A PROOF OF THE NAKAOKA-TODA FORMULA

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If $X_j$ (1 $\leq j$ $\leq r$) are objects we denote the corresponding $r$-tuple $(X_1, X_2, \ldots, X_r)$ by $X$ and the $(r - 1)$-tuple $(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_r)$ by $X(i)$. When $X_j$ (1 $\leq j$ $\leq r$) are based topological spaces $\Pi X$ will denote their topological product and $\Pi^{i}X$ the subspace of $\Pi X$ whose points have at least $i$ coordinates at base points (always denote by $\ast$).

Let $\alpha_j \in \pi_{n_j}(X_j)$ ($n_j \geq 2, 1 \leq j \leq r, r \geq 3$) be elements of homotopy groups then we have

$$\ast\alpha(\text{say}) = \alpha_1\ast\alpha_2\ast\ldots\ast\alpha_r \in \pi_n(\Pi X, \Pi^1 X),$$

where $n = \sum n_j$ and $\ast$ denotes the product of Blakers and Massey [1].

We thus also have

$$\ast\alpha(i) \in \pi_{n-n_i}(\Pi X(i), \Pi^1 X(i)).$$

There is a natural map $\Pi X(i), \Pi^1 X(i) \to \Pi^2 X, \Pi^3 X$ and we denote also by $\ast\alpha(j)$ its image induced in $\pi_{n-n_i}(\Pi^1 X, \Pi^2 X)$. Let $\partial$ denote the homotopy boundary homomorphism in the exact sequence of the triple $(\Pi X, \Pi^1 X, \Pi^2 X)$. We shall prove the formula:

$$\partial*\alpha = \Sigma(1 \leq i \leq r)(-1)^{\varepsilon(i)}[\alpha_i, \ast\alpha(i)] \in \pi_{n-1}(\Pi^1 X, \Pi^2 X), \quad (0.1)$$

where $\varepsilon(1) = 0$, $\varepsilon(i) = n_i(n_i + n_{i-1} + \ldots + n_1)$ ($i > 1$) and where the brackets refer to the generalised Whitehead product of Blakers and Massey [1]. In the case of the universal example 0.1 becomes the formula of Nakaoka and Toda stated in [4] and proved there for $r = 3$. I. M. James¹ has raised the question of its validity for $r > 3$ and as the formula has applications (see [2], [3]) it would seem desirable to have a proof available in the literature. The present argument while inspired by [4] has a few novel features.

(1) DEFINITIONS AND LEMMAS. Let $x = (x_1, x_2, \ldots, x_n)$ denote a point of $n$-dimensional Euclidean space and let

$$V^n = \{x; \Sigma x_i^2 \leq 1\},$$

$$S^{n-1} = \{x; \Sigma x_i^2 = 1\},$$

$$E_+^{n-1} = \{x \in S^n; x_n \geq 0\},$$

$$E_-^{n-1} = \{x \in S^n; x_n \leq 0\},$$

$$D_+^n = \{x \in V^n; x_n \geq 0\},$$

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¹ Math. Reviews 25 # 4521.
We recall that if $Y \subseteq X$ then $X$ is a closed $n$-cell and $Y$ is a face of $X$ if there exists a homeomorphism $f : V^n \to X$ such that $f(E^{n-1}) = Y$. The subset $X^\circ = f(S^{n-1})$ is the boundary of $X$. If $X$ and $Y$ are oriented cells we assign to $X \times Y$ the cross-product of the orientations of $X$ and $Y$.

**Lemma 1.1.** Let $X_i$ be a face of the cell $X$ and $Y_i$ a face of the cell $Y$. Then

$$(X_i \times Y) \cup (X \times Y_i)$$

is a face of $X \times Y$.

A proof of 1.1 may be found in [1] to which the reader may also refer for details concerning orientations. The proofs of the following two lemmas are standard exercises in homotopy theory and will be omitted.

**Lemma 1.2.** Suppose given a simplicial decomposition of a closed $n$-cell $F(n \geq 3)$ and a subcomplex $G$ which is a closed $n$-cell oriented coherently with $F$. If $A$ is a simply-connected subset of a space $Y$ and if $f : F \to Y$ is a map such that $f((F - G) \cup G^\circ) \subseteq A$ then $f : F, F^\circ \to Y, A$ and $f : G, G^\circ \to Y, A$ represent the same element of $\pi_n(Y, A)$.

**Lemma 1.3.** Suppose given a simplicial decomposition of $V^{n+1}(n \geq 3)$ and subcomplexes $F_i (i = 1, 2, \ldots, m)$ which are faces of $V^{n+1}$ with disjoint interiors oriented coherently with $S^n$. Let $A$ be a simply-connected subset of a simply-connected space $Y$, let $f : S^n \to Y$ be a map such that $f((S^n - \bigcup F_i) \cup (\bigcup F_i^\circ)) \subseteq A$, let $f : S^n \to Y$ represent $\alpha \in \pi_n(Y)$ and let $f : F_i, F_i^\circ \to Y, A$ represent $\alpha_i \in \pi_n(Y, A)$ $(i = 1, 2, \ldots, m)$. Then $j\alpha = \Sigma \alpha_i$ where $j : \pi_n(Y) \to \pi_n(Y, A)$ is the injection homomorphism.

