

# Pacific Journal of Mathematics

**A PROOF OF THE NAKAOKA-TODA FORMULA**

KEITH A. HARDIE

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If  $X_j$  ( $1 \leq j \leq r$ ) are objects we denote the corresponding  $r$ -tuple  $(X_1, X_2, \dots, X_r)$  by  $X$  and the  $(r-1)$ -tuple  $(X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_r)$  by  $X(i)$ . When  $X_j$  ( $1 \leq j \leq r$ ) are based topological spaces  $\Pi X$  will denote their topological product and  $\Pi^i X$  the subspace of  $\Pi X$  whose points have at least  $i$  coordinates at base points (always denote by  $*$ ).

Let  $\alpha_j \in \pi_{n_j}(X_j)$  ( $n_j \geq 2, 1 \leq j \leq r, r \geq 3$ ) be elements of homotopy groups then we have

$$\star\alpha(\text{say}) = \alpha_1 \star \alpha_2 \star \dots \star \alpha_r \in \pi_n(\Pi X, \Pi^1 X),$$

where  $n = \sum n_j$  and  $\star$  denotes the product of Blakers and Massey [1]. We thus also have

$$\star\alpha(i) \in \pi_{n-n_i}(\Pi X(i), \Pi^1 X(i)).$$

There is a natural map  $\Pi X(i), \Pi^1 X(i) \rightarrow \Pi^1 X, \Pi^2 X$  and we denote also by  $\star\alpha(j)$  its image induced in  $\pi_{n-n_i}(\Pi^1 X, \Pi^2 X)$ . Let  $\partial$  denote the homotopy boundary homomorphism in the exact sequence of the triple  $(\Pi X, \Pi^1 X, \Pi^2 X)$ . We shall prove the formula:

$$\partial \star\alpha = \Sigma(1 \leq i \leq r)(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)] \in \pi_{n-1}(\Pi^1 X, \Pi^2 X), \quad (0.1)$$

where  $\varepsilon(1) = 0, \varepsilon(i) = n_i(n_1 + n_2 + \dots + n_{i-1})$  ( $i > 1$ ) and where the brackets refer to the generalised Whitehead product of Blakers and Massey [1]. In the case of the universal example 0.1 becomes the formula of Nakaoka and Toda stated in [4] and proved there for  $r = 3$ . I. M. James<sup>1</sup> has raised the question of its validity for  $r > 3$  and as the formula has applications (see [2], [3]) it would seem desirable to have a proof available in the literature. The present argument while inspired by [4] has a few novel features.

(1) DEFINITIONS AND LEMMAS. Let  $x = (x_1, x_2, \dots, x_n)$  denote a point of  $n$ -dimensional Euclidean space and let

$$\begin{aligned} V^n &= \{x; \Sigma x_i^2 \leq 1\}, \\ S^{n-1} &= \{x; \Sigma x_i^2 = 1\}, \\ E_+^{n-1} &= \{x \in S^n; x_n \geq 0\}, \\ E_-^{n-1} &= \{x \in S^n; x_n \leq 0\}, \\ D_+^n &= \{x \in V^n; x_n \geq 0\}, \end{aligned}$$

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$$\begin{aligned} D_-^n &= \{x \in V^n; x_n \leq 0\}, \\ D_1^n &= \{x \in V^n; x_1 \geq 0\}, \\ D_2^n &= \{x \in V^n; x_1 \leq 0\}. \end{aligned}$$

We recall that if  $Y \subseteq X$  then  $X$  is a closed  $n$ -cell and  $Y$  is a face of  $X$  if there exists a homeomorphism  $f: V^n \rightarrow X$  such that  $f(E_+^{n-1}) = Y$ . The subset  $X^0 = f(S^{n-1})$  is the boundary of  $X$ . If  $X$  and  $Y$  are oriented cells we assign to  $X \times Y$  the cross-product of the orientations of  $X$  and  $Y$ .

LEMMA 1.1. *Let  $X_1$  be a face of the cell  $X$  and  $Y_1$  a face of the cell  $Y$ . Then*

$$(X_1 \times Y) \cup (X \times Y_1) \text{ is a face of } X \times Y.$$

A proof of 1.1 may be found in [1] to which the reader may also refer for details concerning orientations. The proofs of the following two lemmas are standard exercises in homotopy theory and will be omitted.

