

Pacific Journal of Mathematics

A REPRESENTATION OF THE BERNOULLI NUMBER B_n

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The function $\sigma_n(\nu)$ and the polynomial $\phi_n(\nu)$ have been defined in [2] and [3] respectively. Let $J_\nu(z)$ be the Bessel function of the first kind, and $j_{\nu, m}$ be the zeros of $z^{-\nu}J_\nu(z)$, then

$$(1) \quad \sigma_n(\nu) = \sum_{m=1}^{\infty} (j_{\nu, m})^{-2n}, \quad n = 1, 2, 3, \dots,$$

$$(2) \quad \phi_n(\nu) = 4^n \prod_{k=1}^n (\nu + k)^{[n/k]} \sigma_n(\nu),$$

where $[x]$ is the greatest integer $\leq x$.

$\sigma_n(\nu)$ is a rational function of ν with rational coefficient. $\phi_n(\nu)$ is a polynomial in ν with positive integral coefficients, and has degree $1 - 2n + \sum_{k=1}^n [n/k]$. All real zeros of $\phi_n(\nu)$ lie in the interval $(-n, -2)$. These polynomials also satisfy certain congruences [3].

Let B_n and G_n be the Bernoulli and Genocchi numbers:

$$(3) \quad B_n = \sum_{k=0}^n \binom{n}{k} B_k, \quad n \neq 1,$$

$$(4) \quad G_n = 2(1 - 2^n)B_n.$$

The symmetric function $\sigma_n(\nu)$ can be expressed in terms of the Bernoulli and Genocchi numbers by means of the following formulas:

$$(5) \quad \sigma_n\left(\frac{1}{2}\right) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} B_n,$$

$$(6) \quad \sigma_n\left(-\frac{1}{2}\right) = (-1)^n \frac{2^{2n-2}}{(2n)!} G_n,$$

where by B_n and G_n we understand the even-suffix numbers B_{2n} and G_{2n} [2].

In a previous paper [4] a structure of $\phi_n(\nu)$ has been given. This in turn leads, through (2), to a corresponding structure of $\sigma_n(\nu)$. And since for $\nu = 1/2$, $\sigma_n(\nu)$ is expressible in terms of the Bernoulli number B_n it is natural to enquire about a structure of B_n corresponding to that of $\sigma_n(\nu)$.

Three formulas from a previous paper [4, (8), (15), (18)] will be used here. They are written down as formulas (7), (8) and (9).

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$$(7) \quad \phi_n(\nu) = \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k \Omega_k(\nu) \phi_k(\nu) \phi_{n-k}(\nu),$$

where $\alpha_k = 2$, $k < \lfloor n/2 \rfloor$, and for $k = \lfloor n/2 \rfloor$,

$$\alpha_k = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$

$$\Omega_k(\nu) = \prod_{s=1}^{n-1} (\nu + s)^{\varepsilon(s,k,n)}, \quad \varepsilon(s,k,n) = \left\lfloor \frac{n}{s} \right\rfloor - \left\lfloor \frac{k}{s} \right\rfloor - \left\lfloor \frac{n-k}{s} \right\rfloor.$$

$$(8) \quad \phi_n(\nu) = \sum_{i=1}^{c(n)} 2^{n_i} \prod_{j=2}^{n-1} (\nu + j)^{n_{ij}},$$

where (i) $c(n)$ is the number of components of $\phi_n(\nu)$,

(ii) at most one $n_i = 0$,

$$(iii) \sum_{i=1}^{c(n)} 2^{n_i} = n^{-1} \binom{2n-2}{n-1},$$

$$(iv) \sum_{j=2}^{n-1} n_{ij} = 1 - 2n + \sum_{s=1}^n \left\lfloor \frac{n}{s} \right\rfloor, \text{ for all } i, \text{ and}$$

(v) given an integer s , $1 < s < n$, $n > 3$, there exists i such that $0 < n_{is} \leq \lfloor n/s \rfloor$.

$$(9) \quad c(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} c(k)c(n-k), \quad c(1) = 1.$$

We shall obtain specific information about certain components of $\phi_n(\nu)$ which will be used later on. We begin with

(10) For $2 < s < n$, $(\nu + s)^{\lfloor n/s \rfloor}$ is a factor of some component of $\phi_n(\nu)$, and if $s = 2$, $(\nu + s)^{\lfloor n/s \rfloor - 1}$ is a factor of a component of $\phi_n(\nu)$, $n > 3$.

