

Pacific Journal of Mathematics

A PROOF OF THE NAKAOKA-TODA FORMULA

KEITH A. HARDIE

A PROOF OF THE NAKAOKA-TODA FORMULA

K. A. HARDIE

If X_j ($1 \leq j \leq r$) are objects we denote the corresponding r -tuple (X_1, X_2, \dots, X_r) by X and the $(r-1)$ -tuple $(X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_r)$ by $X(i)$. When X_j ($1 \leq j \leq r$) are based topological spaces ΠX will denote their topological product and $\Pi^i X$ the subspace of ΠX whose points have at least i coordinates at base points (always denote by $*$).

Let $\alpha_j \in \pi_{n_j}(X_j)$ ($n_j \geq 2, 1 \leq j \leq r, r \geq 3$) be elements of homotopy groups then we have

$$\star\alpha(\text{say}) = \alpha_1 \star \alpha_2 \star \dots \star \alpha_r \in \pi_n(\Pi X, \Pi^1 X),$$

where $n = \sum n_j$ and \star denotes the product of Blakers and Massey [1]. We thus also have

$$\star\alpha(i) \in \pi_{n-n_i}(\Pi X(i), \Pi^1 X(i)).$$

There is a natural map $\Pi X(i), \Pi^1 X(i) \rightarrow \Pi^1 X, \Pi^2 X$ and we denote also by $\star\alpha(j)$ its image induced in $\pi_{n-n_i}(\Pi^1 X, \Pi^2 X)$. Let ∂ denote the homotopy boundary homomorphism in the exact sequence of the triple $(\Pi X, \Pi^1 X, \Pi^2 X)$. We shall prove the formula:

$$\partial \star\alpha = \sum (1 \leq i \leq r) (-1)^{\varepsilon(i)} [\alpha_i, \star\alpha(i)] \in \pi_{n-1}(\Pi^1 X, \Pi^2 X), \quad (0.1)$$

where $\varepsilon(1) = 0$, $\varepsilon(i) = n_i(n_1 + n_2 + \dots + n_{i-1})$ ($i > 1$) and where the brackets refer to the generalised Whitehead product of Blakers and Massey [1]. In the case of the universal example 0.1 becomes the formula of Nakaoka and Toda stated in [4] and proved there for $r = 3$. I. M. James¹ has raised the question of its validity for $r > 3$ and as the formula has applications (see [2], [3]) it would seem desirable to have a proof available in the literature. The present argument while inspired by [4] has a few novel features.

(1) DEFINITIONS AND LEMMAS. Let $x = (x_1, x_2, \dots, x_n)$ denote a point of n -dimensional Euclidean space and let

$$\begin{aligned} V^n &= \{x; \sum x_i^2 \leq 1\}, \\ S^{n-1} &= \{x; \sum x_i^2 = 1\}, \\ E_+^{n-1} &= \{x \in S^n; x_n \geq 0\}, \\ E_-^{n-1} &= \{x \in S^n; x_n \leq 0\}, \\ D_+^n &= \{x \in V^n; x_n \geq 0\}, \end{aligned}$$

Received November 12, 1963.

¹ Math. Reviews 25 # 4521.

$$\begin{aligned}
 D_-^n &= \{x \in V^n; x_n \leq 0\}, \\
 D_1^n &= \{x \in V^n; x_1 \geq 0\}, \\
 D_2^n &= \{x \in V^n; x_1 \leq 0\}.
 \end{aligned}$$

We recall that if $Y \subseteq X$ then X is a closed n -cell and Y is a face of X if there exists a homeomorphism $f: V^n \rightarrow X$ such that $f(E_+^{n-1}) = Y$. The subset $X^0 = f(S^{n-1})$ is the boundary of X . If X and Y are oriented cells we assign to $X \times Y$ the cross-product of the orientations of X and Y .