LEMMA 1.2. *Suppose given a simplicial decomposition of a closed  $n$ -cell  $F (n \geq 3)$  and a subcomplex  $G$  which is a closed  $n$ -cell oriented coherently with  $F$ . If  $A$  is a simply-connected subset of a space  $Y$  and if  $f: F \rightarrow Y$  is a map such that  $f\{(F - G) \cup G^0\} \subseteq A$  then  $f: F, F^0 \rightarrow Y, A$  and  $f: G, G^0 \rightarrow Y, A$  represent the same element of  $\pi_n(Y, A)$ .*

LEMMA 1.3. *Suppose given a simplicial decomposition of  $V^{n+1} (n \geq 3)$  and subcomplexes  $F_i (i = 1, 2, \dots, m)$  which are faces of  $V^{n+1}$  with disjoint interiors oriented coherently with  $S^n$ . Let  $A$  be a simply-connected subset of a simply-connected space  $Y$ , let  $f: S^n \rightarrow Y$  be a map such that  $f\{(S^n - \cup F_i) \cup (\cup F_i^0)\} \subseteq A$ , let  $f: S^n \rightarrow Y$  represent  $\alpha \in \pi_n(Y)$  and let  $f: F_i, F_i^0 \rightarrow Y, A$  represent  $\alpha_i \in \pi_n(Y, A)$  ( $i = 1, 2, \dots, m$ ). Then  $j\alpha = \sum \alpha_i$  where  $j: \pi_n(Y) \rightarrow \pi_n(Y, A)$  is the injection homomorphism.*

Let  $A$  be a simply-connected subset of a space  $Y$ . Let  $f: V^p, S^{p-1} \rightarrow A, *$  and  $g: V^q, S^{q-1}, E_+^{q-1} \rightarrow Y, A, *$  be representatives of  $\alpha \in \pi_p(A)$  and  $\beta \in \pi_q(Y, A)$ . Let

$$h: S^{p-1} \times V^q \cup V^p \times E_+^{q-1}, S^{p-1} \times E_-^{q-1} \cup V^p \times S^{q-2} \rightarrow Y, A$$

be the map such that

$$h(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in V^p \times E_+^{q-1}, \\ g(y) & \text{if } (x, y) \in S^{p-1} \times V^q. \end{cases}$$

Then if  $S^{p-1} \times V^p \cup V^q \times E_+^{q-1}$  is oriented coherently with  $V^p \times V^q$  we recall 3.1 of [1]:

DEFINITION 1.4.  $h$  represents  $[\alpha, \beta] \in \pi_{p+q-1}(Y, A)$ .

(2) *Proof of 0.1.* Let  $\alpha_i$  be represented by a map

$$\psi_i : V^{n_i}, S^{n_i-1} \rightarrow X_i, *$$

with the property that

$$(2.1) \quad \psi_i(D_+^{n_i} \cup D_2^{n_i}) = *.$$

If we denote  $V^{n_1} \times V^{n_2} \times \dots \times V^{n_r}$  by  $V$  and  $V^{n_1} \times V^{n_{i-1}} \times V^{n_{i+1}} \times \dots \times V^{n_r}$  by  $V(i)$  then  $\star\alpha$  and  $\star\alpha(i)$  are represented by maps

$$\begin{aligned} \psi &: V, V^\circ \rightarrow \Pi X, \Pi^1 X, \\ \psi(i) &: V(i), V(i)^\circ \rightarrow \Pi^1 X, \Pi^2 X \end{aligned}$$

such that

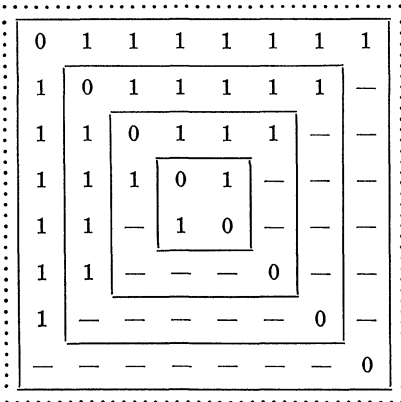
$$(2.2) \quad \begin{aligned} \psi(x_1, \dots, x_r) &= (\psi_1(x_1), \dots, \psi_r(x_r)), \\ \psi(i)(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r) &= (\psi_1(x_1), \dots, \psi_{i-1}(x_{i-1}), *, \psi_{i+1}(x_{i+1}), \dots, \psi_r(x_r)) \quad (x_i \in V^{n_i}). \end{aligned}$$

