mk \otimes k)k)k = (B_mk) uk = ku (B_mk) = wB_mk\). Hence, by (B1),

\[
(1.25) (B_mk) \otimes k = wB_m.
\]

If we substitute (1.24) and (1.25) in (1.22), we obtain \([wk, wB_mB_mk] = [k, B_{mn}k]\). Hence, from (1.3) and (B1), we have \(B_mB_n = B_{mn}\). We now show that \(P^*(B_m \cup B_n) = P^*B_{mn}\). From (1.4), (1, m) \((n, 1) = (n \times m, m \times n)\). Hence, it follows from (1.21), (B1), and (B2) that \([k, B_mk][k, B_nk] = [k, B_mk].\) Thus, by virtue of (1.4), \(((B_mk) \otimes (B_mk))k, (B_mk \otimes (B_mk)k)k = [B_{mn}k, B_{mn}k].\) Hence, by (1.3) and (B1), \((B_mk) \otimes (B_mk) = \rho^* B_{mn}\) where \(\rho^*\) is a unit of \(P^*\). Thus, by (1.2), \(B_mk B_mk = ((B_mk) \otimes (B_mk))B_mk = \rho^* B_{mn}B_mk = \rho^* B_{mn}B_{mn}k.\) There-
fore, by (B1) and (1.9), \( \rho' (B_n u B_m) = \rho^* B_{n m} \) where \( \rho' \) is a unit of \( P^* \). Hence \( P^*(B_n u B_m) = P*B_{n m} \).

**Theorem 1.2.** Let \( S, P, S^* \), and \( P^* \) be as in Theorem 1.1. Let \( \Omega \) be the set of isomorphisms of \( P \) onto \( P^* \). Define \((m, n)M_N = [mN, nN]\) for \( N \) in \( \Omega \). Then \( \{M_N : N \text{ in } \Omega \} \) is the complete set of isomorphisms of \( S \) onto \( S^* \). Hence, \( N \rightarrow M_N \) is a one-to-one correspondence between the isomorphisms of \( P \) onto \( P^* \) and those of \( S \) onto \( S^* \) and \( S \) is isomorphic to \( S^* \) if and only if \( P \) is isomorphic to \( P^* \). The group of automorphisms of \( P \) is isomorphic to the group of automorphisms of \( S \).

**Proof.** We first show that \( P^* (aNubN) \subseteq P^* ((avb)N) \) for \( a, b \) in \( P \) and for any isomorphism \( N \) of \( P \) onto \( P^* \). It is easy to see that \( Pa \subseteq Pb \) if and only if \( P^*(aN) \subseteq P^*(bN) \). Since \( aNubN = zN \) for some \( z \) in \( P \), \( P^*zN = P^*(aN) \cap P^*(bN) \subseteq P^*(aN) \), \( P^*(bN) \) by the definition of \( u \). Thus, \( Pz \subseteq P(ab) \) by the definition of \( v \) and the desired result follows. Therefore, by Theorem 1.1, \( M_N \) is a homomorphism of \( S \) into \( S^* \). To show it is one-to-one let \((m, n)M_N = (p, q)M_N \), i.e. \([mN, nN] = [pN, qN]\). Thus, using (1.3), we may show that \( mN = (\rho' p)N \) and \( nN = (\rho' q)N \) where \( \rho' \) is a unit of \( P \). Thus, by (1.3), \((m, n) = (p, q)\). Clearly, \( M_N \) maps \( S \) onto \( S^* \). Conversely, let \( M \) be an isomorphism of \( S \) onto \( S^* \). By Theorem 1.1, \((m, n)M = [(mN)k, (nN)k]\) where \( k \) in \( P^* \) and \( N \) is a homomorphism of \( P \) into \( P^* \). Now, it follows from (1.6), (B1), and (B2) that \((1, 1)M = [k, k] = [1^*, 1^*]\) where \( 1^* \) is the identity of \( P^* \). Thus, by (1.3), \( k \) is a unit of \( P^* \). Now, let \( nA = k^{-1} (nN)k \) for all \( n \) in \( P \). It is easily seen that \( A \) is a homomorphism of \( P \) into \( P^* \). Now, by (B1), (B2), and (1.3), we have

\[
(1.26) \quad (m, 1)M = [(mN)k, k] = [k^{-1}(mN)k, 1^*] = [mA, 1^*] \\
(1, m)M = [k, (mN)k] = [1^*, k^{-1}(mN)k] = [1^*, mA].
\]