Consider the first part of the statement. We observe that if $2 < s < n$, the statement is true for $n = 4, 5, 6, 7$ (see [3]). Assume the statement to be true for $k = 4, 5, \dots, n-1$. Take the k th term of (7), $T_k = \alpha_k \Omega_k(\nu) \phi_k(\nu) \phi_{n-k}(\nu)$, $k \geq 4$, $n \geq 8$. Then some component of $\phi_k(\nu) \phi_{n-k}(\nu)$ has a factor $(\nu + s)^{\lfloor k/s \rfloor + \lfloor (n-k)/s \rfloor}$. However, $\Omega_k(\nu)$ has a factor $(\nu + s)$ if and only if $\varepsilon(s,k,n) = 1$. Therefore, some component of T_k which is a component of $\phi_n(\nu)$ has a factor

$$(\nu + s)^{\lfloor k/s \rfloor + \lfloor (n-k)/s \rfloor + \varepsilon(s,k,n)} = (\nu + s)^{\lfloor n/s \rfloor}.$$

The second part of the statement may be proved by a similar method.

The following may be obtained from (10)

$$(11) \quad \begin{aligned} \max(n_{ij}) &= \lfloor n/j \rfloor, \quad 2 < j < n, \\ &= \lfloor n/2 \rfloor - 1, \quad j = 2. \end{aligned}$$

(12) For $s > 2$, and m such that $(2m + 1)s + m < n$, the product

$$\prod(n, m) \equiv \prod_{\lambda=0}^m \{\nu + (2\lambda + 1)s + \lambda\}^{[n/(2\lambda+1)s+\lambda]}$$

is a factor of some component of $\phi_n(\nu)$.

Proof. We shall use induction. Define the set of integers

$$I_m = \{\text{integers } x: (2m + 1)s + m < x < (2m + 3)s + m + 1\}, \\ m = 0, 1, 2, \dots .$$

If $n \in I_0$, $\prod(n, 0) = (\nu + s)^{[n/s]}$ and $(\nu + s)^{[n/s]}$ is a factor of some component of $\phi_n(\nu)$ by (10). Assume that for $k \leq m - 1$, $n \in I_k$ implies $\prod(n, k)$ is a factor of some component of $\phi_n(\nu)$. Let $n \in I_m$, and suppose $n = (2m + 1)s + m + i$, $1 \leq i \leq 2s$. Then $n - 2i = (2m + 1)s + m - i \in I_{m-1}$. Take formula (7), and consider the $(2i)$ -th term,

$$T_{2i} = \alpha_{2i} \Omega_{2i}(\nu) \phi_{2i}(\nu) \phi_{n-2i}(\nu) .$$

By induction hypothesis there are components V_1 of $\phi_{2i}(\nu)$ and V_2 of $\phi_{n-2i}(\nu)$ such that \prod_1 and \prod_2 are factors of V_1 and V_2 respectively, where

$$\prod_1 = \prod_{\lambda=0}^p \{\nu + (2\lambda + 1)s + \lambda\}^{[2i/(2\lambda+1)s+\lambda]} ,$$

$$\prod_2 = \prod_{\lambda=0}^{m-1} \{\nu + (2\lambda + 1)s + \lambda\}^{[n-2i/(2\lambda+1)s+\lambda]} ,$$

and $(2p + 1)s + p < 2i$, $(2m - 1)s + m - 1 < n - 2i$. Since the term T_{2i} yields a component of $\phi_n(\nu)$, we have that $\alpha_{2i} \Omega_{2i}(\nu) \prod_1 \prod_2$ is a factor of $\alpha_{2i} \Omega_{2i} V_1 V_2 = V$, where V is a component of $\phi_n(\nu)$. However,

$$\Omega_{2i}(\nu) = \prod_{r=1}^{n-1} (\nu + r)^{\varepsilon(r, 2i, n)} .$$

Hence after a simplification, we obtain

$$\alpha_{2i} \Omega_{2i}(\nu) \prod_1 \prod_2 = P(\nu) \prod(n, m) ,$$

where $P(\nu)$ is a polynomial in ν of degree ≥ 0 . Thus the term T_{2i} yields a component V of $\phi_n(\nu)$ such that $\prod(n, m)$ is a factor of V .