LEMMA 1.1. *Let X_1 be a face of the cell X and Y_1 a face of the cell Y . Then*

$$(X_1 \times Y) \cup (X \times Y_1) \text{ is a face of } X \times Y.$$

A proof of 1.1 may be found in [1] to which the reader may also refer for details concerning orientations. The proofs of the following two lemmas are standard exercises in homotopy theory and will be omitted.

LEMMA 1.2. *Suppose given a simplicial decomposition of a closed n -cell $F(n \geq 3)$ and a subcomplex G which is a closed n -cell oriented coherently with F . If A is a simply-connected subset of a space Y and if $f: F \rightarrow Y$ is a map such that $f\{(F - G) \cup G^\circ\} \subseteq A$ then $f: F, F^\circ \rightarrow Y, A$ and $f: G, G^\circ \rightarrow Y, A$ represent the same element of $\pi_n(Y, A)$.*

LEMMA 1.3. *Suppose given a simplicial decomposition of $V^{n+1}(n \geq 3)$ and subcomplexes $F_i(i = 1, 2, \dots, m)$ which are faces of V^{n+1} with disjoint interiors oriented coherently with S^n . Let A be a simply-connected subset of a simply-connected space Y , let $f: S^n \rightarrow Y$ be a map such that $f\{(S^n - \cup F_i) \cup (\cup F_i^\circ)\} \subseteq A$, let $f: S^n \rightarrow Y$ represent $\alpha \in \pi_n(Y)$ and let $f: F_i, F_i^\circ \rightarrow Y, A$ represent $\alpha_i \in \pi_n(Y, A)$ ($i = 1, 2, \dots, m$). Then $j\alpha = \sum \alpha_i$ where $j: \pi_n(Y) \rightarrow \pi_n(Y, A)$ is the injection homomorphism.*

Let A be a simply-connected subset of a space Y . Let $f: V^p, S^{p-1} \rightarrow A, *$ and $g: V^q, S^{q-1}, E_+^{q-1} \rightarrow Y, A, *$ be representatives of $\alpha \in \pi_p(A)$ and $\beta \in \pi_q(Y, A)$. Let

$$h: S^{p-1} \times V^q \cup V^p \times E_+^{q-1}, S^{p-1} \times E_+^{q-1} \cup V^p \times S^{q-2} \rightarrow Y, A$$

be the map such that

$$h(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in V^p \times E_+^{q-1}, \\ g(y) & \text{if } (x, y) \in S^{p-1} \times V^q. \end{cases}$$

Then if $S^{p-1} \times V^p \cup V^q \times E_+^{q-1}$ is oriented coherently with $V^p \times V^q$ we recall 3.1 of [1]:

DEFINITION 1.4. h represents $[\alpha, \beta] \in \pi_{p+q-1}(Y, A)$.

(2) Proof of 0.1. Let α_i be represented by a map

$$\psi_i : V^{n_i}, S^{n_i-1} \rightarrow X_i, *$$

with the property that

$$(2.1) \quad \psi_i(D_+^{n_i} \cup D_2^{n_i}) = * .$$

If we denote $V^{n_1} \times V^{n_2} \times \dots \times V^{n_r}$ by V and $V^{n_1} \times V^{n_{i-1}} \times V^{n_{i+1}} \times \dots \times V^{n_r}$ by $V(i)$ then $\star\alpha$ and $\star\alpha(i)$ are represented by maps

$$\begin{aligned} \psi &: V, V^\circ \rightarrow \Pi X, \Pi^1 X, \\ \psi(i) &: V(i), V(i)^\circ \rightarrow \Pi^1 X, \Pi^2 X \end{aligned}$$

