

# Pacific Journal of Mathematics

**ON THE NORMAL BUNDLE OF A MANIFOLD**

MARK MAHOWALD

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In the Michigan lecture notes of 1940 [8] Whitney proved that any manifold in the cobordism class of  $P_2$  cannot be embedded in  $R^4$  with a normal field while non-orientable manifolds in the trivial cobordism class may or may not have a normal field. We will give a new proof of this result using some of the recent notions of differential topology. As one would expect, Whitney's theorem is a special case of a more general theorem and for the statement of this theorem we introduce some notation.

Let  $M^n$  be a compact smooth  $n$ -manifold. Let  $\bar{w}_i$  be the dual Stiefel Whitney classes of  $M^n$ .

DEFINITION. Let  $\sigma(M^n) = 0$  if  $\bar{w}_1 \cdot \bar{w}_{n-1} = 0$  and  $\sigma(M^n) = 1$  if  $\bar{w}_1 \cdot \bar{w}_{n-1} \neq 0$ .

Clearly  $\sigma(M^n)$  is just a Stiefel Whitney number [6]. Note also that by a result of Massey [5],  $\sigma(M^n) = 0$  unless  $n = 2^j$ .

THEOREM 1. *For any embedding of  $M^n$  in  $R^{2n}$  the (twisted) Euler class is congruent to  $2\sigma \pmod{4}$ .*

This result is a slight sharpening of the theorem of Massey [4]; the proof is given in § 4 after some preliminary results in §§ 2 and 3.

Let  $\chi$  be the Euler characteristic of  $M^2$ . In Whitney's theorem the role of  $\sigma$  in Theorem 1 is played by  $\chi$ . It is not hard to verify that for 2-dimension manifolds  $\sigma = \chi \pmod{2}$ . In addition, for 2-dimensional manifolds we can prove (section 6)

THEOREM 2. *For each  $k$  and each value of  $\sigma$  there is a manifold  $M^2$  and an embedding of  $M^2$  in  $R^4$  with twisted Euler class  $2\sigma + 4k$ .*

We have not been able to show that a single manifold has an embedding for each  $k$ . Whitney exhibited two embeddings of the Klein bottle, one with a trivial Euler class and one with a non-trivial one.

We also have this weaker result (section 7) for other values of  $n$ .

THEOREM 3. *For every even  $n$  there exists a manifold  $M^n$  and an embedding of  $M^n$  in  $R^{2n}$  with no normal field.*

It is known that if  $n \neq 2^j$  and  $n > 3$ , then every  $n$ -manifold embeds

in  $R^{2n-1}$ . Hence this result asserts in addition that some  $n$ -manifolds have inequivalent embeddings in  $R^{2n}$ .

It is interesting to note that the principal lemma yielding Theorem 1 also gives a new proof of the following slightly strengthened version of a result of Levine [2] and Mahowald [3].

**THEOREM 4.** *Suppose  $M^n$  is orientable in addition. If there exists a class  $d$  of dimension  $(n - k - 1)/2$  such that  $d \cup Sq^k d \cup \bar{w}_k \neq 0$ , then  $M^n$  does not embed in  $R^{n+k+1}$ .*

In [3] only the application of this result to give  $-P_n$  does not embed in  $R^{2n-2}$  if  $n = 2^j + 1$ —is given.

2. Some lemmas. In this section we will derive some information about a particular secondary cohomology operation. Let  $K$  be a semi-simplicial complex and let  $u \in C^{2k}(K; Z)$  such that  $\delta u = 2v$ . If  $w$  is an integer (a mod  $j$ ) cocycle we write  $[w]$  ( $[w]_j$ ) for the cohomology class containing  $w$ . We have the following results, some of which are well known.

2.1.  $Sq^1[u]_2 = [v]_2$  and  $\beta_2[u]_2 = [v]$  where  $\beta_j$  is the Bockstein coboundary connected with the sequence  $0 \rightarrow Z \rightarrow Z \rightarrow Z_j \rightarrow 0$ .

2.2. If  $\mathfrak{p}$  is the Pontriagin square operation  $\mathfrak{p}: H^{2k}(K; Z_2) \rightarrow H^{4k}(K; Z_4)$  then  $\mathfrak{p}([u]_2) = [u \cup u + u \cup_1 \delta u]_4$ .