Thus, from (1.26) and (1.3), we have \( mA = nA \) implies \( m = n \). Let \( a \) be in \( P^* \). Then, by the remarks on page 3, it follows that \([1^*, a] = (1, m)M \) for some \( m \) in \( P \). Hence, by (1.26) and (1.3), \( a = mA \). Therefore \( A \) is an isomorphism of \( P \) onto \( P^* \). From (1.26) and (1.8), we have \((m, n)M = [mA, nA]\). Thus, \( M = M_A \).

**Section 2.** A reduction of the homomorphism problem by an application of Schreier extensions.

We first will briefly review the work of Rédei [6] on the Schreier extension theory for semigroups (we actually give the right-left dual of his construction.). Let \( G \) be a semigroup with identity \( e \). We con-
sider a congruence relation $n$ on $G$ and call the corresponding division of $G$ into congruence classes a compatible class division of $G$. The class $H$ containing the identity is said to be the main class of the division. $H$ is easily shown to be a subsemigroup of $G$. The division is called right normal it and only if the classes are of the form,

$$\text{(2.1)} \quad Ha_1, Ha_2, \cdots (a_i = e)$$

and $h_1 a_i = h_2 a_i$ with $h_1, h_2$ in $H$ implies $h_1 = h_2$. The system (2.1) is shown to be uniquely determined by $H$. $H$ is then called a right normal divisor of $G$ and $G/n$ is denoted by $G/H$.

Let $G$, $H$, and $S$ be semigroups with identity. Then, if there exists a right normal divisor $H'$ of $G$ such that $H \cong H'$ and $S \cong G/H'$, $G$ is said to be a Schreier extension of $H$ by $S$.

Now, let $H$ and $S$ be semigroups with identities $E$ and $e$ respectively. Consider $H \times S$ under the following multiplication:

$$\text{(2.2)} \quad (A, a) (B, b) = (AB^{\alpha}a^{\beta}b, ab) \quad (A, B \text{ in } H; \ a, b \text{ in } S)$$

in which

$$a^{\beta}, B^{\alpha} \text{ (in } H)$$

designate functions of the arguments $a, b$ and $B, a$ respectively, and are subject to the conditions

$$\text{(2.3)} \quad a^e = E, \ e^a = E, \ B^e = B, \ E^a = E .$$

We call $H \times S$ under this multiplication a Schreier product of $H$ and $S$ and denote it by $HoS$.

Redéi's main theorem states:

**THEOREM 2.1 (Rédei).** A Schreier product $G = HoS$ is a semigroup if and only if

$$\text{(2.4)} \quad (AB)^c = A^eB^{a^c} \quad (A, B \text{ in } H; \ c \text{ in } S)$$

$$\text{(2.5)} \quad (B^a)^{e^{a^c}} = e^{aB^{\alpha^c}} \quad (B \text{ in } H; \ a, c \text{ in } S)$$

$$\text{(2.6)} \quad (a^b)^{(e^{a^c})} = e^{c(\alpha a)^c} \quad (a, b, c \text{ in } S)$$

are valid. These semigroups (up to an isomorphism) are all the Schreier extensions of $H$ by $S$ and indeed the elements $(A, e)$ form a right normal divisor $H'$ of $G$ for which

$$\text{(2.7)} \quad G/H' \cong S \ (H'(E, a) \rightarrow a) \quad H' \cong H \ ((A, e) \rightarrow A)$$

are valid.
THEOREM 2.2 Let \( U \) be a group with identity \( E \) and let \( S \) be a semigroup satisfying \( B_1 \) and \( B_2 \) (denote its identity by \( e \)) and suppose \( S \) has a trivial group of units. Then every Schreier extension \( P = UoS \) of \( U \) by \( S \) satisfies \( B_1 \) and \( B_2 \) (the identity is \((E, e)\)) and the group of units of \( P \) is \( U' = \{(A, e) : A \in U\} \cong U \). Furthermore \( L \) is a congruence relation on \( P \) and \( P/L \cong S \). \( P \) satisfies \( B_3 \) if and only if \( S \) satisfies \( B_3 \).