such that

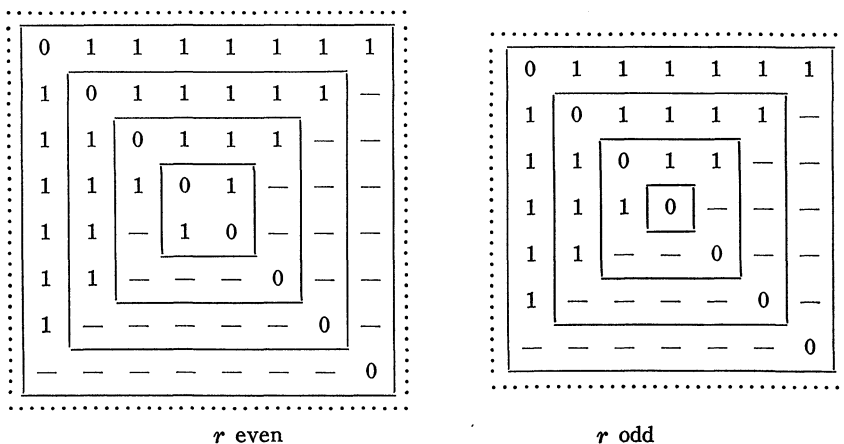
$$(2.2) \quad \begin{aligned} \psi(x_1, \dots, x_r) &= (\psi_1(x_1), \dots, \psi_r(x_r)) , \\ \psi(i)(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r) &= (\psi_1(x_1), \dots, \psi_{i-1}(x_{i-1}), *, \psi_{i+1}(x_{i+1}), \dots, \psi_r(x_r)) \quad (x_i \in V^{n_i}) . \end{aligned}$$

Let $\rho_i : V^{n_i} \times V(i) \rightarrow V$ be the map such that

$$\rho_i(x_i, (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r)) = (x_1, x_2, \dots, x_r) .$$

As an easy consequence of our orientation convention we obtain:

LEMMA 2.3. The degree of ρ_i is $(-1)^{\varepsilon(i)}$.



The proof of 0.1 depends on the construction of certain closed cells $G_i \cong V(i)$ ($1 \leq i \leq r$). Consider the two infinite arrays illustrated

in the diagram. They contain between them exactly one centrally situated $r \times r$ matrix. Let $\eta(i, k, r)$ denote the symbol in the (i, k) position of this matrix. We define

$$G_i = \prod D_{\eta(i,k,r)}^{n_k},$$

where topological product \prod is taken over all values of k (in ascending order) except those for which $\eta(i, k, r) = 0$.

EXAMPLES If $r = 5$ then $G_2 = D_1^{n_1} \times D_1^{n_3} \times D_1^{n_4} \times D_2^{n_5}$.

If $r = 6$ then $G_4 = D_1^{n_1} \times D_2^{n_2} \times D_1^{n_3} \times D_2^{n_5} \times D_2^{n_6}$.

Certainly $G_i \subseteq V(i)$. We shall refer later to the following property of the G_i which is obvious from the diagram.

LEMMA 2.4. *If $i < j \leq r$ then there is an integer k with $i \neq k \neq j$ such that G_i has a factor $D_1^{n_k}$ and G_j a factor $D_-^{n_k}$.*

The proof of the following lemma we postpone.

LEMMA 2.5. *For each $i = 1, 2, \dots, r$, there exists a face τ_i of G_i and of $V(i)$ such that if G_i has a factor $D_1^{n_k}$ then the projection of τ_i on $D_1^{n_k}$ does not intersect $D_-^{n_k}$ and such that if G_i has a factor $D_-^{n_k}$ then the projection of τ_i on $D_-^{n_k}$ does not intersect $D_1^{n_k}$.*

In view of 2.1 and 2.5 we have $\psi(i)(\tau_i) = *$. Moreover 2.1 and 2.2 imply that

$$\psi(i)\{(V(i) - G_i) \cup G_i^\circ\} \subseteq \Pi^2 X.$$

Thus applying 1.2 (we may assume $\Pi^2 X$ simply-connected for this is certainly so in the case of the universal example) we obtain that

$$(2.6) \quad (\psi(i) | G_i) : G_i, G_i^\circ, \tau_i \rightarrow \Pi^1 X, \Pi^2 X, *$$

represents $\star\alpha(i)$.