2.3. If  $a \in H^i(X; Z)$  then let  $\bar{a}$  be its mod 2 restriction. Then

$$\beta_4 \mathfrak{p}([u]_2) = [v \cup_1 v + u \cup v]$$

and

$$\overline{\beta_4 \mathfrak{p}([u]_2)} = Sq^{2k} Sq^1 [u]_2 + [u]_2 \cup [v]_2 .$$

*Proof.* By the coboundary formula [7] which also holds in s.s.c. we have  $\delta(u \cup u + u \cup_1 \delta u) = 4(v \cup_1 v + u \cup v)$ . This gives the first statement and the second now follows by definition.

2.4. If  $u \cup u + \delta \mathfrak{p}$  is an integer cocycle then  $u \cup_1 v$  is a mod 2 cocycle and  $Sq^1([u \cup_1 v]) = Sq^{2k} Sq^1 [u]_2 + [u]_2 \cup [v]$ .

*Proof.* By the coboundary formula we have

$$\begin{aligned} \delta(u \cup_1 v) &= u \cup v - v \cup u + \delta u \cup_1 v \\ &= 2(u \cup v) + 2(v \cup_1 v) \end{aligned}$$

since  $\delta(u \cup u) = 0$  implies  $u \cup v + v \cup u = 0$ . Now 2.1 completes the proof.

2.5. If  $u \cup u = 2b + \delta c$ , then  $b + u \cup_1 v$  is a mod 2 cocycle and

$$Sq^1[b + u \cup_1 v]_2 = Sq^2 Sq^1[u]_2 + [u]_2 \cup Sq^1[u]_2 .$$

*Proof.* Note that  $\delta(u \cup u) = 2(v \cup u + u \cup v) = 2\delta b$ . Hence

$$v \cup u + u \cup v = \delta b$$

and the result follows as in 2.4.

In 2.4 we require that  $u \cup u + \delta p$  is an integer cocycle, that is, we require that  $\beta_2[u \cup u] = 0$ . The universal example for such a class  $u$  is obtained by considering a fibering  $p: X \rightarrow K(A_2, 2k)$  with fiber  $K(Z_2, 4k)$  and  $k$ -invariant  $2\beta_4 p(\alpha)$  where  $\alpha$  is the fundamental class of  $K(Z_2, 2k)$ . Let  $\alpha' = p^*(\alpha)$ . Then by 2.4,  $\alpha' \cup_1 Sq^1 \alpha'$  is a cocycle and not a coboundary (since  $\alpha' \cup Sq^1 \alpha' \neq 0$ ). Let  $\varepsilon = \alpha' \cup_1 Sq^1 \alpha'$ .

Let  $SA$  be the suspension of  $A$  and let  $s: H^j(A) \rightarrow H^{j+1}(SA)$  be the suspension isomorphism. There is a natural map  $f: SK(Z_2, 2k-1) \rightarrow X$  such that  $f^*$  is an isomorphism in dimension  $2k$ .

2.6. With the above notation there is a class  $\beta \in p^* H^*(K(Z_2, 2k); Z_2)$  (that is a primary operation) such that  $f^*(\beta + \varepsilon) = s(\alpha \cup Sq^1 \alpha)$  where  $s: H^j(K(Z_2, 2k-1)) \simeq H^{j+1}(SK(Z_2, 2k-1))$ . If  $\beta$  satisfies the above equation then  $\beta + Sq^{2k}$  will do so too.

*Proof.* As a vector space  $H^{4k}(SK; Z_2)$  is generated by

$$f^* p^* H^{4k}(K(Z_2, 2k)) \text{ and } s(\alpha \cup Sq^1 \alpha) .$$

Hence  $f^*(\varepsilon) = \lambda s(\alpha \cup Sq^1 \alpha) + \beta$  where  $\lambda = 0$  or  $1$  and  $\beta$  satisfies the theorem. By direct computation we see that

$$Sq^1 s(\alpha \cup Sq^1 \alpha) = Sq^{2k} Sq^1 s \alpha \notin f^* p^* Sq^1 H^{4k}(K(Z_2, 2k); Z_2) .$$

But by 2.4  $Sq^1 f^*(\varepsilon) = Sq^{2k} Sq^1 s \alpha$ . Since

$$Sq^1 \lambda s(\alpha \cup Sq^1 \alpha) + Sq^1 \beta = Sq^{2k} Sq^1 s \alpha$$

if and only if  $\lambda = 1$  and  $Sq^1 \beta = 0$  we are finished.

