Pacific Journal of Mathematics

# A PROOF OF THE NAKAOKA-TODA FORMULA

KEITH A. HARDIE

Vol. 14, No. 4 August 1964

# A PROOF OF THE NAKAOKA-TODA FORMULA

K. A. HARDIE

If $X_j$ $(1 \leqq j \leqq r)$ are objects we denote the corresponding $r$-tuple $(X_1, X_2, \cdots, X_r)$ by $X$ and the $(r-1)$-tuple $(X_1, X_2, \cdots, X_{i-1}, X_{i+1}, \cdots, X_r)$ by $X(i)$. When $X_j$ $(1 \leqq j \leqq r)$ are based topological spaces $\Pi X$ will denote their topological product and $\Pi^i X$ the subspace of $\Pi X$ whose points have at least $i$ coordinates at base points (always denote by $*$).

Let $\alpha_j \in \pi_{n_j}(X_j)$ $(n_j \geqq 2, 1 \leqq j \leqq r, r \geqq 3)$ be elements of homotopy groups then we have

$$\star\alpha(\text{say}) = \alpha_1 \star \alpha_2 \star \cdots \star \alpha_r \in \pi_n(\Pi X, \Pi^1 X) ,$$

where $n = \Sigma n_j$ and $\star$ denotes the product of Blakers and Massey [1]. We thus also have

$$\star\alpha(i) \in \pi_{n-n_i}(\Pi X(i), \Pi^1 X(i)) .$$

There is a natural map $\Pi X(i), \Pi^1 X(i) \to \Pi^1 X, \Pi^2 X$ and we denote also by $\star\alpha(j)$ its image induced in $\pi_{n-n_i}(\Pi^1 X, \Pi^2 X)$. Let $\partial$ denote the homotopy boundary homomorphism in the exact sequence of the triple $(\Pi X, \Pi^1 X, \Pi^2 X)$. We shall prove the formula:

$$\partial\star\alpha = \Sigma(1 \leqq i \leqq r)(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)] \in \pi_{n-1}(\Pi^1 X, \Pi^2 X) , \qquad (0.1)$$

where $\varepsilon(1) = 0$, $\varepsilon(i) = n_i(n_1 + n_2 + \cdots + n_{i-1})$ $(i > 1)$ and where the brackets refer to the generalised Whitehead product of Blakers and Massey [1]. In the case of the universal example 0.1 becomes the formula of Nakaoka and Toda stated in [4] and proved there for $r = 3$. I. M. James[1] has raised the question of its validity for $r > 3$ and as the formula has applications (see [2], [3]) it would seem desirable to have a proof available in the literature. The present argument while inspired by [4] has a few novel features.

(1) DEFINITIONS AND LEMMAS. Let $x = (x_1, x_2, \cdots, x_n)$ denote a point of $n$-dimensional Euclidean space and let

$$\begin{aligned}
V^n &= \{x; \Sigma x_i^2 \leqq 1\} , \\
S^{n-1} &= \{x; \Sigma x_i^2 = 1\} , \\
E_+^{n-1} &= \{x \in S^n; x_n \geqq 0\} , \\
E_-^{n-1} &= \{x \in S^n; x_n \leqq 0\} , \\
D_+^n &= \{x \in V^n; x_n \geqq 0\} ,
\end{aligned}$$

Received November 12, 1963.

1 Math. Reviews 25 # 4521.

$$D_-^n = \{x \in V^n; x_n \leqq 0\} ,$$
$$D_1^n = \{x \in V^n; x_1 \geqq 0\} ,$$
$$D_2^n = \{x \in V^n; x_1 \leqq 0\} .$$

We recall that if $Y \subseteqq X$ then $X$ is a closed *n-cell* and $Y$ is a *face* of $X$ if there exists a homeomorphism $f: V^n \to X$ such that $f(E_+^{n-1}) = Y$. The subset $X^0 = f(S^{n-1})$ is the *boundary* of $X$. If $X$ and $Y$ are oriented cells we assign to $X \times Y$ the cross-product of the orientations of $X$ and $Y$.

