Pacific Journal of Mathematics

LINEAR TRANSFORMATIONS ON GRASSMAN SPACES

ROY WESTWICK

Vol. 14, No. 3 July 1964

LINEAR TRANSFORMATIONS ON GRASSMANN SPACES

R. WESTWICK

1. Let U denote an n-dimensional vector space over an algebraically closed field F, and let G_{nr} denote the set of nonzero pure r-vectors of the Grassmann product space $\bigwedge^r U$. Let T be a linear transformation of $\bigwedge^r U$ which sends G_{nr} into G_{nr}. In this note we prove that T is nonsingular, and then, by using the results of Wei-Liang Chow in [1], we determine the structure of T.

For each $z = x_1 \wedge \cdots \wedge x_r \in G_{nr}$, we let $[z]$ denote the r-dimensional subspace of U spanned by the vectors $x_1, \cdots, x_r$. By Lemma 5 of [1], two independent elements z_1 and z_2 of G_{nr} span a subspace all of whose nonzero elements are in G_{nr} if and only if $\dim([z_1] \cap [z_2]) = r - 1$; that is, if and only if $[z_1]$ and $[z_2]$ are adjacent. If $V \subseteqq \bigwedge^r U$ is a subspace such that each nonzero vector in V is in G_{nr} and if V is maximal (that is, not contained in a larger such subspace) then $\{[z] \mid z \in V, z \neq 0\}$ is a maximal set of pairwise adjacent r-dimensional subspaces of U. These sets of subspaces are of two types; namely, the set of all r-dimensional subspaces of U containing a common $(r-1)$-dimensional subspace, and the set of all r-dimensional subspaces of an $(r+1)$-dimensional subspace of U. We adopt the usual convention of calling these sets of subspaces maximal sets of the first and second kind respectively. We will let A_r denote the set of those maximal V which determine a set of pairwise adjacint subspaces of the first kind, and we will let B_r denote the set of those maximal V which determine a set of pairwise adjacent subspaces of the second kind.

2. In this section we prove that if T sends each member of B_r into a member of B_r then T is nonsingular.

Let $U_1, \cdots, U_t$ be k-dimensional pairwise adjacent subspaces of U and let $z_i \in G_{nk}$ be such that $[z_i] = U_i$ for $i = 1, \cdots, t$. Then $\{U_1, \cdots, U_t\}$ is said to be independent if and only if $\{z_1, \cdots, z_t\}$ is an independent subset of $\bigwedge^k U$. We note the following facts concerning an independent set $\{U_1, \cdots, U_t\}$. If it is of the first kind (in the sense of the previous section) then there is an independent set of vectors $\{x_1, \cdots, x_{k-1}, y_1, \cdots, y_t\}$ of U such that for $i = 1, \cdots, t$, $U_i = \langle x_1, \cdots, x_{k-1}, y_i\rangle \cdot \langle \cdots \rangle$ denotes the linear subspace spanned by the vectors enclosed. If it is of the second kind, then there is an independent set of vectors $\{x_1, \cdots, x_{k+1}\}$ such that $U_i = \langle x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}\rangle$, for $i = 1, \cdots, t$. It is easily

Received July 2, 1963. The author is indebted to M. Marcus for his encouragement and help.

deduced from this that $\dim(\bigwedge^r U_1 + \cdots + \bigwedge^r U_t)$ is equal to $t\binom{k-1}{r-1} + \binom{k-1}{r}$ or $\sum_{i=0}^{t-1}\binom{k-i}{r-1}$ according as the set of subspaces $\{U_i\}$ is of the first or second kind. We adopt the usual convention that $\binom{m}{n} = 0$ if $m < n$. Finally, if the set $\{U_1, \cdots, U_t\}$ is not independent, then for some i, $\bigwedge^r U_i \subseteqq \bigwedge^r U_1 + \cdots + \bigwedge^r U_{i-1}$. In fact, the choice of i such that $\{z_1, \cdots, z_{i-1}\}$ is independent and $z_i \in \langle z_1, \cdots, z_{i-1}\rangle$ will do.