(13) $V(n) \equiv 2^{n-2} \prod_{r=2}^{[n/2]} (\nu + r)^{[n/r]-1}$, $n \geq 2$, is a component and the only component of $\phi_n(\nu)$ with the greatest numerical factor 2^{n-2} .

Proof. First we shall show that $V(n)$ is a component of $\phi_n(\nu)$. Observe that for $n = 2, 3, 4$, $V(n)$ is a component of $\phi_n(\nu)$. Assume: $V(m)$ is a component of $\phi_m(\nu)$, $2 \leq m \leq n - 1$. Consider the first term

T_1 of (7): $T_1 = 2\Omega_1(\nu)\phi_{n-1}(\nu)$. There is a component $V(n-1)$ of $\phi_{n-1}(\nu)$ such that

$$V(n-1) = 2^{n-3} \prod_{r=2}^{\lceil n-1/2 \rceil} (\nu+r)^{\lfloor n-1/r \rfloor - 1}.$$

Hence $2\Omega_1(\nu)V(n-1)$ is a component of $\phi_n(\nu)$. Substituting the expression for $\Omega_1(\nu)$, we obtain

$$2\Omega_1(\nu)V(n-1) = 2^{n-2} \prod_{r=2}^{\lceil n/2 \rceil} (\nu+r)^{\lfloor n/r \rfloor - 1} = V(n).$$

The second part of the statement that $V(n)$ is the only component of $\phi_n(\nu)$ with the greatest numerical factor 2^{n-2} may be proved by induction.

$$(14) \quad V_1(n) \equiv \frac{(\nu+3)V(n)}{4(\nu+2)}, \quad n \geq 4, \text{ is a component of } \phi_n(\nu).$$

This may be proved by considering the first term T_1 of (7) and using induction.

$$(15) \quad \text{For } \nu = 1/2, \text{ the value of } V_1(n) \text{ is less than the value of any other component of } \phi_n(\nu).$$

Proof. Take the k th term T_k of (7),

$$T_k = \alpha_k \Omega_k(\nu) \phi_k(\nu) \phi_{n-k}(\nu).$$

$V_1(n)$ is obtained from T_1 . For $k = 2, 3$ and $\nu = 1/2$, the smallest components of T_k correspond to the smallest components of $\phi_{n-k}(\nu)$, because $\alpha_k \Omega_k(\nu) \phi_k(\nu)$ is constant. We observe that for $n = 4, 5, 6, 7$, $V_1(n)$ is less than any other component of $\phi_n(\nu)$, $\nu = 1/2$. Assume that for $\nu = 1/2$, $V_1(m)$ is less than any other component of $\phi_m(\nu)$, $4 \leq m < n$. Using the induction hypothesis it is seen that for $\nu = 1/2$, $V_1(n)$ is less than any component obtained from T_k , $k = 2, 3$. For $k \geq 4$, $n \geq 8$, $\alpha_k \Omega_k(\nu) V_1(k) V_1(n-k)$ is a component of $\phi_n(\nu)$ and its value at $\nu = 1/2$ is less than the value of any other component obtained from T_k . Thus among all components of $\phi_n(\nu)$ there is a set S of exactly $\lceil n/2 \rceil$ minimum components

$$S = \left\{ \alpha_k \Omega_k(\nu) V_1(k) V_1(n-k) : 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \right\}.$$

Obviously $V_1(n) \in S$. We claim: $V_1(n)$ is less than any other element of S . It suffices to show that

$$\lim_{\nu \rightarrow 1/2} \frac{V_1(n)}{\alpha_k \Omega_k(\nu) V_1(k) V_1(n-k)} < 1, \quad k \neq 1.$$

A verification of this inequality is left to the reader.