LEMMA 1.1. *Let $X_1$ be a face of the cell $X$ and $Y_1$ a face of the cell $Y$. Then*

$$(X_1 \times Y) \cup (X \times Y_1) \text{ is a face of } X \times Y .$$

A proof of 1.1 may be found in [1] to which the reader may also refer for details concerning orientations. The proofs of the following two lemmas are standard exercises in homotopy theory and will be omitted.

LEMMA 1.2. *Suppose given a simplicial decomposition of a closed $n$-cell $F (n \geqq 3)$ and a subcomplex $G$ which is a closed $n$-cell oriented coherently with $F$. If $A$ is a simply-connected subset of a space $Y$ and if $f: F \to Y$ is a map such that $f\{(F - G) \cup G^\circ\} \subseteqq A$ then $f: F, F^\circ \to Y, A$ and $f: G, G^\circ \to Y, A$ represent the same element of $\pi_n(Y, A)$.*

LEMMA 1.3. *Suppose given a simplicial decomposition of $V^{n+1}(n \geqq 3)$ and subcomplexes $F_i (i = 1, 2, \cdots, m)$ which are faces of $V^{n+1}$ with disjoint interiors oriented coherently with $S^n$. Let $A$ be a simply-connected subset of a simply-connected space $Y$, let $f: S^n \to Y$ be a map such that $f\{(S^n - \cup F_i) \cup (\cup F_i^\circ)\} \subseteqq A$, let $f: S^n \to Y$ represent $\alpha \in \pi_n(Y)$ and let $f: F_i, F_i^\circ \to Y, A$ represent $\alpha_i \in \pi_n(Y, A)$ $(i = 1, 2, \cdots, m)$. Then $j\alpha = \Sigma \alpha_i$ where $j: \pi_n(Y) \to \pi_n(Y, A)$ is the injection homomorphism.*

Let $A$ be a simply-connected subset of a space $Y$. Let $f: V^p, S^{p-1} \to A, *$ and $g: V^q, S^{q-1}, E_+^{q-1} \to Y, A, *$ be representatives of $\alpha \in \pi_p(A)$ and $\beta \in \pi_q(Y, A)$. Let

$$h: S^{p-1} \times V^q \cup V^p \times E_+^{q-1}, S^{p-1} \times E_-^{q-1} \cup V^p \times S^{q-2} \to Y, A$$

be the map such that

$$h(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in V^p \times E_+^{q-1} , \\ g(y) & \text{if } (x, y) \in S^{p-1} \times V^q . \end{cases}$$

Then if $S^{p-1} \times V^p \cup V^q \times E_+^{q-1}$ is oriented coherently with $V^p \times V^q$ we recall 3.1 of [1]:

DEFINITION 1.4. $h$ represents $[\alpha, \beta] \in \pi_{p+q-1}(Y, A)$.

(2) *Proof of* 0.1. Let $\alpha_i$ be represented by a map

$$\psi_i : V^{n_i}, S^{n_i-1} \to X_i, *$$

with the property that

$$\psi_i(D_+^{n_i} \cup D_2^{n_i}) = * . \tag{2.1}$$

If we denote $V^{n_1} \times V^{n_2} \times \cdots \times V^{n_r}$ by $V$ and $V^{n_1} \times V^{n_{i-1}} \times V^{n_{i+1}} \times \cdots \times V^{n_r}$ by $V(i)$ then $\star\alpha$ and $\star\alpha(i)$ are represented by maps

$$\psi : V, V^\circ \to \Pi X, \Pi^1 X ,$$
$$\psi(i) : V(i), V(i)^\circ \to \Pi^1 X, \Pi^2 X$$

such that

$$\psi(x_1, \cdots, x_r) = (\psi_1(x_1), \cdots, \psi_r(x_r)) ,$$
$$\psi(i)(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r) = (\psi_1(x_1), \cdots, \psi_{i-1}(x_{i-1}), *, \psi_{i+1}(x_{i+1}), \cdots, \psi_r(x_r)) \quad (x_i \in V^{n_i}) . \tag{2.2}$$

Let $\rho_i : V^{n_i} \times V(i) \to V$ be the map such that

$$\rho_i(x_i, (x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_r)) = (x_1, x_2, \cdots, x_r) .$$

As an easy consequence of our orientation convention we obtain:

