LINEAR TRANSFORMATIONS ON GRASSMAN SPACES

ROY WESTWICK
1. Let $U$ denote an $n$-dimensional vector space over an algebraically closed field $F$, and let $G_{nr}$ denote the set of nonzero pure $r$-vectors of the Grassmann product space $\Lambda^r U$. Let $T$ be a linear transformation of $\Lambda^r U$ which sends $G_{nr}$ into $G_{nr}$. In this note we prove that $T$ is nonsingular, and then, by using the results of Wei-Liang Chow in [1], we determine the structure of $T$.

For each $z = x_1 \wedge \cdots \wedge x_r \in G_{nr}$, we let $[z]$ denote the $r$-dimensional subspace of $U$ spanned by the vectors $x_1, \ldots, x_r$. By Lemma 5 of [1], two independent elements $z_1$ and $z_2$ of $G_{nr}$ span a subspace all of whose nonzero elements are in $G_{nr}$ if and only if $\dim ([z_1] \cap [z_2]) = r - 1$; that is, if and only if $[z_1]$ and $[z_2]$ are adjacent. If $V \subseteq \Lambda^r U$ is a subspace such that each nonzero vector in $V$ is in $G_{nr}$ and if $V$ is maximal (that is, not contained in a larger such subspace) then $\{[z] \mid z \in V, z \neq 0\}$ is a maximal set of pairwise adjacent $r$-dimensional subspaces of $U$. These sets of subspaces are of two types; namely, the set of all $r$-dimensional subspaces of $U$ containing a common $(r - 1)$-dimensional subspace, and the set of all $r$-dimensional subspaces of an $(r + 1)$-dimensional subspace of $U$. We adopt the usual convention of calling these sets of subspaces maximal sets of the first and second kind respectively. We will let $A_r$ denote the set of those maximal $V$ which determine a set of pairwise adjacent subspaces of the first kind, and we will let $B_r$ denote the set of those maximal $V$ which determine a set of pairwise adjacent subspaces of the second kind.

2. In this section we prove that if $T$ sends each member of $B_r$ into a member of $B_r$ then $T$ is nonsingular.

Let $U_i, \ldots, U_t$ be $k$-dimensional pairwise adjacent subspaces of $U$ and let $z_i \in G_{nk}$ be such that $[z_i] = U_i$ for $i = 1, \ldots, t$. Then $\{U_1, \ldots, U_t\}$ is said to be independent if and only if $\{z_1, \ldots, z_t\}$ is an independent subset of $\Lambda^k U$. We note the following facts concerning an independent set $\{U_1, \ldots, U_t\}$. If it is of the first kind (in the sense of the previous section) then there is an independent set of vectors $\{x_1, \ldots, x_{k-1}, y_1, \ldots, y_t\}$ of $U$ such that for $i = 1, \ldots, t$, $U_i = \langle x_1, \ldots, x_{k-1}, y_i, \cdots \rangle$ denotes the linear subspace spanned by the vectors enclosed. If it is of the second kind, then there is an independent set of vectors $\{x_1, \cdots, x_{k+1}\}$ such that $U_i = \langle x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}\rangle$, for $i = 1, \cdots, t$. It is easily

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deduced from this that \( \dim (\bigwedge^r U_1 + \cdots + \bigwedge^r U_l) \) is equal to \( t \binom{k - 1}{r} \) or \( \sum_{i=0}^{l} \binom{k - i}{r} \) according as the set of subspaces \( \{U_i\} \) is of the first or second kind. We adopt the usual convention that \( \binom{m}{n} = 0 \) if \( m < n \). Finally, if the set \( \{U_1, \ldots, U_l\} \) is not independent, then for some \( i, \bigwedge^r U_i \subseteq \bigwedge^r U_1 + \cdots + \bigwedge^r U_{i-1} \). In fact, the choice of \( i \) such that \( \{z_i, \ldots, z_{i-1}\} \) is independent and \( z_i \in \langle z_1, \ldots, z_{i-1} \rangle \) will do. We require the

**Lemma 1.** Let \( \{U_1, \ldots, U_{s+1}\} \) be a set of pairwise adjacent \( k \)-dimensional subspaces of \( U \). Suppose further that the set is independent and is of the second kind. Let \( V \subseteq \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1} \) be a subspace with dimension \( \binom{k - s}{r - s} \), where \( s \leq r \leq k \). Then there is a set \( \{V_1, \ldots, V_s\} \) of pairwise adjacent \( k \)-dimensional subspaces of \( U \) such that \( V \cap (\bigwedge^r V_1 + \cdots + \bigwedge^r V_s) \neq \{0\} \).