In 2.5 we required that  $u \cup u \equiv 0 \pmod{2}$ . The universal example for such a class  $u$  is given by a fiber space  $p_1: Y \rightarrow K(Z_2, 2k)$  with  $K(Z_2, 4k-1)$  as the fiber and  $Sq^{2k}$  as the  $k$ -invariant. Since there is no homotopy in dimension  $4k$  we have, letting  $[u]_2 = p_1^* \alpha$ :

2.7. The class  $\mu = [b + u \cup_1 v] \in H^{4k}(Y; Z_2)$  is not spherical and

hence is the universal example of a nontrivial natural cohomology operation which we write as  $\mu$  too.

Let  $g: SK(Z_2, 2k - 1) \rightarrow Y$  be the natural map inducing an isomorphism  $g^*$  in dimension  $2k$ . By an argument identical to the proof of 2.6 we have 2.8. In the above notation  $g^*(\mu + \beta') = s(\alpha \cup Sq^1\alpha)$  where  $\beta' \in p_1^*H^*(K(Z_2, 2k), Z_2)$ . If  $\beta'$  satisfies the above equation then  $\beta' + Sq^{2k}$  will do so too.

3. Let  $\gamma_n$  be the universal  $n$ -plane bundle and let  $I$  be the trivial line bundle. The base space of  $I$  will usually be clear from the context. If  $\nu$  is any  $n$ -plane bundle we let  $T(\nu)$  be the Thom complex and  $U \in H^n(T; Z_2)$  be the Thom class. Recall that in  $T$ ,  $U \cup U$  is equal to  $U \cup \bar{w}_n$  which is the restriction mod 2 of an integer class  $U \cup \chi$  where  $\chi$  is the twisted Euler class (of order 2 if  $n$  is odd). Hence  $\beta_2 Sq^n U = 0$ . By usual obstruction theory, letting  $n = 2k$ , we see that there exists a map  $g: T(\gamma_{2k}) \rightarrow X$  such that  $g^*$  is an isomorphism in dimension  $2k$ .

LEMMA 3.1. *In the above notation we can find a  $\beta$  satisfying 2.6 such that  $g^*(\beta + \epsilon) = U \cup \bar{w}_{n-1} \cup \bar{w}_1$ ,  $n = 2k$ .*

*Proof.* Consider the diagram:

$$\begin{array}{ccc} ST(\gamma_{n-1}) \cong T(\gamma_{n-1} \oplus I) & \xrightarrow{g'} & SK(Z_2, n - 1) \\ \downarrow i & & \downarrow f \\ T(\gamma_n) & \xrightarrow{g} & X \end{array}$$

where  $i$  is the map induced by the natural inclusion of  $\gamma_{n-1} \oplus I$  in  $\gamma_n$ , and  $g'$  is defined by requiring  $g'^*(s\alpha) = U'$ , the Thom class of  $T(\gamma_{n-1} \oplus I)$ . Letting  $\beta$  be the class of 2.6, we have  $g'^*f^*(\beta + \epsilon) = s(U_{n-1} \cup U_{n-1} \cup \bar{w}_1) = U' \cup \bar{w}_{n-1} \cup \bar{w}_1$  where  $U_{n-1}$  is the Thom class of  $T(\gamma_{n-1})$ . Hence  $g^*(\beta + \epsilon) = U \cup \bar{w}_{n-1} \cup \bar{w}_1 + \alpha$  where  $\alpha \in \ker i^*$ . But  $\ker i^*$  is generated by  $Sq^n U = U \cup \bar{w}_n$ . Therefore 2.6 completes the proof.

#### 4. Proof of Theorem 1.

NOTATION. In the remaining sections it will be convenient to use a dot for the cup product.