Let  $\rho_i : V^{n_i} \times V(i) \rightarrow V$  be the map such that

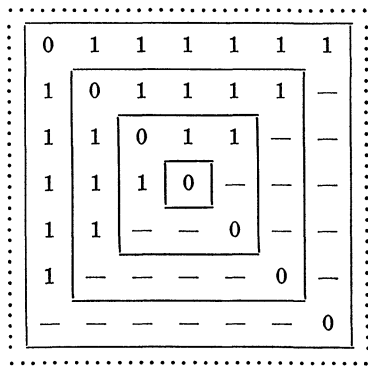
$$\rho_i(x_i, (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r)) = (x_1, x_2, \dots, x_r).$$

As an easy consequence of our orientation convention we obtain:

LEMMA 2.3. *The degree of  $\rho_i$  is  $(-1)^{\varepsilon(i)}$ .*



$r$  even



$r$  odd

The proof of 0.1 depends on the construction of certain closed cells  $G_i \subseteq V(i)$  ( $1 \leq i \leq r$ ). Consider the two infinite arrays illustrated

in the diagram. They contain between them exactly one centrally situated  $r \times r$  matrix. Let  $\eta(i, k, r)$  denote the symbol in the  $(i, k)$  position of this matrix. We define

$$G_i = \prod D_{\eta(i,k,r)}^{n_k},$$

where topological product  $\prod$  is taken over all values of  $k$  (in ascending order) except those for which  $\eta(i, k, r) = 0$ .

EXAMPLES If  $r = 5$  then  $G_2 = D_1^{n_1} \times D_1^{n_3} \times D_1^{n_4} \times D_-^{n_5}$ .

If  $r = 6$  then  $G_4 = D_1^{n_1} \times D_-^{n_2} \times D_1^{n_3} \times D_-^{n_5} \times D_-^{n_6}$ .

Certainly  $G_i \subseteq V(i)$ . We shall refer later to the following property of the  $G_i$  which is obvious from the diagram.

LEMMA 2.4. *If  $i < j \leq r$  then there is an integer  $k$  with  $i \neq k \neq j$  such that  $G_i$  has a factor  $D_1^{n_k}$  and  $G_j$  a factor  $D_-^{n_k}$ .*

The proof of the following lemma we postpone.

LEMMA 2.5. *For each  $i = 1, 2, \dots, r$ , there exists a face  $\tau_i$  of  $G_i$  and of  $V(i)$  such that if  $G_i$  has a factor  $D_1^{n_k}$  then the projection of  $\tau_i$  on  $D_1^{n_k}$  does not intersect  $D_-^{n_k}$  and such that if  $G_i$  has a factor  $D_-^{n_k}$  then the projection of  $\tau_i$  on  $D_-^{n_k}$  does not intersect  $D_1^{n_k}$ .*

In view of 2.1 and 2.5 we have  $\psi(i)(\tau_i) = *$ . Moreover 2.1 and 2.2 imply that

$$\psi(i)\{(V(i) - G_i) \cup G_i^\circ\} \subseteq \Pi^2 X.$$

Thus applying 1.2 (we may assume  $\Pi^2 X$  simply-connected for this is certainly so in the case of the universal example) we obtain that

$$(2.6) \quad (\psi(i) | G_i) : G_i, G_i^\circ, \tau_i \rightarrow \Pi^1 X, \Pi^2 X, *$$

represents  $\star\alpha(i)$ .