Let $A$ be a simply-connected subset of a space $Y$. Let $f : V^p, S^{p-1} \to A, -$ and $g : V^q, S^{q-1}, E^q_+ \to Y, A, -$ be representatives of $\alpha \in \pi_p(A)$ and $\beta \in \pi_q(Y, A)$. Let

$$h : S^{p-1} \times V^q \cup V^p \times E^q_+ \to Y, A$$

be the map such that

$$h(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in V^p \times E^q_+, \\ g(y) & \text{if } (x, y) \in S^{p-1} \times V^q. \end{cases}$$
Then if \( S^{n-1} \times V^p \cup V^q \times E_{+}^{q-1} \) is oriented coherently with \( V^p \times V^q \) we recall 3.1 of [1]:

**Definition 1.4.** \( h \) represents \([\alpha, \beta] \in \pi_{p+q-1}(Y, A)\).

(2) **Proof of 0.1.** Let \( \alpha \) be represented by a map

\[ \psi_i : V^{n_i}, S^{n_i-1} \to X_i, * \]

with the property that

(2.1) \[ \psi_i(D_{+}^{n_i} \cup D_{-}^{n_i}) = * . \]

If we denote \( V^{n_1} \times V^{n_2} \times \cdots \times V^{n_r} \) by \( V \) and \( V^{n_1} \times V^{n_1} \times \cdots \times V^{n_1} \times \cdots \times V^{n_r} \) by \( V(i) \) then \(*\alpha\) and \(*\alpha(i)\) are represented by maps

\[ \psi : V, V^o \to I(X), I(X), \]

\[ \psi(i) : V(i), V(i)^o \to I(X), I(X) \]

such that

\[ \psi(x_1, \cdots, x_r) = (\psi_1(x_1), \cdots, \psi_r(x_r)) , \]

(2.2) \[ \psi(i)(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r) \]

\[ = (\psi_1(x_1), \cdots, \psi_{i-1}(x_{i-1}), *, \psi_{i+1}(x_{i+1}), \cdots, \psi_r(x_r)) \ (x_i \in V^{n_i}) . \]

Let \( \rho_i : V^{n_i} \times V(i) \to V \) be the map such that

\[ \rho_i(x_i, (x_1, \cdots, x_{i-1}, x_{i-1}, \cdots, x_r)) = (x_1, x_2, \cdots, x_r) . \]

As an easy consequence of our orientation convention we obtain:

**Lemma 2.3.** The degree of \( \rho_i \) is \((-1)^{i(i)}\).

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & - \\
1 & 1 & 0 & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & 0 & 1 & - & - & - \\
1 & 1 & - & 1 & 0 & - & - & - \\
1 & 1 & - & - & 0 & - & - & - \\
1 & - & - & - & - & 0 & - & - \\
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & - \\
1 & 1 & 0 & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & 0 & 1 & - & - & - \\
1 & 1 & - & 1 & 0 & - & - & - \\
1 & 1 & - & - & 0 & - & - & - \\
1 & - & - & - & - & 0 & - & - \\
\end{array}
\]

\( r \) even \hspace{1cm} \( r \) odd

The proof of 0.1 depends on the construction of certain closed cells \( G_i \subseteq V(i) \ (1 \leq i \leq r) \). Consider the two infinite arrays illustrated
in the diagram. They contain between them exactly one centrally situated $r \times r$ matrix. Let $\gamma(i, k, r)$ denote the symbol in the $(i, k)$ position of this matrix. We define

$$G_i = \Pi D_{\gamma(i, k, r)}^k,$$

where topological product $\Pi$ is taken over all values of $k$ (in ascending order) except those for which $\gamma(i, k, r) = 0$.

**EXAMPLES** If $r = 5$ then $G_2 = D_1^{n1} \times D_1^{n3} \times D_1^{n4} \times D_1^{n5}$.

If $r = 6$ then $G_4 = D_1^{n1} \times D_1^{n3} \times D_1^{n5} \times D_1^{n6}$.

Certainly $G_i \subseteq V(i)$. We shall refer later to the following property of the $G_i$ which is obvious from the diagram.

**LEMMA 2.4.** If $i < j \leq r$ then there is an integer $k$ with $i \neq k \neq j$ such that $G_i$ has a factor $D_i^{s_k}$ and $G_j$ a factor $D_i^{s_k}$.

The proof of the following lemma we postpone.