Let  $M^n$  be embedded in  $R^{2n}$  and let  $T(\gamma)$  be the Thom complex of the normal bundle. By [6]  $M^n$  has a normal field if  $n = 1 \pmod 2$  (it even embeds in  $R^{2n-1}$ ) so we suppose  $n$  is even. The group  $H^{2n}(T(n); Z) = Z$  and is generated by a class  $b$  such that  $2jb = U \cdot \lambda$

( $\bar{w}_n$  is zero, hence  $\lambda$  is zero mod 2). The cohomology operation  $\mu$  is defined on  $U$  and by 2.7 and 3.1 we have  $\mu(U) = [U \cdot \bar{w}_1 \cdot \bar{w}_{n-1} + j\bar{b}]_2$ . Since the top cohomology class of the Thom complex of a normal bundle to an embedding is spherical [6],  $\mu(U) = 0$ . Therefore  $j\bar{b} = U \cdot \bar{w}_1 \cdot \bar{w}_{n-1}$  (mod 2).

5. **Proof of Theorem 4.** Suppose we have an embedding of the kind described. Let  $E$  and  $E_0$  be the normal disk and sphere bundle respectively. Consider the sequence

$$T(\eta) = E/E_0 \xrightarrow{\tau} SE_0 \xrightarrow{Sf} SK(Z_2, j) \xrightarrow{g} Y$$

where  $g$  is defined in the paragraph just before 2.8 and  $Sf$  is the suspension of the map  $f: E_0 \rightarrow K(Z_2, j)$  satisfying  $f^*(\alpha) = a \cdot d$  where  $a$  is any class such that  $\tau^*(sa) = U$ . The map  $\tau$  is the natural map.<sup>1</sup> Let  $\lambda = fSf\tau$ . Clearly  $\lambda$  is a defining map for  $\mu$ . We have  $g^*\mu = s(\alpha \cdot Sq^1\alpha)$  by 2.8. By direct computation  $f^*(\alpha \cdot Sq^1\alpha) = a \cdot \bar{w}_k \cdot d \cdot Sq^1d + b$  where  $b$  is in  $\ker \tau^*$ . Finally  $\lambda^*(\mu) = U \cdot \bar{w}_k \cdot d \cdot Sq^1d$  which is in the top cohomology class of  $T(\eta)$  and hence must be zero. This contradiction proves the theorem.

6. **Proof of Theorem 2.** Let  $f': S^4 \rightarrow T(\gamma^2)$  be any map. By Theorem 36 [6] the map  $f'$  is homotopic to a map  $f: S^4 \rightarrow T(\gamma^2)$  which is transverse regular on  $G_{2,k}$  (the grassmann manifold of 2 planes in  $R^{2+k}$  which, if  $k > 3$ , is universal for classifying 2 plane bundles over 2-manifolds. Then  $f^{-1}(G_{2,k}) = M^2$  is a sub-manifold of  $S^4$  and  $f/M^2: M^2 \rightarrow G_{2,k}$  is the classifying map of the normal bundle to an embedding of  $M^2$  in  $R^4 \subset S^4$ . All that remains is to investigate the structure of  $\pi_4(T(\gamma^2))$ .

LEMMA 6.1. *The first few homotopy groups of  $T(\gamma^2)$  are*

$i$	1	2	3	4
$\pi_i(T(\gamma^2))$	0	$Z_2$	0	$Z$ .

*The  $k$ -invariant with which the  $Z$  group is added is  $2\beta_{k,p}(\alpha)$  where  $\alpha$  is the fundamental class of  $K(Z_2, 2)$ .*

REMARK. It is interesting to note that this portion of the Postnikov tower for  $T(\gamma^2)$  is the same as the corresponding portion for  $\tilde{G}_n$ ,  $n > 4$  where  $\tilde{G}_n$  is the classifying space for oriented  $n$ -plane bundles. Indeed the  $k$ -invariants computed in [1] agree with these

<sup>1</sup> If we realize  $E/E_0$  by adding a cone over  $E_0$  to  $E$ , then  $E$  is naturally embedded in  $E \cup_c E_0$  and  $\tau: E \cup_c E_0 \rightarrow E \cup_c E_0/E$ .

given here. The class  $w_4 \in H^4(\tilde{G}_n; Z_2)$  is associated with  $U \cdot w_1^2$  in  $H^4(T(\gamma^2); Z_2)$  while  $w_2^2$  and  $U \cdot w_2$  are similarly associated.