LEMMA 2.3. *The degree of* $\rho_i$ is $(-1)^{\varepsilon(i)}$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | — |
| 1 | 1 | 0 | 1 | 1 | 1 | — | — |
| 1 | 1 | 1 | 0 | 1 | — | — | — |
| 1 | 1 | — | 1 | 0 | — | — | — |
| 1 | 1 | — | — | — | 0 | — | — |
| 1 | — | — | — | — | — | 0 | — |
| — | — | — | — | — | — | — | 0 |

$r$ even

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | — |
| 1 | 1 | 0 | 1 | 1 | — | — |
| 1 | 1 | 1 | 0 | — | — | — |
| 1 | 1 | — | — | 0 | — | — |
| 1 | — | — | — | — | 0 | — |
| — | — | — | — | — | — | 0 |

$r$ odd

The proof of 0.1 depends on the construction of certain closed cells $G_i \subseteqq V(i)$ $(1 \leqq i \leqq r)$. Consider the two infinite arrays illustrated in the diagram. They contain between them exactly one centrally situated $r \times r$ matrix. Let $\eta(i, k, r)$ denote the symbol in the $(i, k)$ position of this matrix. We define

$$G_i = \Pi D^{n_k}_{\eta(i,k,r)} ,$$

where topological product $\Pi$ is taken over all values of $k$ (in ascending order) except those for which $\eta(i, k, r) = 0$.

EXAMPLES If $r = 5$ then $G_2 = D^{n_1}_1 \times D^{n_3}_1 \times D^{n_4}_1 \times D^{n_5}_-$.
If $r = 6$ then $G_4 = D^{n_1}_1 \times D^{n_2}_- \times D^{n_3}_1 \times D^{n_5}_- \times D^{n_6}_-$.

Certainly $G_i \subseteqq V(i)$. We shall refer later to the following property of the $G_i$ which is obvious from the diagram.

LEMMA 2.4. *If $i < j \leqq r$ then there is an integer $k$ with $i \neq k \neq j$ such that $G_i$ has a factor $D^{n_k}_1$ and $G_j$ a factor $D^{n_k}_-$.*

The proof of the following lemma we postpone.

LEMMA 2.5. *For each $i = 1, 2, \cdots, r$, there exists a face $\tau_i$ of $G_i$ and of $V(i)$ such that if $G_i$ has a factor $D^{n_k}_1$ then the projection of $\tau_i$ on $D^{n_k}_1$ does not intersect $D^{n_k}_-$ and such that if $G_i$ has a factor $D^{n_k}_-$ then the projection of $\tau_i$ on $D^{n_k}_-$ does not intersect $D^{n_k}_1$.*

In view of 2.1 and 2.5 we have $\psi(i)(\tau_i) = *$. Moreover 2.1 and 2.2 imply that

$$\psi(i)\{(V(i) - G_i) \cup G_i^\circ\} \subseteqq \Pi^2 X .$$

Thus applying 1.2 (we may assume $\Pi^2 X$ simply-connected for this is certainly so in the case of the universal example) we obtain that

$$(\psi(i) \mid G_i) : G_i, G_i^\circ, \tau_i \to \Pi^1 X, \Pi^2 X, * \quad \text{represents } \star\alpha(i) . \tag{2.6}$$

We now define

$$F_i = \rho_i(S^{n_i - 1} \times G_i \cup V^{n_i} \times \tau_i) \qquad (1 \leqq i \leqq r)$$

and prove later:

LEMMA 2.7. *The $F_i$ are faces of $V$ with disjoint interiors. The map $(\psi\rho_i \mid \rho_i^{-1} F_i)$ has the property that*

$$(\psi\rho_i \mid \rho_i^{-1} F_i)(x, y) = \begin{cases} \psi_i(x) & \text{if } (x, y) \in V^{n_i} \times \tau_i , \\ \psi(i)(y) & \text{if } (x, y) \in S^{n_i - 1} \times G_i . \end{cases}$$

If we orient $F_i$ coherently with $V$ and $\rho_i^{-1} F_i$ coherently with $V^{n_i} \times V(i)$, 1.4 implies that $(\psi\rho_i \mid \rho_i^{-1}F_i)$ represents $[\alpha_i, \star\alpha(i)]$.