Let (8) be multiplied by $2^{2-n}(\nu + 2)^{1-[n/2]}$. Then considering (7), induction yields the following

$$(16) \quad n - [n/2] - 1 \geq n_i - n_{i2}.$$

THEOREM. *The Bernoulli number B_n has the following representation :*

$$(17) \quad B_n = \frac{(-1)^{n-1}(2n)!}{20 \cdot 6^n \cdot (2n + 1)} \sum_{i=1}^{c(n)} (2^{r_i} a_i)^{-1},$$

- where
1. $\sum_{i=1}^{c(n)} (2^{r_i} a_i)^{-1} \equiv \begin{cases} 30 & \text{if } n = 1, \\ 5 & \text{if } n = 2, \\ 1 & \text{if } n = 3; \text{ for } n > 3, \end{cases}$
 2. $a_i = \prod_{m=1}^{n-2} (2m + 3)^{i_m}$, $\sum_{m=1}^{n-2} i_m = n - 3$, $0 \leq i_m < \left[\frac{n}{2} \right]$,
 3. $2^{r_1} a_1 = \frac{4}{7} \cdot 5 \cdot 7 \cdot 9 \cdot \dots \cdot (2n - 1)$,
 $\frac{4}{7} \cdot 5 \cdot 7 \cdot 9 \cdot \dots \cdot (2n - 1) > 2^{r_i} a_i > 7^{n-3}$, $i > 1$,
 4. $r_1 = 2$, $r_2 = 0$; $r_i \neq 0$, $i \neq 2$,
 5. $\sum_i 2^{-r_i} = 2^{2-n} \cdot n^{-1} \binom{2n-2}{n-1}$,
 6. *the g.c.d. $(2^{r_1} a_1, 2^{r_2} a_2, \dots) = 1$, and*
 7. *given an odd integer s , $5 < s \leq 2n - 1$, there is i such that $s^{\lfloor 2n/s-1 \rfloor}$ divides a_i ; if $s = 5$ then $s^{\lfloor 2n/s-1 \rfloor - 1}$ divides a_i , for some i .*

Proof. Substitute (2) in (8) and let $\nu = 1/2$, then in view of (5) the following is obtained after some simplification

$$B_n = \frac{(-1)^{n-1}(2n)!}{20 \cdot 6^n \cdot (2n + 1)} \sum_{i=1}^{c(n)} \left\{ 2^{r_i} \cdot 5^{-1} \prod_{k=2}^{n-1} (2k + 1)^{[n/k] - n_{ik}} \right\}^{-1},$$

where $r_i = n - 2 - n_i \geq 0$ by (13). Note that

$$\prod_{k=2}^{n-1} (2k + 1)^{[n/k] - n_{ik}}$$

is divisible by 5 for each i , because by (11) $[n/2] - n_{i2} \geq 1$. And $-1 + \sum_{k=2}^{n-1} \{[n/k] - n_{ik}\} = n - 3$ by (8, (iv)). Therefore, we may write

$$a_i \equiv 5^{-1} \prod_{k=2}^{n-1} (2k + 1)^{[n/k] - n_{ik}} = \prod_{m=1}^{n-2} (2m + 3)^{i_m},$$

where $\sum_{m=1}^{n-2} i_m = n - 3$, $0 \leq i_m < [n/2]$ by (11).

$$\begin{aligned} \text{and } i_1 &= [n/2] - 1 - n_{i_2}, \\ i_m &= [n/h] - n_{ih}, \quad h = m + 1, \quad m > 1. \end{aligned}$$

Thus property 2 is verified.

By (13) and (14), $V(n)$ and $V_1(n)$ are components of $\phi_n(\nu)$. If the components of $\phi_n(\nu)$ are ordered in such a way that $V_1(n)$ is the first and $V(n)$ is the second component, then for $\nu = 1/2$, the values of $V_1(n)$ and $V(n)$ correspond to $2^{r_1}a_1$ and $2^{r_2}a_2$. By actual calculation it is seen that $2^{r_1}a_1 = 4/7 \cdot 5 \cdot 7 \cdot 9 \cdots (2n - 1)$, $r_1 = 2$, $r_2 = 0$. Therefore, by (15) $2^{r_i}a_i < 4/7 \cdot 5 \cdot 7 \cdot 9 \cdots (2n - 1)$, $i > 1$. Since $r_i = n - 2 - n_i$, it follows from (13) that $r_i \neq 0$, if $i \neq 2$. By (16),

$$r_i = n - 2 - n_i \geq [n/2] - 1 - n_{i_2} = i_1.$$

Hence for each i ,

$$\begin{aligned} 2^{r_i}a_i &= 2^{r_i} \sum_{m=1}^{n-2} (2m + 3)^{i_m} \\ &= 2^{r_i - i_1} 10^{i_1} \sum_{m=2}^{n-2} (2m + 3)^{i_m} > 7^{n-3}. \end{aligned}$$