**Proof.** Let \( m = \binom{k - s}{r - s} \) and let \( \{z_1, \ldots, z_m\} \) be a basis of \( V \). Choose an independent set of vectors \( \{x_1, \ldots, x_{k+1}\} \) of \( U \) such that for \( i = 1, \ldots, s + 1 \), \( U_i = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1} \rangle \). We can write

\[
  z_i = z_i^1 + x_1 \wedge \cdots \wedge x_{s-1} \wedge x_s \wedge z_i^3 + x_1 \wedge \cdots \wedge x_{s-1} \wedge x_{s+1} \wedge z_i^3
\]

where

\[
  z_i^1 \in \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} \quad \text{and} \quad z_i^3 \in \bigwedge^r \langle x_{s+2}, \ldots, x_{k+1} \rangle
\]

for \( i = 1, \ldots, m \). In the case that \( s = 1 \), we take \( z_i^1 \in \bigwedge^r \langle x_3, \ldots, x_{k+1} \rangle \). In the case that \( s = r \), we take \( z_i^1, z_i^3 \in F \). If \( \{z_i^1, \ldots, z_m^1\} \) or \( \{z_i^3, \ldots, z_m^3\} \) is dependent, then we can form a linear combination of \( z_1, \ldots, z_m \) which will be in \( \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s+1} \) or \( \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} + \bigwedge^r U_s \), respectively. If, on the other hand, both sets are independent then each is a basis of \( \bigwedge^{r-s} \langle x_{s+2}, \ldots, x_{k+1} \rangle \) since \( \dim (\bigwedge^{r-s} \langle x_{s+2}, \ldots, x_{k+1} \rangle) = \binom{k - s}{r - s} = m \). Let \( z_i^1 = \sum_{j=1}^{m} a_{ij} z_j^1 \), \( i = 1, \ldots, m \). Choose \( \lambda \neq 0 \) and \( b_i \in F \), not all equal to zero, such that

\[
  \lambda b_j = \sum_{i=1}^{m} b_i a_{ij}, \quad j = 1, \ldots, m.
\]

Then

\[
  0 \neq \sum_{j=1}^{m} b_j z_j = \sum_{j=1}^{m} z_j^1 + \sum_{j=1}^{m} x_1 \wedge \cdots \wedge x_{s-1} \wedge (x_s + \lambda^{-1} x_{s+1}) \wedge b_j z_j^3
\]

\[
  \in \bigwedge^r U_1 + \cdots + \bigwedge^r U_{s-1} + \bigwedge^r V_1
\]

where \( V_1 = \langle x_1, \ldots, x_{s-1}, x_s + \lambda^{-1} x_{s+1}, x_{s+2}, \ldots, x_{k+1} \rangle \). The subspaces...
U_1, \ldots, U_{s-1}, V_1$ are pairwise adjacent and so the Lemma is proved.

The nonsingularity of $T$ is now proved as follows. Let $W$ be a subspace of $U$. We prove, by induction on the dimension of $W$, that $T$ is one-to-one on $\Lambda^r W$ and that the image of $\Lambda^r W$ under $T$ is $\Lambda^r W'$ for some subspace $W'$ of $U$ with $\dim(W) = \dim(W')$. When $\dim(W) = r + 1$ this is clear since we are assuming that $B_r$ is sent into $B_r$ by $T$. Suppose that the statement has been proved for $k$-dimensional subspaces, and consider a $(k+1)$-dimensional subspace $W$ of $U$. Let $s$ be the largest integer such that for any set $\{W_1, \ldots, W_s\}$ of pairwise adjacent $k$-dimensional subspaces of $W$, $T$ is one-to-one on $\Lambda^r W_1 + \cdots + \Lambda^r W_s$. If $s \geq r + 1$ then $T$ is one-to-one on $\Lambda^r W$, since in this case, for an independent set $\{W_1, \ldots, W_s\}$ we must have $\Lambda^r W = \Lambda^r W_1 + \cdots + \Lambda^r W_s$. Suppose then that $1 \leq s \leq r$ and let $\{U_1, \ldots, U_{s+1}\}$ be any set of $s + 1$ pairwise adjacent $k$-dimensional subspaces of $W$. If the set is dependent then $T$ is one-to-one $\Lambda^r U_1 + \cdots + \Lambda^r U_{s+1}$, since we may drop one of the terms. Therefore we assume that the set is independent. Choose $k$-dimensional subspaces $U_1', \ldots, U_{s+1}'$ such that $T(\Lambda^r U_i) = \Lambda^r U_i'$ for $i = 1, \ldots, s + 1$. For each $j \leq s$, $T$ maps $\Lambda^r U_1 + \cdots + \Lambda^r U_j$ onto $\Lambda^r U_1' + \cdots + \Lambda^r U_j'$. Therefore, since $T$ is one-to-one on $\Lambda^r U_1 + \cdots + \Lambda^r U_{s+1}$, the set $\{U_1', \ldots, U_{s+1}'\}$ is independent. Furthermore, the set $\{U_1', \ldots, U_{s+1}'\}$ is also independent. If not, then the image under $T$ of both $\Lambda^r U_1 + \cdots + \Lambda^r U_s$ and $\Lambda^r U_1 + \cdots + \Lambda^r U_{s+1}$ is $\Lambda^r U_1' + \cdots + \Lambda^r U_s'$. But then the dimension of the null space of $T$ in $\Lambda^r U_1 + \cdots + \Lambda^r U_{s+1}$ is at least as large as the difference in the dimensions of $\Lambda^r U_1 + \cdots + \Lambda^r U_{s+1}$ and $\Lambda^r U_1 + \cdots + \Lambda^r U_s$, that is, $\left( \binom{k-s}{r-s} \right)$. We apply Lemma 1 to contradict the choice of $s$.