*Proof of the lemma.* Since the Thom class of  $T(\gamma^2)$  is also the fundamental class and since  $Sq^1 U \neq 0$ , the Hurewicz isomorphism theorem proves that  $\pi_2(T(\gamma^2)) = Z_2$ . Now  $H^3(T(\gamma^2); J) = Z_2$  if  $J = Z$  or  $Z_{2k}$  for any  $k$  and zero for other  $Z_p$ . Hence any homotopy group in dimension 3 must be attached with a nontrivial  $k$ -invariant. But  $H^4(K(Z_2, 2); Z_2)$  is generated by  $Sq^2 \alpha$  and  $Sq^2 U = U \cdot w_2$  in  $H^*(T(\gamma^2))$  and so  $\pi^2(T(\gamma^2)) = 0$ .

Now  $H^4(T(\gamma^2); Z) = Z$ , generated by  $U \cdot \chi$  where  $\chi$  is the twisted Euler class. Hence the rank of  $\pi_4(T(\gamma^2))$  is 1. Since the restriction mod 2 of  $U \cdot \chi$  is  $Sq^2 U$ , the  $Z$  component is attached with a nontrivial  $k$ -invariant. Finally  $H^5(K(Z_2, 2); Z) = Z_4$  generated by  $\beta_4 p(\alpha)$  and  $\overline{(\beta_4 p(\alpha))} = Sq^2 Sq^1 \alpha + \alpha Sq^1 \alpha$  (see 2.3) and since  $Sq^2 Sq^1 U + U \cdot U w_1 = U \cdot w_2 \cdot w_1 \neq 0$  the  $k$ -invariant for the  $Z$  component can not be  $\beta_4 p(\alpha)$ . Therefore it must be  $2\beta_4 p(\alpha)$ .

Let  $p: X \rightarrow K(Z_2, 2)$  be the fiber map having  $2\beta_4 p(\alpha)$  as  $k$ -invariant and  $K(Z, 4)$  as fiber. By 2.4 we see that  $H^4(X; Z_2) = Z_2 + Z_2$  generated by a new class  $\alpha' \cup_1 Sq^1 \alpha'$  and by  $Sq^2 \alpha'$  where  $\alpha' = p^* \alpha$ . Hence the natural map  $f: T(\gamma^2) \rightarrow X$  induces an isomorphism  $f^*: H^i(X) \rightarrow H^i(T(\gamma^2))$ , for all coefficient groups if  $i \leq 4$ . To complete the proof of the lemma we note that  $f^*$  is also an isomorphism in dimension 5.

Now we can complete the proof of Theorem 2. Since the order of the  $k$ -invariant is 2,  $f'^*(U \cdot \chi) = 2j \mathfrak{N}$  where  $\mathfrak{N}$  is a generator of  $H^4(S^4; Z)$  and  $j = [f']$ , the homotopy class of  $f'$  in  $\pi_4$  under some identification with the integers. Let  $\eta$  be the normal bundle for the embedding of  $M^2$  in  $R^4$  constructed above. Then the composite

$$S^4 \xrightarrow{\lambda_1} T(\eta) \xrightarrow{\lambda_2} T(\gamma^2)$$

(where  $\lambda_2$  is the natural map and  $\lambda_1$  is obtained by collapsing the complement of a normal neighborhood of  $M^2$  to a point) is just  $f'$ . Since  $\lambda_1^*$  is an isomorphism in dimension 4, the twisted Euler class of the embedding is  $2j$  times the twisted fundamental cohomology class.

**7. Proof of Theorem 3.** Let  $T(\gamma^n)$  be the Thom complex of the universal  $n$ -plane bundle,  $n$  even. Then  $H_n(T(\gamma^n); Z) = Z_2$  generated by the cycle dual to the Thom class  $U$ . Since  $T(\gamma_n)$  is  $(n - 1)$ -connected, we have  $\pi_n(T(\gamma^n)) = Z_2$ . Therefore by Serre's theorem, ([6], page 109)  $\text{rank } H^{2n}(T(\gamma^n); Z) = \text{rank } \pi_{2n}(T(\gamma^n))$ . In particular there is a map  $f: S^{2n} \rightarrow T(\gamma^n)$  such that  $f^*(U \cdot \chi) \neq 0$  where  $\chi$  is the twisted Euler class. Now following the argument of § 6 we construct the desired manifold.

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