Properties 3 and 4 are proved. Property 5 is derived from (8, (iii));

$$\sum_i 2^{-r_i} = \sum_i 2^{2-n+n_i} = 2^{2-n} \sum_i 2^{n_i} = 2^{2-n} n^{-1} \binom{2n-2}{n-1}.$$

Concerning property 6, in view of 4, it suffices to prove that $\text{g.c.d.}(a_1, a_2, \dots) = 1$. Note that each a_i is a product of odd integers. By (12), $\prod(n, m)$ is a factor of a component, say V_p , of $\phi_n(\nu)$. However,

$$V_p 2^{2-n} \prod_{k=2}^{n-1} (\nu + k)^{-[n/k]} = \{P(\nu)\}^{-1},$$

where $P(\nu)$, a product of linear factors, is a polynomial in ν of degree > 0 . $P(\nu)$ is not divisible by any factor of $\prod(n, m)$. For $\nu = 1/2$, $\prod(n, m)$ is divisible by all odd factors $q(2s + 1)$, $q = 1, 3, 5, \dots$, which are less than n . Therefore, for $\nu = 1/2$, $P(\nu)$ is not divisible by any factor $q(2s + 1)$. Since $P(\nu)$, for $\nu = 1/2$, corresponds to some a_i the latter does not contain any factor $q(2s + 1)$. Thus for each $s > 2$, there is a_i which is not divisible by $q(2s + 1)$, $q = 1, 3, 5, \dots$. Hence the $\text{g.c.d.}(a_1, a_2, \dots) = 1$.

Suppose $s = 2m + 1$. Take a component V' of $\phi_n(\nu)$ which does not have the factor $(\nu + m)$. It may be shown that there exists such a component V' . Then

$$V'2^{2-n} \prod_{k=2}^{n-1} (\nu + k)^{[n/k]} = \{Q(\nu)\}^{-1},$$

where the polynomial $Q(\nu)$ has a factor $(\nu + m)^{[n/m]}$, $m > 2$. For $\nu = 1/2$, $Q(\nu)$ corresponds to some a_i and $(\nu + m)^{[n/m]}$ corresponds to the factor $(2m + 1)^{[n/m]}$ of a_i . However, if $m = 2$ than $5^{[n/2]-1}$ is a factor of a_i for some i . This completes the proof of the theorem.

We remark that the Genocchi number G_n and the numbers defined by L . Carlitz (see [1]).

$$a_r = 2^{2r} r! (r - 1)! \sigma_r(0),$$

may be expressed in a manner similar to (17). In fact, for the numbers a_r we have the following

$$(18) \quad a_r = \{(r - 1)!\}^2 \sum_{i=1}^{c(r)} 2^{ri} \prod_{k=2}^{r-1} k^{k_i k - [r/k]}.$$

A list of first few Bernoulli numbers expressed according to the theorem is given below.

$$B_1 = \frac{2!}{20 \cdot 6 \cdot 3} (30),$$

$$B_2 = -\frac{4!}{20 \cdot 6^2 \cdot 5} (5),$$

$$B_3 = \frac{6!}{20 \cdot 6^3 \cdot 7} (1),$$

$$B_4 = -\frac{8!}{20 \cdot 6^4 \cdot 9} \left(\frac{1}{2^2 \cdot 5} + \frac{1}{7} \right),$$

$$B_5 = \frac{10!}{20 \cdot 6^5 \cdot 11} \left(\frac{1}{2^2 \cdot 5 \cdot 9} + \frac{1}{7 \cdot 9} + \frac{1}{2 \cdot 5 \cdot 7} \right),$$

$$B_6 = -\frac{12!}{20 \cdot 6^6 \cdot 13} \left(\frac{1}{2^2 \cdot 5 \cdot 9 \cdot 11} + \frac{1}{7 \cdot 9 \cdot 11} + \frac{1}{2 \cdot 5 \cdot 7 \cdot 9} + \frac{1}{2 \cdot 5 \cdot 7 \cdot 11} \right. \\ \left. + \frac{1}{2^2 \cdot 5 \cdot 7^2} + \frac{1}{2^3 \cdot 5^2 \cdot 9} \right),$$