Conversely, let \( P \) be a semigroup satisfying \( B_1 \) and \( B_2 \) on which \( L \) is a congruence relation. Let \( U \) be the group of units of \( P \). Then \( U \) is a right normal divisor of \( P \) and \( P/U \cong P/L \). Thus, \( P \) is a Schreier extension of \( U \) by \( P/L \). \( P/L \) satisfies \( B_1 \) and \( B_2 \) and has a trivial group of units.

REMARK. Hence if \( P \) is any semigroup satisfying \( B_1 \) and \( B_2 \) with group of units \( U \) such that \( L \) is a congruence relation on \( P \), we will write \( P = (U, P/L, \alpha^\#, A^\#) \) in conjunction with Theorem 2.1 and 2.2. (We note that \( L \) is a right regular equivalence relation on any semigroup) \( \alpha^\#, A^\# \) will be called the function pair belonging to \( P \).

REMARK. A theorem of Rees [8, Theorem 3.3] is a special case of the above theorem.

Proof. It follows easily from (2.2) and (2.3) that \( P \) satisfies \( B_1 \) and has identity \((E, e)\). From Theorem 2.1, \( U' \cong U \). Now, suppose \((A, a)\) is a unit of \( P \). Then, \((A, a) \ (B, b) = (E, e)\) for some \((B, b)\) in \( P \). Hence by (2.2), \( ab = e \). Thus, by (B1), (B2), and the fact that the group of units of \( S \) is \( e \), \( a = b = e \), and \((A, a)\) in \( U' \). From (2.2) and (2.3), every element of \( U' \) is a unit of \( P \).

Next, we determine the principal left ideals of \( P \). From (2.2), we have

\[
P(A, a) = \{(BA^\#b^\#, ba) : B \in U, b \in S\}
\]

= \{(C, ba) : C \in U, b \in S\}.

Since \( P(A, a) \) just depends on \( a \), we may write \( P(A, a) = P_\# \) for all \( A \) in \( U \).

Next, we show that

\[
(A, a) \ L (B, b) \text{ if and only if } a = b .
\]

Now, from (2.8), \((A, a) \ L (B, b) \) implies \( b = xa \) and \( a = yb \) for some \( x, y \) in \( S \) Thus, by B1, \( xy = yx = e \), and since \( S \) has a trivial group of units, \( x = y = e \). Thus, \( a = b \). The converse is evident from (2.8). It follows easily from (2.9) and (2.2) that \( L \) is a congruence relation. \( L_{(E, a)} \) will denote the \( L \)-class of \( P \) containing \((E, a)\). It is easily seen
that the mapping \( L(x, a) \rightarrow a \) is an isomorphism of \( P/L \) onto \( S \). Now suppose \( S \) satisfies B3, i.e. \( a, b \) in \( S \) implies there exists \( c \) in \( S \) such that

\[
(2.10) \quad Sa \cap Sb = Sc.
\]

From (2.10) and (2.8),

\[
(2.11) \quad P_a \cap P_b = P_c
\]

and \( P \) satisfies B3. If \( P \) satisfies B3, it follows from (2.8) and (2.11) that \( S \) satisfies B3.

Now, let \( P \) be a semigroup satisfying B1 and B2 with group of units \( U \) on which \( L \) is a congruence relation. By (1.12) (this is shown without using B3) \( U \) is the congruence class mod \( L \) containing the identity 1 of \( P \), i.e. \( U \) is the main class of the compatible class division of \( P \) given by \( L \). If \( a \) in \( P \), \( L_a = Ua \) from (1.12). If \( \rho_a = \rho \alpha \) \( a \) where \( \rho \), \( \rho_2 \) in \( U \), then \( \rho = \rho_2 \) by B1. Thus, \( U \) is a right normal divisor of \( P \) and \( P/U \cong P/L \). Hence, \( P \) is a Schreier extension of \( U \) by \( P/L \). By virtue of (1.12) and (B1), \( P/L \) satisfies B1.

Let \( a \rightarrow \bar{a} \) be the natural homomorphism of \( P \) onto \( P/L \). Then, \( \bar{1} \) is the identity of \( P/L \). Let \( \bar{a} \) be a unit of \( P \). Then, by (1.12), (B1), and (B2), \( a \) is in \( U \). Hence, \( \bar{a} = \bar{1} \). Therefore, \( P/L \) has a trivial group of units.