Since $\rho_i$ is of degree $(-1)^{\varepsilon(i)}$, $(\psi \mid F_i)$ represents $(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)]$ and hence applying 1.3 the formula 0.1 follows in view of the commutativity in the diagram

$$\begin{array}{ccc} \pi_n(\Pi X, \Pi^1 X) & \xrightarrow{\partial} & \pi_{n-1}(\Pi^1 X, \Pi^2 X) \\ \Big\downarrow d & \nearrow j & \\ \pi_{n-1}(\Pi^1 X) & & \end{array}$$

where $d$ denotes the boundary homomorphism in the homotopy sequence of the pair $(\Pi X, \Pi^1 X)$.

*Proof of* 2.5. Let $D_0^n$ and $D_\times^n$ denote the subsets

$$D_0^n = \left\{x \in V^n;\, x_1 \geqq \frac{1}{2} \text{ and } x_n \geqq \frac{1}{2}\right\},$$

$$D_\times^n = \left\{x \in V^n;\, x_1 \leqq \frac{1}{2} \text{ and } x_n \leqq \frac{1}{2}\right\}.$$

Let $D \subseteqq G_i$ have a factor $D_0^{n_k}$ for every factor $D_1^{n_k}$ of $G_i$ and a factor $D_\times^{n_k}$ for every factor $D_-^{n_k}$ of $G_i$. Then certainly $\tau_i = D \cap V(i)^\circ$ has the desired property.

*Proof of* 2.7. If $\sigma_i$ is the face of $G_i$ complementary to $\tau_i$ then it may be observed that $F_i$ is the face of $\rho_i(V^{n_i} \times G_i)$ complementary to $\rho_i(V^{n_i} \times \sigma_i)$. Thus

$$F_i^\circ = \rho_i(S^{n_i-1} \times \sigma_i \cup V^{n_i} \times \tau_i^\circ).$$

Suppose $i < j$ and let

$$H = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(S^{n_j-1} \times G_j),$$
$$H' = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(V^{n_j} \times \tau_j),$$
$$H'' = \rho_i(V^{n_i} \times \tau_j) \cap \rho_j(S^{n_j-1} \times G_j).$$

2.7 will follow when we have proved that $H \subseteqq F_i^\circ \cap F_j^\circ$, $H' = \emptyset$ and $H'' = \emptyset$. Since the images of $H$ under the projections into $V^{n_i}$ and $V^{n_j}$ are contained in $S^{n_i-1}$ and $S^{n_j-1}$ respectively we have

$$H \subseteqq \rho_i(S^{n_i-1} \times G_i^\circ) \cap \rho_i(S^{n_j-1} \times G_j^\circ).$$

2.4 asserts the existence of an integer $k$ with $i \neq k \neq j$ such that $G_i$ has a factor $D_1^{n_k}$ and $G_j$ a factor $D_-^{n_k}$. Hence 2.5 implies that

$$H \cap \rho_i(S^{n_i-1} \times \tau_i) = H \cap \rho_j(S^{n_j-1} \times \tau_j) = \emptyset$$

and hence that

$$H \subseteqq \rho_i(S^{n_i-1} \times (G_i^\circ - \tau_i)) \cap \rho_j(S^{n_j-1} \times (G_j^\circ - \tau_j)) \subseteqq F_i^\circ \cap F_j^\circ .$$

2.5 also implies that $H' = H'' = \emptyset$ which completes the proof of 2.6.

## REFERENCES


1. A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Ann. of Math., **58** (1953), 295-324.
2. K. A. Hardie, *On a construction of E. C. Zeeman*, J. London Math. Soc., **35** (1960), 452-64.
3. ———, *Higher Whitehead products*, Quart. J. Math. Oxford (2), **12** (1961), 241-9.
4. M. Nakaoka and H. Toda, *On Jacobi identity for Whitehead products*, J. Inst. Poly tech. Osaka City Univ. A **5** (1954), 1-13.

# PACIFIC JOURNAL OF MATHEMATICS




Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should by typewritten (double spaced), and on submission, must be accompanied by a separate author's résumé. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal but they are not owners or publishers and have no responsibility for its content or policies.

# Pacific Journal of Mathematics

Vol. 14, No. 4 August, 1964