It follows that $T$ is one-to-one on all of $\Lambda^r W$. Finally, let $\{W_1, \ldots, W_{k+1}\}$ be an independent set of $k$-dimensional pairwise adjacent subspaces of $W$ (necessarily of the second kind). Let $W_i'$ be chosen so that $T(\Lambda^r W_i) = \Lambda^r W_i'$. It follows easily that $\{W_1', \ldots, W_{k+1}'\}$ is of the second kind also, so that the image of $\Lambda^r W$ is $\Lambda^r W'$ where $W'$ is the $(k+1)$-dimensional subspace of $U$ containing $W_1', \ldots, W_{k+1}'$. By taking $W = U$ we see that $T$ is one-to-one on $\Lambda^r U$.

3. It is necessary to investigate whether a general $T$ does necessarily send each element of $B_r$ into $B_r$. For the cases $n > 2r, n < 2r$, this is proved directly, using Lemma 2. The case $n = 2r$ requires a more delicate argument, given at the end of this section; there it is shown that if some element of $B_r$ is sent into $B_r$ by $T$, then $T$ sends $B_r$ into $B_r$.

**Lemma 2.** Let $r < n$ and let $V_1$ and $V_2$ be in $A_r$ such that $V_1 \cap V_2 \neq \{0\}$. Then, if $V \subseteq V_1 + V_2$ and $\dim(V) = n - r$, we have $V \cap G_{sr} \neq \phi$. 
Proof. Let \( U_i \) be the \((r - 1)\)-dimensional subspace of \( U \) determined by \( V_i \) for \( i = 1, 2 \). Since \( V_1 \cap V_2 \neq \emptyset \), either \( U_1 = U_2 \) or \( \dim (U_1 \cap U_2) = r - 2 \).

If \( U_1 = U_2 \) then \( V_1 = V_2 \), so that in this case it is clear that \( V \cap G_{nr} \neq \emptyset \).

Suppose that \( \dim (U_1 \cap U_2) = r - 2 \) and let \( \{x_1, \ldots, x_{r-2}\} \) be a basis of this intersection. Choose \( y_i \) such that \( U_i = \langle x_1, \ldots, x_{r-2}, y_i \rangle \) for \( i = 1, 2 \). Choose \( u_i \) and \( v_i \) in \( U \), \( i = 1, \ldots, n - r \), such that

\[
\{z_i = x_1 \land \cdots \land x_{r-2} \land (y_1 \land u_i + y_2 \land v_i) \mid i = 1, \ldots, n - r\}
\]

forms a basis of \( V \). If

\[
\{x_1, \ldots, x_{r-2}, y_1, y_2, v_1, \ldots, v_{n-r}\} \quad \text{or} \quad \{x_1, \ldots, x_{r-2}, y_1, y_2, u_1, \ldots, u_{n-r}\}
\]

is dependent, then there is a linear combination of the \( z_i \) which is in \( V_1 \) or \( V_2 \) respectively. If, on the other hand, both sets are independent, then they are both bases for \( U \) and we may write

\[
u_i = w_i + c_i y_2 + \sum_{j=1}^{n-r} a_{ij} v_j, \quad i = 1, \ldots, n - r,
\]

where \( w_i \in \langle x_1, \ldots, x_{r-2}, y_1 \rangle \) and \( c_i, a_{ij} \in F \). We note that \( \det (a_{ij}) \neq 0 \) so we can choose \( \lambda \neq 0 \) and \( b_j \) for \( j = 1, \ldots, n - r \), not all zero, such that \( \lambda b_j = \sum_{i=1}^{n-r} b_i a_{ij} \). Then

\[
0 \neq \sum_{j=1}^{n-r} b_j z_j = x_1 \land \cdots \land x_{r-2} \land (y_1 + \lambda^{-1} y_2) \land \left[ \sum_{j=1}^{n-r} b_j c_j \right] y_2 + \lambda \left[ \sum_{j=1}^{n-r} b_j v_j \right]
\]

is an element of \( V \cap G_{nr} \). This proves the Lemma.