We now define

$$F_i = \rho_i(S^{n_i-1} \times G_i \cup V^{n_i} \times \tau_i) \quad (1 \leq i \leq r)$$

and prove later:

LEMMA 2.7. *The F_i are faces of V with disjoint interiors. The map $(\psi\rho_i | \rho_i^{-1}F_i)$ has the property that*

$$(\psi\rho_i | \rho_i^{-1}F_i)(x, y) = \begin{cases} \psi_i(x) & \text{if } (x, y) \in V^{n_i} \times \tau_i, \\ \psi(i)(y) & \text{if } (x, y) \in S^{n_i-1} \times G_i. \end{cases}$$

If we orient F_i coherently with V and $\rho_i^{-1}F_i$ coherently with $V^{n_i} \times V(i)$,

1.4 implies that $(\psi\rho_i | \rho_i^{-1}F_i)$ represents $[\alpha_i, \star\alpha(i)]$.

Since ρ_i is of degree $(-1)^{\varepsilon(i)}$, $(\psi | F_i)$ represents $(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)]$ and hence applying 1.3 the formula 0.1 follows in view of the commutativity in the diagram

$$\begin{array}{ccc} \pi_n(\Pi X, \Pi^1 X) & \xrightarrow{\partial} & \pi_{n-1}(\Pi^1 X, \Pi^2 X) \\ \downarrow d & \nearrow j & \\ \pi_{n-1}(\Pi^1 X) & & \end{array}$$

where d denotes the boundary homomorphism in the homotopy sequence of the pair $(\Pi X, \Pi^1 X)$.

Proof of 2.5. Let D_0^n and D_{\times}^n denote the subsets

$$D_0^n = \left\{ x \in V^n; x_1 \geq \frac{1}{2} \text{ and } x_n \geq \frac{1}{2} \right\},$$

$$D_{\times}^n = \left\{ x \in V^n; x_1 \leq \frac{1}{2} \text{ and } x_n \leq \frac{1}{2} \right\}.$$

Let $D \subseteq G_i$ have a factor $D_0^{n_k}$ for every factor $D_1^{n_k}$ of G_i and a factor $D_{\times}^{n_k}$ for every factor $D_{-}^{n_k}$ of G_i . Then certainly $\tau_i = D \cap V(i)^{\circ}$ has the desired property.

Proof of 2.7. If σ_i is the face of G_i complementary to τ_i then it may be observed that F_i is the face of $\rho_i(V^{n_i} \times G_i)$ complementary to $\rho_i(V^{n_i} \times \sigma_i)$. Thus

$$F_i^{\circ} = \rho_i(S^{n_i-1} \times \sigma_i \cup V^{n_i} \times \tau_i^{\circ}).$$

Suppose $i < j$ and let

$$H = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(S^{n_j-1} \times G_j),$$

$$H' = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(V^{n_j} \times \tau_j),$$

$$H'' = \rho_i(V^{n_i} \times \tau_j) \cap \rho_j(S^{n_j-1} \times G_j).$$

2.7 will follow when we have proved that $H \subseteq F_i^{\circ} \cap F_j^{\circ}$, $H' = \emptyset$ and $H'' = \emptyset$. Since the images of H under the projections into V^{n_i} and V^{n_j} are contained in S^{n_i-1} and S^{n_j-1} respectively we have

$$H \subseteq \rho_i(S^{n_i-1} \times G_i^{\circ}) \cap \rho_j(S^{n_j-1} \times G_j^{\circ}).$$

2.4 asserts the existence of an integer k with $i \neq k \neq j$ such that G_i has a factor $D_1^{n_k}$ and G_j a factor $D_{-}^{n_k}$. Hence 2.5 implies that

$$H \cap \rho_i(S^{n_i-1} \times \tau_i) = H \cap \rho_j(S^{n_j-1} \times \tau_j) = \emptyset$$

and hence that

$$H \subseteq \rho_i(S^{n_i-1} \times (G_i^\circ - \tau_i)) \cap \rho_j(S^{n_j-1} \times (G_j^\circ - \tau_j)) \subseteq F_i^\circ \cap F_j^\circ .$$

2.5 also implies that $H' = H'' = \emptyset$ which completes the proof of 2.6.