$$B_7 = \frac{14!}{20 \cdot 6^7 \cdot 15} \left(\frac{1}{2^2 \cdot 5 \cdot 9 \cdot 11 \cdot 13} + \frac{1}{7 \cdot 9 \cdot 11 \cdot 13} + \frac{1}{2 \cdot 5 \cdot 7 \cdot 9 \cdot 11} \right. \\ \left. + \frac{1}{2 \cdot 5 \cdot 7 \cdot 11 \cdot 13} + \frac{1}{2 \cdot 5 \cdot 7 \cdot 9 \cdot 13} + \frac{1}{2 \cdot 5 \cdot 7^2 \cdot 9} \right. \\ \left. + \frac{1}{2^2 \cdot 5^2 \cdot 7 \cdot 11} + \frac{1}{2^2 \cdot 5 \cdot 7^2 \cdot 13} + \frac{1}{2^3 \cdot 5^2 \cdot 7 \cdot 9} \right. \\ \left. + \frac{1}{2^3 \cdot 5^2 \cdot 9 \cdot 11} + \frac{1}{2^3 \cdot 5^2 \cdot 9 \cdot 13} \right).$$

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Homer Franklin Bechtell, Jr., <i>Pseudo-Frattini subgroups</i>	1129
Thomas Kelman Boehme and Andrew Michael Bruckner, <i>Functions with convex means</i>	1137
Lutz Bungart, <i>Boundary kernel functions for domains on complex manifolds</i>	1151
L. Carlitz, <i>Rings of arithmetic functions</i>	1165
D. S. Carter, <i>Uniqueness of a class of steady plane gravity flows</i>	1173
Richard Albert Dean and Robert Harvey Oehmke, <i>Idempotent semigroups with distributive right congruence lattices</i>	1187
Lester Eli Dubins and David Amiel Freedman, <i>Measurable sets of measures</i>	1211
Robert Pertsch Gilbert, <i>On class of elliptic partial differential equations in four variables</i>	1223
Harry Gonshor, <i>On abstract affine near-rings</i>	1237
Edward Everett Grace, <i>Cut points in totally non-semi-locally-connected continua</i>	1241
Edward Everett Grace, <i>On local properties and G_δ sets</i>	1245
Keith A. Hardie, <i>A proof of the Nakaoka-Toda formula</i>	1249
Lowell A. Hinrichs, <i>Open ideals in $C(X)$</i>	1255
John Rolfe Isbell, <i>Natural sums and abelianizing</i>	1265
G. W. Kimble, <i>A characterization of extremals for general multiple integral problems</i>	1283
Nand Kishore, <i>A representation of the Bernoulli number B_n</i>	1297
Melven Robert Krom, <i>A decision procedure for a class of formulas of first order predicate calculus</i>	1305
Peter A. Lappan, <i>Identity and uniqueness theorems for automorphic functions</i>	1321
Lorraine Doris Lavalley, <i>Mosaics of metric continua and of quasi-Peano spaces</i>	1327
Mark Mahowald, <i>On the normal bundle of a manifold</i>	1335
J. D. McKnight, <i>Kleene quotient theorems</i>	1343
Charles Kimbrough Megibben, III, <i>On high subgroups</i>	1353
Philip Miles, <i>Derivations on B^* algebras</i>	1359
J. Marshall Osborn, <i>A generalization of power-associativity</i>	1367
Theodore G. Ostrom, <i>Nets with critical deficiency</i>	1381
Elvira Rapaport Strasser, <i>On the defining relations of a free product</i>	1389
K. Rogers, <i>A note on orthogonal Latin squares</i>	1395
P. P. Saworotnow, <i>On continuity of multiplication in a complemented algebra</i>	1399
Johanan Schonheim, <i>On coverings</i>	1405
Victor Lenard Shapiro, <i>Bounded generalized analytic functions on the torus</i>	1413
James D. Stafney, <i>Arens multiplication and convolution</i>	1423
Daniel Sterling, <i>Coverings of algebraic groups and Lie algebras of classical type</i>	1449
Alfred B. Willcox, <i>Šilov type C algebras over a connected locally compact abelian group. II</i>	1463
Bertram Yood, <i>Faithful $*$-representations of normed algebras. II</i>	1475
Alexander Zabrodsky, <i>Covering spaces of paracompact spaces</i>	1489