We require the

LEMMA 1. *Let $\{U_1, \cdots, U_{s+1}\}$ be a set of pairwise adjacent k-dimensional subspaces of U. Suppose further that the set is independent and is of the second kind. Let $V \subseteqq \bigwedge^r U_1 \cdots + \bigwedge^r U_{s+1}$ be a subspace with dimension $\binom{k-s}{r-s}$, where $s \leqq r \leqq k$. Then there is a set $\{V_1, \cdots, V_s\}$ of pairwise adjacent k-dimensional subspaces of U such that $V \cap (\bigwedge^r V_1 + \cdots + \bigwedge^r V_s) \neq \{0\}$.*

Proof. Let $m = \binom{k-s}{r-s}$ and let $\{z_1, \cdots, z_m\}$ be a basis of V. Choose an independent set of vectors $\{x_1, \cdots, x_{k+1}\}$ of U such that for $i = 1, \cdots, s+1$, $U_i = \langle x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}\rangle$. We can write

$$z_i = z_1^i + x_1 \wedge \cdots \wedge x_{s-1} \wedge x_s \wedge z_2^i + x_1 \wedge \cdots \wedge x_{s-1} \wedge x_{s+1} \wedge z_3^i$$

where

$$z_1^i \in \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} \quad \text{and} \quad z_2^i, z_3^i \in \bigwedge^{r-s} \langle x_{s+2}, \cdots, x_{k+1}\rangle$$

for $i = 1, \cdots, m$. In the case that $s = 1$, we take $z_1^i \in \bigwedge^r \langle x_3, \cdots, x_{k+1}\rangle$. In the case that $s = r$, we take $z_2^i, z_3^i \in F$. If $\{z_2^1, \cdots, z_2^m\}$ or $\{z_3^1, \cdots, z_3^m\}$ is dependent, then we can form a linear combination of $z_1, \cdots, z_m$ which will be in $\bigwedge^r U_1 + \cdots \bigvee^r{}_{s-1} + \bigwedge^r U_{s+1}$ or $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} + \bigwedge^r U_s$ respectively. If, on the other hand, both sets are independent then each is a basis of $\bigwedge^{r-s}\langle x_{s+2}, \cdots, x_{k+1}\rangle$ since $\dim(\bigwedge^{r-s}\langle x_{s+2}, \cdots, x_{k+1}\rangle) = \binom{k-s}{r-s} = m$. Let $z_2^i = \sum_{j=1}^m a_{ij} z_3^j$, $i = 1, \cdots, m$. Choose $\lambda \neq 0$ and $b_i \in F$, not all equal to zero, such that

$$\lambda b_j = \sum_{i=1}^m b_i a_{ij}\,, \qquad j = 1, \cdots, m\,.$$

Then

$$0 \neq \sum_{j=1}^m b_j z_j = \sum_{j=1}^m z_1^j + \sum_{j=1}^m x_1 \wedge \cdots \wedge x_{s-1} \wedge (x_s + \lambda^{-1} x_{s+1}) \wedge b_j z_2^j$$
$$\in \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} + \bigwedge^r V_1$$

where $V_1 = \langle x_1 \cdots, x_{s-1}, x_s + \lambda^{-1}x_{s+1}, x_{s+2}, \cdots, x_{k+1}\rangle$. The subspaces

$U_1, \cdots, U_{s-1}, V_1$ are pairwise adjacent and so the Lemma is proved.