REFERENCES

1. A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Ann. of Math., **58** (1953), 295-324.
2. K. A. Hardie, *On a construction of E. C. Zeeman*, J. London Math. Soc., **35** (1960), 452-64.
3. ———, *Higher Whitehead products*, Quart. J. Math. Oxford (2), **12** (1961), 241-9.
4. M. Nakaoka and H. Toda, *On Jacobi identity for Whitehead products*, J. Inst. Polytech. Osaka City Univ. A **5** (1954), 1-13.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

ROBERT OSSERMAN
Stanford University
Stanford, California

M. G. ARSOVE
University of Washington
Seattle 5, Washington

J. DUGUNDJI
University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and on submission, must be accompanied by a separate author's résumé. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 14, No. 4

August, 1964

Homer Franklin Bechtell, Jr., <i>Pseudo-Frattini subgroups</i>	1129
Thomas Kelman Boehme and Andrew Michael Bruckner, <i>Functions with convex means</i>	1137
Lutz Bungart, <i>Boundary kernel functions for domains on complex manifolds</i>	1151
L. Carlitz, <i>Rings of arithmetic functions</i>	1165
D. S. Carter, <i>Uniqueness of a class of steady plane gravity flows</i>	1173
Richard Albert Dean and Robert Harvey Oehmke, <i>Idempotent semigroups with distributive right congruence lattices</i>	1187
Lester Eli Dubins and David Amiel Freedman, <i>Measurable sets of measures</i>	1211
Robert Pertsch Gilbert, <i>On class of elliptic partial differential equations in four variables</i>	1223
Harry Gonshor, <i>On abstract affine near-rings</i>	1237
Edward Everett Grace, <i>Cut points in totally non-semi-locally-connected continua</i>	1241
Edward Everett Grace, <i>On local properties and G_δ sets</i>	1245
Keith A. Hardie, <i>A proof of the Nakaoka-Toda formula</i>	1249
Lowell A. Hinrichs, <i>Open ideals in $C(X)$</i>	1255
John Rolfe Isbell, <i>Natural sums and abelianizing</i>	1265
G. W. Kimble, <i>A characterization of extremals for general multiple integral problems</i>	1283
Nand Kishore, <i>A representation of the Bernoulli number B_n</i>	1297
Melven Robert Krom, <i>A decision procedure for a class of formulas of first order predicate calculus</i>	1305
Peter A. Lappan, <i>Identity and uniqueness theorems for automorphic functions</i>	1321
Lorraine Doris Lavalley, <i>Mosaics of metric continua and of quasi-Peano spaces</i>	1327
Mark Mahowald, <i>On the normal bundle of a manifold</i>	1335
J. D. McKnight, <i>Kleene quotient theorems</i>	1343
Charles Kimbrough Megibben, III, <i>On high subgroups</i>	1353
Philip Miles, <i>Derivations on B^* algebras</i>	1359
J. Marshall Osborn, <i>A generalization of power-associativity</i>	1367
Theodore G. Ostrom, <i>Nets with critical deficiency</i>	1381
Elvira Rapaport Strasser, <i>On the defining relations of a free product</i>	1389
K. Rogers, <i>A note on orthogonal Latin squares</i>	1395
P. P. Saworotnow, <i>On continuity of multiplication in a complemented algebra</i>	1399
Johanan Schonheim, <i>On coverings</i>	1405
Victor Lenard Shapiro, <i>Bounded generalized analytic functions on the torus</i>	1413
James D. Stafney, <i>Arens multiplication and convolution</i>	1423
Daniel Sterling, <i>Coverings of algebraic groups and Lie algebras of classical type</i>	1449
Alfred B. Willcox, <i>Šilov type C algebras over a connected locally compact abelian group. II</i>	1463
Bertram Yood, <i>Faithful $*$-representations of normed algebras. II</i>	1475
Alexander Zabrodsky, <i>Covering spaces of paracompact spaces</i>	1489