**Theorem 2.3.** Let \( P = (U, P/L, a^e, A^e) \) and \( P^* = (U^e, P^*/L^*, b^e, B^e) \) be semigroups satisfying B1 and B2 on which \( L \) and \( L^* \) are congruence relations. \( U \) and \( a^e, A^e \) denote the unit group and function pair of \( P \). \( U^e \) and \( b^e, B^e \) denote the unit group and function pair of \( P^* \). \( P/L \) is the factor semigroup of \( P \) mod \( L \) and \( P^*/L^* \) is the factor semigroup of \( P^* \) mod \( L^* \). Let \( f \) be a homomorphism of \( U \) into \( U^e \), \( g \) be a homomorphism of \( P/L \) into \( P^*/L^* \), and \( h \) be a function of \( P/L \) into \( U^e \). Suppose \( f, g \) and \( h \) are subject to the following conditions:

\[
(2.12) \quad (ah)(bh)(ag)(bg) = (a^e f)(ab)h
\]

\[
(2.13) \quad (bh)(Af)(bg) = (A^e f)(bh).
\]

For each \( (A, a) \) in \( P \) define

\[
(2.14) \quad (A, a)M = [(Af)(ah), ag]
\]

where the square brackets denote elements of \( P^* \). Then \( M \) is a homomorphism of \( P \) into \( P^* \). Conversely, every homomorphism of \( P \) into \( P^* \) is obtained in this fashion. \( M \) is an isomorphism if and
only if $f$ and $g$ are isomorphisms.

Proof. Clearly, $M$ is single valued. From (2.14), (2.2), (2.4), (2.13) and (2.12), we have


Thus, $M$ is a homomorphism of $P$ into $P^*$. Conversely, let $M$ be any homomorphism of $P$ into $P^*$. It follows from B1 and B2 that $UM \subseteq U$.

Thus, $M$ is a homomorphism of $P$ into $P^*$. Conversely, let $M$ be any homomorphism of $P$ into $P^*$. It follows from B1 and B2 that $UM \subseteq U$. Thus, by Theorem 2.2, we may let

$$(A, e)M = [Af, e^*]$$

where $e$ and $e^*$ denote the identities of $P/L$ and $P^*/L^*$ respectively. Clearly, $f$ is a mapping of $U$ into $U^*$. It follows easily from (2.15), (2.2) and (2.3) that $f$ is a homomorphism of $U$ into $U^*$. Let $E$ be the identity of $U$. Then,

$$(E, a)M = [ah, ag] .$$

Clearly, $h$ is a function of $P/L$ into $U^*$ and $g$ is a function of $P/L$ into $P^*/L^*$. From (2.2) and (2.3), $(A, a) = (A, e)(E, a)$. Thus, by (2.16), (2.15), (2.2), and (2.3)


From (2.2) and (2.3), we have $(E, a)(E, b) = (a^b, ab)$. Thus, by (2.17), we have $[ah, ag] [bh, bg] = [(a^b f)(ab)h, (ab)g]$. Therefore, by (2.2)

$$(ah)(bh)^x[(ag)^x, (ag)(bg)] = [(a^b f)(ab)h, (ab)g] .$$

From (2.18), it follows that $g$ is a homomorphism and (2.12) is satisfied. From (2.2) and (2.3), we have $(E, b)(A, e) = (A^b, b)$. Thus, from (2.17) and (2.15), $[bh, bg][Af, e^*] = [(Af)(bh), bg]$. Hence, (2.13) follows from (2.2) and (2.3).