The nonsingularity of T is now proved as follows. Let W be a subspace of U. We prove, by induction on the dimension of W, that T is one-to-one on $\bigwedge^r W$ and that the image of $\bigwedge^r W$ under T is $\bigwedge^r W'$ for some subspace W' of U with $\dim(W) = \dim(W')$. When $\dim(W) = r + 1$ this is clear since we are assuming that B_r is sent into B_r by T. Suppose that the statement has been proved for k-dimensional subspaces, and consider a $(k+1)$-dimensional subspace W of U. Let s be the largest integer such that for any set $\{W_1, \cdots, W_s\}$ of pairwise adjacent k-dimensional subspaces of W, T is one-to-one on $\bigwedge^r W_1 + \cdots + \bigwedge^r W_s$. If $s \geqq r + 1$ then T is one-to-one on $\bigwedge^r W$, since in this case, for an independent set $\{W_1, \cdots, W_s\}$ we must have $\bigwedge^r W = \bigwedge^r W_1 + \cdots + \bigwedge^r W_s$. Suppose then that $1 \leqq s \leqq r$ and let $\{U_1, \cdots, U_{s+1}\}$ be any set of $s+1$ pairwise adjacent k-dimensional subspaces of W. If the set is dependent then T is one-to-one $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1}$ since we may drop one of the terms. Therefore we assume that the set is independent. Choose k-dimensional subspaecs $U_1', \cdots, U_{s+1}'$ such that $T(\bigwedge^r U_i) = \bigwedge^r U_i'$ for $i = 1, \cdots, s+1$. For each $j \leqq s$, T maps $\bigwedge^r U_1 + \cdots + \bigwedge^r U_j$ onto $\bigwedge^r U_1' + \cdots + \bigwedge^r U_j'$. Therefore, since T is one-to-one on $\bigwedge^r U_1 + \cdots + \bigwedge^r U_s$, the set $\{U_1', \cdots, U_s'\}$ is independent. Furthermore, the set $\{U_1', \cdots, U_{s+1}'\}$ is also independent. If not, then the image under T of both $\bigwedge^r U_1 + \cdots + \bigwedge^r U_s$ and $\bigwedge^r U_1 + \cdots \bigwedge^r U_{s+1}$ is $\bigwedge^r U_1' + \cdots + \bigwedge^r U_s'$. But then the dimension of the null space of T in $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1}$ is at least as large as the difference in the dimensions of $\bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1}$ and $\bigwedge^r U_1 + \cdots + \bigwedge^r U_s$, that is, $\binom{k-s}{r-s}$. We apply Lemma 1 to contradict the choice of s. It follows that T is one-to-one on all of $\bigwedge^r W$. Finally, let $\{W_1, \cdots, W_{k+1}\}$ be an independent set of k-dimensional pairwise adjacent subspaces of W (necessarily of the second kind). Let W_i' be chosen so that $T(\bigwedge^r W_i) = \bigwedge^r W_i'$. It follows easily that $\{W_1', \cdots, W_{k+1}'\}$ is of the second kind also, so that the image of $\bigwedge^r W$ is $\bigwedge^r W'$ where W' is the $(k+1)$-dimensional subspace of U containing $W_1', \cdots, W_{k+1}'$. By taking $W = U$ we see that T is one-to-one on $\bigwedge^r U$.

3. It is necessary to investigate whether a general T does necessarily send each element of B_r into B_r. For the cases $n > 2r$, $n < 2r$, this is proved directly, using Lemma 2. The case $n = 2r$ requires a more delicate argument, given at the end of this section; there it is shown that if some element of B_r is sent into B_r by T, then T sends B_r into B_r.

LEMMA 2. *Let $r < n$ and let V_1 and V_2 be in A_r such that $V_1 \cap V_2 \neq \{0\}$. Then, if $V \subseteqq V_1 + V_2$ and* $\dim(V) = n - r$, *we have $V \cap G_{nr} \neq \phi$.*

Proof. Let U_i be the $(r-1)$-dimensional subspace of U determined by V_i for $i = 1, 2$. Since $V_1 \cap V_2 \neq \{0\}$, either $U_1 = U_2$ or $\dim (U_1 \cap U_2) = r - 2$.

If $U_1 = U_2$ then $V_1 = V_2$, so that in this case it is clear that $V \cap G_{nr} \neq \phi$.

Suppose that $\dim (U_1 \cap U_2) = r - 2$ and let $\{x_1, \cdots, x_{r-2}\}$ be a basis of this intersection. Choose y_i such that $U_i = \langle x_1, \cdots, x_{r-2}, y_i \rangle$ for $i = 1, 2$. Choose u_i and v_i in U, $i = 1, \cdots, n - r$, such that

$$\{z_i = x_1 \wedge \cdots \wedge x_{r-2} \wedge (y_1 \wedge u_i + y_2 \wedge v_i) \mid i = 1, \cdots, n - r\}$$

forms a basis of V. If

$$\{x_1, \cdots, x_{r-2}, y_1, y_2, v_1, \cdots, v_{n-r}\} \quad \text{or} \quad \{x_1, \cdots, x_{r-2}, y_1, y_2, u_1, \cdots, u_{n-r}\}$$

is dependent, then there is a linear combination of the z_i which is in V_1 or V_2 respectively. If, on the other hand, both sets are independent, then they are both bases for U and we may write

$$u_i = w_i + c_i y_2 + \sum_{j=1}^{n-r} a_{ij} v_j \ , \qquad i = 1, \cdots, n - r \ ,$$

where $w_i \in \langle x_1, \cdots, x_{r-2}, y_1 \rangle$ and $c_i, a_{ij} \in F$. We note that $\det (a_{ij}) \neq 0$ so we can choose $\lambda \neq 0$ and b_j for $j = 1, \cdots, n - r$, not all zero, such that $\lambda b_j = \sum_{i=1}^{n-r} b_i a_{ij}$. Then

$$0 \neq \sum_{j=1}^{n-r} b_j z_j = x_1 \wedge \cdots \wedge x_{r-2} \wedge (y_1 + \lambda^{-1} y_2) \wedge \left[\left(\sum_{j=1}^{n-r} b_j c_j \right) y_2 + \lambda \sum_{j=1}^{n-r} b_j v_j \right]$$

is an element of $V \cap G_{nr}$. This proves the Lemma.