We now define

$$F_i = \rho_i(S^{n_i-1} \times G_i \cup V^{n_i} \times \tau_i) \quad (1 \leq i \leq r)$$

and prove later:

LEMMA 2.7. *The  $F_i$  are faces of  $V$  with disjoint interiors. The map  $(\psi\rho_i | \rho_i^{-1}F_i)$  has the property that*

$$(\psi\rho_i | \rho_i^{-1}F_i)(x, y) = \begin{cases} \psi_i(x) & \text{if } (x, y) \in V^{n_i} \times \tau_i, \\ \psi(i)(y) & \text{if } (x, y) \in S^{n_i-1} \times G_i. \end{cases}$$

If we orient  $F_i$  coherently with  $V$  and  $\rho_i^{-1}F_i$  coherently with  $V^{n_i} \times V(i)$ ,

1.4 implies that  $(\psi\rho_i | \rho_i^{-1}F_i)$  represents  $[\alpha_i, \star\alpha(i)]$ .

Since  $\rho_i$  is of degree  $(-1)^{\varepsilon(i)}$ ,  $(\psi | F_i)$  represents  $(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)]$  and hence applying 1.3 the formula 0.1 follows in view of the commutativity in the diagram

$$\begin{array}{ccc} \pi_n(\Pi X, \Pi^1 X) & \xrightarrow{\partial} & \pi_{n-1}(\Pi^1 X, \Pi^2 X) \\ \downarrow d & \nearrow j & \\ \pi_{n-1}(\Pi^1 X) & & \end{array}$$

where  $d$  denotes the boundary homomorphism in the homotopy sequence of the pair  $(\Pi X, \Pi^1 X)$ .

*Proof of 2.5.* Let  $D_0^n$  and  $D_{\times}^n$  denote the subsets

$$D_0^n = \left\{ x \in V^n; x_1 \geq \frac{1}{2} \text{ and } x_n \geq \frac{1}{2} \right\},$$

$$D_{\times}^n = \left\{ x \in V^n; x_1 \leq \frac{1}{2} \text{ and } x_n \leq \frac{1}{2} \right\}.$$

Let  $D \subseteq G_i$  have a factor  $D_0^{n_k}$  for every factor  $D_1^{n_k}$  of  $G_i$  and a factor  $D_{\times}^{n_k}$  for every factor  $D_{-}^{n_k}$  of  $G_i$ . Then certainly  $\tau_i = D \cap V(i)^\circ$  has the desired property.

*Proof of 2.7.* If  $\sigma_i$  is the face of  $G_i$  complementary to  $\tau_i$  then it may be observed that  $F_i$  is the face of  $\rho_i(V^{n_i} \times G_i)$  complementary to  $\rho_i(V^{n_i} \times \sigma_i)$ . Thus

$$F_i^\circ = \rho_i(S^{n_i-1} \times \sigma_i \cup V^{n_i} \times \tau_i^\circ).$$

Suppose  $i < j$  and let

$$H = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(S^{n_j-1} \times G_j),$$

$$H' = \rho_i(S^{n_i-1} \times G_i) \cap \rho_j(V^{n_j} \times \tau_j),$$

$$H'' = \rho_i(V^{n_i} \times \tau_j) \cap \rho_j(S^{n_j-1} \times G_j).$$

2.7 will follow when we have proved that  $H \subseteq F_i^\circ \cap F_j^\circ$ ,  $H' = \emptyset$  and  $H'' = \emptyset$ . Since the images of  $H$  under the projections into  $V^{n_i}$  and  $V^{n_j}$  are contained in  $S^{n_i-1}$  and  $S^{n_j-1}$  respectively we have

$$H \subseteq \rho_i(S^{n_i-1} \times G_i^\circ) \cap \rho_j(S^{n_j-1} \times G_j^\circ).$$

2.4 asserts the existence of an integer  $k$  with  $i \neq k \neq j$  such that  $G_i$  has a factor  $D_1^{n_k}$  and  $G_j$  a factor  $D_{-}^{n_k}$ . Hence 2.5 implies that

$$H \cap \rho_i(S^{n_i-1} \times \tau_i) = H \cap \rho_j(S^{n_j-1} \times \tau_j) = \emptyset$$

and hence that

$$H \subseteq \rho_i(S^{n_i-1} \times (G_i^\circ - \tau_i)) \cap \rho_j(S^{n_j-1} \times (G_j^\circ - \tau_j)) \subseteq F_i^\circ \cap F_j^\circ .$$

2.5 also implies that  $H' = H'' = \emptyset$  which completes the proof of 2.6.

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