For \( n < 2r \) the image under \( T \) of an element of \( B_r \) is an element of \( B_r \). For \( n < 2r \) this is clearly so since the subspaces of \( \bigwedge^r U \) in \( B_r \) have dimension \( r + 1 \), which is greater than the dimension \( (n - r + 1) \) of the subspaces in \( A_r \).

For \( n > 2r \) we proceed as follows. The image of an \( A_r \) is an \( A_r \). Suppose that the image of a \( W \in B_r \) is a subspace of a \( V \in A_r \). Choose two elements \( V_1 \) and \( V_2 \) of \( A_r \) such that \( V_1 \cap V_2 \neq \emptyset \) and \( \dim (V_1 \cap W) = \dim (V_2 \cap W) = 2 \). One does this by choosing \( V_1 \) and \( V_2 \) so that the \((r - 1)\)-dimensional subspaces of \( U \) determined by them are adjacent subspaces of the \((r + 1)\)-dimensional subspace determined by \( W \). Now, \( T(V_i) = T(V_i) = V \) since each is in \( A_r \) and each intersects \( V \) in at least two dimensions. Therefore \( T(V_1 + V_2) = V \) and so the null space of \( T \) in \( V_1 + V_2 \) has dimension equal to \((2n - 2r + 1) - (n - r + 1) = n - r \). By Lemma 2, it follows that the null space of \( T \) intersects \( G_{nr} \) which contradicts the hypothesis that \( T \) sends \( G_{nr} \) into \( G_{nr} \).
In the case that \( n = 2r \) the image of a \( B_r \) may be an \( A_r \) since the dimensions are equal. However, we prove that if some \( B_r \) is sent into a \( B_r \) by \( T \), then the image of each \( B_r \) is a \( B_r \). Suppose not. Then we can choose \((r + 1)\)-dimensional subspaces \( W_1 \) and \( W_2 \) of \( U \) such that \( T(\Lambda^r W_1) \in A_r \) and \( T(\Lambda^r W_2) \in B_r \). Furthermore, we can choose \( W_1 \) and \( W_2 \) adjacent, so that \( \dim(W_1 \cap W_2) = r \). Choose three distinct elements \( V_1, V_2, \) and \( V_3 \) of \( A_r \) such that the \((r - 1)\)-dimensional subspaces of \( U \) determined by these elements are contained in \( W_1 \cap W_2 \). Then \( \dim(V_i \cap \Lambda^r W_j) = 2 \) for \( i = 1, 2, 3 \) and \( j = 1, 2 \), so that \( T(V_i) \) intersects \( T(\Lambda^r W_j) \) in at least two dimensions for each \( i, j \). This implies that each \( T(V_i) \) is equal to one of \( T(\Lambda^r W_j) \) and so two of them are equal. The argument of the previous paragraph now leads to a contradiction.

4. By essentially the same argument as used by Chow in [1] to prove his Theorem 1, we can prove that; if \( S \) is a nonsingular linear transformation of \( \Lambda^r U \) sending \( G_{nr} \) into \( G_{nr} \), and if the image of each \( B_r \) is a \( B_r \), then \( S \) is a compound. (By a compound we mean a linear transformation of \( \Lambda^r U \) which is induced by a linear transformation of \( U \).)

In the case that \( n \neq 2r \) it follows that \( T \) is necessarily a compound. For \( n = 2r \), \( T \) is a compound if some \( B_r \) is sent into a \( B_r \). If we let \( T_0 \) denote a linear transformation of \( \Lambda^r U \) induced by a correlation of the \( r \)-dimensional subspaces of \( U \), then \( T_0 \) is nonsingular and sends \( G_{nr} \) onto \( G_{nr} \). The image of each \( A_r \), under \( T_0 \) is a \( B_r \). Therefore, if a \( B_r \) is sent by \( T \) into an \( A_r \), the \( T_0 T \) is a compound. We have proved the

**Theorem.** Let \( U \) be an \( n \)-dimensional vector space over an algebraically closed field and let \( T \) be a linear transformation of \( \Lambda^r U \) which sends \( G_{nr} \) into \( G_{nr} \). Then \( T \) is a compound except, possibly, when \( n = 2r \), in which case \( T \) may be the composite of a compound and a linear transformation induced by a correlation of the \( r \)-dimensional subspaces of \( U \).

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