For $n \neq 2r$ the image under T of an element of B_r is an element of B_r. For $n < 2r$ this is clearly so since the subspaces of $\bigwedge^r U$ in B_r have dimension $r + 1$, which is greater than the dimension $(n - r + 1)$ of the subspaces in A_r.

For $n > 2r$ we proceed as follows. The image of an A_r is an A_r. Suppose that the image of a $W \in B_r$ is a subspace of a $V \in A_r$. Choose two elements V_1 and V_2 of A_r such that $V_1 \cap V_2 \neq \{0\}$ and $\dim (V_1 \cap W) = \dim (V_2 \cap W) = 2$. One does this by choosing V_1 and V_2 so that the $(r-1)$-dimensional subspaces of U determined by them are adjacent subspaces of the $(r+1)$-dimensional subspace determined by W. Now, $T(V_1) = T(V_2) = V$ since each is in A_r and each intersects V in at least two dimensions. Therefore $T(V_1 + V_2) = V$ and so the null space of T in $V_1 + V_2$ has dimension equal to $(2n - 2r + 1) - (n - r + 1) = n - r$. By Lemma 2, it follows that the null space of T intersects G_{nr} which contradicts the hypothesis that T sends G_{nr} into G_{nr}.

In the case that $n = 2r$ the image of a B_r may be an A_r since the dimensions are equal. However, we prove that if some B_r is sent into a B_r by T, then the image of each B_r is a B_r. Suppose not. Then we can choose $(r + 1)$-dimensional subspaces W_1 and W_2 of U such that $T(\bigwedge^r W_1) \in A_r$ and $T(\bigwedge^r W_2) \in B_r$. Furthermore, we can choose W_1 and W_2 adjacent, so that $\dim(W_1 \cap W_2) = r$. Choose three distinct elements V_1, V_2, and V_3 of A_r such that the $(r - 1)$-dimensional subspaces of U determined by these elements are contained in $W_1 \cap W_2$. Then $\dim(V_i \cap \bigwedge^r W_j) = 2$ for $i = 1, 2, 3$ and $j = 1, 2$, so that $T(V_i)$ intersects $T(\bigwedge^r W_j)$ in at least two dimensions for each i, j. This implies that each $T(V_i)$ is equal to one of $T(\bigwedge^r W_j)$ and so two of them are equal. The argument of the previous paragraph now leads to a contradiction.

4. By essentially the same argument as used by Chow in [1] to prove his Theorem 1, we can prove that; if S is a nonsingular linear transformation of $\bigwedge^r U$ sending G_{nr} into G_{nr}, and if the image of each B_r is a B_r, then S is a compound. (By a compound we mean a linear transformation of $\bigwedge^r U$ which is induced by a linear transformation of U.)

In the case that $n \neq 2r$ it follows that T is necessarily a compound. For $n = 2r$, T is a compound if some B_r is sent into a B_r. If we let T_0 denote a linear transformation of $\bigwedge^r U$ induced by a correlation of the r-dimensional subspaces of U, then T_0 is nonsingular and sends G_{nr} onto G_{nr}. The image of each A_r under T_0 is a B_r. Therefore, if a B_r is sent by T into an A_r, the T_0T is a compound. We have proved the

THEOREM. *Let U be an n-dimensional vector space over an algebraically closed field and let T be a linear transformation of $\bigwedge^r U$ which sends G_{nr} into G_{nr}. Then T is a compound except, possibly, when $n = 2r$, in which case T may be the composite of a compound and a linear transformation induced by a correlation of the r-dimensional subspaces of U.*

REFERENCE

1. Wei-Liang Chow, *On the Geometry of Algebraic Homogeneous Spaces*, Annals of Math., **50** (1949), 32-67.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should by typewritten (double spaced), and on submission, must be accompanied by a separate author's résumé. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 14, No. 3 July, 1964