Suppose $M$ is an isomorphism of $P$ onto $P^*$. Therefore, by (2.14) $(A, a)M = [(Af)(ah), ag]$ where $f$ is a homomorphism of $U$ into $U^*$, $h$ is a single valued mapping of $P/L$ into $U^*$ and $g$ is a homomorphism $P/L$ into $P^*/L^*$. It is easy to see that $UM = U^*$. Thus, by virtue of theorem 2.2, if $B$ in $U^*$, there exists $A$ in $U$ such that $(A, e)M = [B, e^*]$. Thus, by (2.15), $Af = B$ and $f$ maps $U$ onto $U^*$. By (2.15), $f$ is one-to-one and hence is an isomorphism of $U$ onto $U^*$. To show $g$ is one-to-one, let
There exists $x$ in $U^*$ such that
\[(\text{2.20}) \quad x(bh) = ah.\]

Now, by (2.2) and (2.3), $(xf^{-1}, e)(E, b) = (xf^{-1}, b)$. Hence, by (2.15), (2.2), (2.3), (2.19) and (2.20), $(xf^{-1}, b)M = [x, e^*][bh, bg] = [x(bh), bg] = [ah, ag] = (E, a)M$. Hence, $a = b$. It follows immediately from (2.14) that $g$ maps $P/L$ onto $P*/L^*$ and hence $g$ is an isomorphism of $P/L$ onto $P*/L^*$.

Conversely, suppose there exists an isomorphism $f$ of $U$ onto $U^*$, an isomorphism $g$ of $P/L$ onto $P*/L^*$ and a single valued mapping $h$ of $P/L$ into $U^*$ such that (2.12) and (2.13) are satisfied. Therefore, by (2.14), $(A, a)M = [(Af)(ah), ag]$ is a homomorphism of $P$ into $P^*$. It is easily seen that $M$ is one-to-one. Let $[B, b]$ be in $P^*$. Now there exists $a$ in $P/L$ such that $b = ag$ and $A$ in $U$ such that $(Af)(ah) = B$. Hence $(A, a)M = [B, b]$ by (2.14).

**Remark.** If $ah = E^*$, where $E^*$ is the identity of $U^*$, then (2.12) and (2.13) simplify greatly:
\[(\text{2.12}') \quad (ag)^{b^*} = a^*f,\]
\[(\text{2.13}') \quad (Af)^{b^*} = A^*f.\]

Professor Clifford remarks that we can bring this about by making a new choice of representative elements in $P$ or in $P^*$, respectively, in the following two cases: if the range of $h$ is contained in the range of $f$; or if $ag = a'g$ ($a, a'$ in $P/L$) implies $ah = a'h$.

Section 3. Examples. We give some examples to illustrate the theory.

**Example 1.** The bicyclic semigroup "C" [2, p. 43] consists of all pairs of nonnegative integers with multiplication given by
\[(3.1) \quad (i, j)(k, s) = (i + k - \min (j, k) j + s - \min (j, k)),\]

A complete set of endomorphisms of "C" is given by
\[(3.2) \quad (i, j)M_{(t, k)} = (ti + k, tj + k)(i, j \text{ are nonnegative integers})\]

where $(t, k)$ runs through all ordered pairs of nonnegative integers.

The only automorphism of 'C' is the identity.

**Example 2.** Let $G$ be any group of order greater than or equal to two with identity $E$. Let $I_0$ be the nonnegative integers under
the usual addition. Consider $P = G \times I_0$ under the following multiplication.

\[(3.3)\quad (A, a)(B, b) = (AB^a, a + b)\]

where $B^a = B$ if $a = 0$

$B^a = E$ if $a \neq 0$.

$P$ is a semigroup satisfying (B1), (B2), (B3) which is not left cancellative. Let $S$ be the semigroup corresponding to $P$ in Clifford’s main theorem. Let $h$ be a mapping of $I_0$ into $G$ such that $oh = E$ and $ah = (a + b)h$ for all $a \neq 0$. Let $f$ be an automorphism of $G$. Then,

\[(3.4)\quad ((A, a), (B, b))M = (((Af)(ah), a), ((Bf)(bh), b)) \quad \text{where } (A, a), (B, b) \in P \text { is an automorphism of } S. \quad \text{Conversely every automorphism of } S \text{ is obtained in this fashion.} \]

One obtains similar results if $I_0$ is replaced by the positive part of any lattice ordered group.

**Example 3.** Let $G^+$ be the positive part of any lattice ordered group $G$. Let $S$ be the semigroup corresponding to $G^+$ in Clifford’s main theorem. Then there exists a one-to-one correspondence between the automorphisms $M$ of $S$ and the order preserving automorphisms $N$ of $G$. This correspondence is given by

\[(m, n)M = (mN, nN) \quad (m \text{ and } n \text{ in } G^+).\]

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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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