**LEMMA 2.5.** For each $i = 1, 2, \ldots, r$, there exists a face $\tau_i$ of $G_i$ and of $V(i)$ such that if $G_i$ has a factor $D^{s_k}_i$ then the projection of $\tau_i$ on $D_i^{s_k}$ does not intersect $D_i^{s_k}$ and such that if $G_i$ has a factor $D_i^{s_k}$ then the projection of $\tau_i$ on $D_i^{s_k}$ does not intersect $D_i^{s_k}$.

In view of 2.1 and 2.5 we have $\psi(i)(\tau_i) = \ast$. Moreover 2.1 and 2.2 imply that

$$\psi(i)((V(i) - G_i) \cup G_i) \subseteq \Pi^2 X.$$

Thus applying 1.2 (we may assume $\Pi^2 X$ simply-connected for this is certainly so in the case of the universal example) we obtain that

$$(\psi(i) | G_i) : G_i, G_i^\circ, \tau_i \rightarrow \Pi^1 X, \Pi^2 X,*$$

represents $\ast \alpha(i)$.

We now define

$$F_i = \rho_i(S^{n_i - 1} \times G_i \cup V^{n_i} \times \tau_i) \quad (1 \leq i \leq r)$$

and prove later:

**LEMMA 2.7.** The $F_i$ are faces of $V$ with disjoint interiors. The map $(\psi \rho_i | \rho_i^{-1} F_i)$ has the property that

$$(\psi \rho_i | \rho_i^{-1} F_i)(x, y) = \begin{cases} 
\psi_i(x) & \text{if } (x, y) \in V^{n_i} \times \tau_i, \\
\psi(i)(y) & \text{if } (x, y) \in S^{n_i - 1} \times G_i.
\end{cases}$$

If we orient $F_i$ coherently with $V$ and $\rho_i^{-1} F_i$ coherently with $V^{n_i} \times V(i)$,
1.4 implies that $(\psi \rho_i \mid \rho_i^{-1} F_i)$ represents $[\alpha_i, \ast \alpha(i)]$.

Since $\rho_i$ is of degree $(-1)^{e(i)}$, $(\psi \mid F_i)$ represents $(-1)^{e(i)}[\alpha_i, \ast \alpha(i)]$ and hence applying 1.3 the formula 0.1 follows in view of the commutativity in the diagram

$$\pi_n(\Pi X, \Pi^1 X) \xrightarrow{\partial} \pi_{n-1}(\Pi^1 X, \Pi^2 X) \xrightarrow{d} \pi_{n-1}(\Pi^1 X)$$

where $d$ denotes the boundary homomorphism in the homotopy sequence of the pair $(\Pi X, \Pi^1 X)$.

**Proof of 2.5.** Let $D_o^n$ and $D_x^n$ denote the subsets

$$D_o^n = \left\{ x \in V^n; x_1 \geq \frac{1}{2} \text{ and } x_n \geq \frac{1}{2} \right\},$$

$$D_x^n = \left\{ x \in V^n; x_1 \leq \frac{1}{2} \text{ and } x_n \leq \frac{1}{2} \right\}.$$

Let $D \subseteq G_i$ have a factor $D_o^{n_k}$ for every factor $D_i^{n_k}$ of $G_i$ and a factor $D_x^{n_k}$ for every factor $D_x^{n_k}$ of $G_i$. Then certainly $\tau_i = D \cap V(i)^\circ$ has the desired property.

**Proof of 2.7.** If $\sigma_i$ is the face of $G_i$ complementary to $\tau_i$ then it may be observed that $F_i$ is the face of $\rho_i(V^n \times G_i)$ complementary to $\rho_i(V^n \times \sigma_i)$. Thus

$$F_i^\circ = \rho_i(S^{n_i-1} \times \sigma_i \cup V^n \times \tau_i^\circ).$$

Suppose $i < j$ and let

$$H = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(S^{n_j-1} \times G_j),$$

$$H' = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(V^n \times \tau_j),$$

$$H'' = \rho_i(V^n \times \tau_j) \cap \rho_j(S^{n_j-1} \times G_j).$$

2.7 will follow when we have proved that $H \subseteq F_i^\circ \cap F_j^\circ$, $H' = \emptyset$ and $H'' = \emptyset$. Since the images of $H$ under the projections into $V^n$ and $V^n$ are contained in $S^{n_i-1}$ and $S^{n_j-1}$ respectively we have

$$H \subseteq \rho_i(S^{n_i-1} \times G_i^\circ) \cap \rho_i(S^{n_j-1} \times G_j^\circ).$$

2.4 asserts the existence of an integer $k$ with $i \neq k \neq j$ such that $G_i$ has a factor $D_i^{n_k}$ and $G_j$ a factor $D_j^{n_k}$. Hence 2.5 implies that
and hence that
\[ H \cap \rho_i(S^{n-1} \times \tau_i) = H \cap \rho_j(S^{n-1} \times \tau_j) = \emptyset \]

2.5 also implies that \( H' = H'' = \emptyset \) which completes the proof of 2.6.

**REFERENCES**

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