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ON HIGH SUBGROUPS

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One of the purposes of this paper is to answer the following three questions:

(1) What groups G with $G^1 = 0$ are direct summands of all groups containing them as high subgroups?

(2) If G is a Σ -group, are all high subgroups of G endomorphic images of G (see [3] and [4])?

(3) If G is a torsion Σ -group, is every subgroup of G a Σ -group (see [3])?

The answer is affirmative to (2) and negative to (3). However an affirmative answer can be given to (3) when $|G^1| \leq \aleph_0$.

All groups in this paper will be assumed to be additively written abelian groups. For the most part, the notation and terminology of [2] will be followed. If G is a group, G_t will denote the torsion subgroup of G and G^1 the subgroup of elements of infinite height, that is, $G^1 = \bigcap_{n=1}^{\infty} nG$. A torsion group is said to be closed if each p -primary component is a closed p -group (see [2], pp. 114-117). A mixed abelian group is said to split if it decomposes into a direct sum of a torsion and torsion free group. By the n -adic topology on the group G , we shall mean the topology defined by taking as neighborhoods of 0 the subgroups nG for each positive integer n . A subgroup H of G is said to be a high subgroup if H is maximal in G with respect to $H \cap G^1 = 0$. If H is a high subgroup of G , then H is pure in G and G/H is divisible (see [3]). If all high subgroups of G are direct sums of cyclic groups, then G is said to be a Σ -group. If one high subgroup of G is a direct sum of cyclic groups, then all high subgroups of G are isomorphic and G is a Σ -group (see [4]).

1. High subgroups. Let G be an arbitrary abelian group and let D be a minimal divisible group containing G^1 . Then let K be the amalgamated sum of G and D over G^1 , that is, K is the abelian group generated by the elements of G and D subject only to $G \cap D = G^1$. (K can be realized as $(G + D)/L$ where L is the subgroup of $G + D$ consisting of all elements of the form $(x, -x)$ with $x \in G^1$.) It then follows that $K/G = \{G, D\}/G \cong D/(G \cap D) = D/G^1$, and similarly that $K/D \cong G/G^1$.

LEMMA 1. *If D is minimal divisible containing G^1 and if K is the amalgamated sum of G and D over G^1 , then*

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- (i) G is a pure subgroup of K ;
- (ii) $K = H + D$ with $H \cong G/G^1$;
- (iii) $H \cap G$ is a high subgroup of G ;
- (iv) $K = \{H, G\}$; and
- (v) G is a subdirect sum of H and D .

Proof. If $g \in G$ and $nk = g$ for some $k \in K$, then we write $k = g_1 + d$ with $g_1 \in G$ and $d \in D$. Then $nd = g - ng_1$ is an element of $G \cap D = G^1$ and hence there is a $g_2 \in G$ such that $ng_2 = g - ng_1$, that is, $g = n(g_1 + g_2)$ and we conclude that G is pure in K .

(ii) is immediate since divisible subgroups are always direct summands and, as observed above, $K/D \cong G/G^1$.

In order to show that $H \cap G$ is high in G we need only prove for every $g \in G \setminus H$ that $\{H \cap G, g\} \cap G^1 \neq 0$. If $g \in G \setminus H$, we write $g = h + d$ with $h \in H$, $d \in D$ and $d \neq 0$. Since D is minimal divisible containing G^1 , for some integer n , nd is a nonzero element in G^1 . Then $nh = n(g - d) \in H \cap G$ and $nd = ng - nh$ is a nonzero element of $\{H \cap G, g\} \cap G^1$.

Let p be an arbitrary prime. Since $G^1[p] = D[p]$, $K[p] \cong \{H, G\}$. Assume that we have established that $K[p^n]$ is contained in $\{H, G\}$. In order to show that $K[p^{n+1}]$ is contained in $\{H, G\}$, we need only consider elements in $D[p^{n+1}]$. If $d \in D[p^{n+1}]$, then $p^n d \in G^1$ and therefore there is a $g \in G$ such that $d - g \in K[p^n] \cong \{H, G\}$, that is, $d \in \{H, G\}$ and we conclude that $K[p^{n+1}] \cong \{H, G\}$. Clearly then the torsion subgroup of K is contained in $\{H, G\}$. To complete the proof that $K = \{H, G\}$, we show that $D \cong \{H, G\}$. Indeed if $d \in D$ with $d \neq 0$, then nd is a nonzero element of G^1 for some integer n . Then there is a $g \in G$ such that $d - g \in K[n] \cong \{H, G\}$ and therefore $d \in \{H, G\}$.

Since $K = \{H, G\} = \{D, G\}$, for each $h \in H$ there is a $g \in G$ and a $d \in D$ such that $g = h + d$; and similarly, for each $d \in D$ there is a $g \in G$ and an $h \in H$ such that $g = h + d$. Thus, G is a subdirect sum of H and D and the kernels are obviously $H \cap G$ and $D \cap G = G^1$.

Lemma 1 suggests a useful method for constructing groups with certain properties. Indeed, it can be shown without difficulty that if D is minimal divisible containing the group A and if H is a group without elements of infinite height having a pure subgroup B such that $H/B \cong D/A$, then any subdirect sum G of H and D with kernels B and A is a pure subgroup of $H + D$ such that

- (i) $G^1 = A$,
- (ii) $G/G^1 \cong H$, and
- (iii) B is a high group of G .

THEOREM 1. *Let M be an abelian group without elements of*

infinite height. Then M is a direct summand of every group containing it as a high subgroup if and only if M_i is closed.

Proof. Suppose that M_i is closed and that G contains M as a high subgroup with $G/M = D$. The G can be represented as a subdirect sum of H and D where $H \cong G/G^1$ and $M = H \cap G$. Therefore $D/G^1 \cong H/M \cong (H/M_i)/(M/M_i)$ and since M/M_i is a torsion free pure subgroup of H/M_i and D/G^1 is torsion, $H/M_i = J/M_i + M/M_i$. But since M_i is closed and is pure in the torsion group J , $J = L + M_i$ with $L \cong D/G^1$. Thus $H = L + M$ and since M is a direct summand of H , M is necessarily a direct summand of G .

Suppose now that M_i is not closed. Let M^* be the n -adic completion of M . Then M is a pure subgroup of M^* with M^*/M divisible. Since M_i^* is closed, $\{M, M_i^*\}/M$ is nonzero and is moreover the torsion subgroup of M^*/M . Let $H = \{M, M_i^*\}$ and choose a direct sum A of cyclic groups such that if D is minimal divisible containing A , then $D/A \cong H/M$. If G is a subdirect sum of H and D with kernels M and A , then G will be a reduced group having M as a high subgroup.

THEOREM 2. *If some high subgroup of G splits, then G/G^1 splits.*

Proof. Let $T + F$ be a high subgroup of G where T is torsion and F is torsion free. Let D be minimal divisible containing G^1 and let K be the amalgamated sum of G and D over G^1 . Then if $K = H + D$, we may assume that $G \cap H = T + F$. Then $H/(T + F) = H/(H \cap G) \cong \{H, G\}/G = K/G \cong D/G^1$. Therefore since D/G^1 is torsion and $(T + F)/T$ is a torsion free pure subgroup of H/T , $H/T = M/T + (T + F)/T$ for some subgroup M of H . Hence $H = M + F$ where M is necessarily a torsion group since M/T is torsion, that is, H splits and $H \cong G/G^1$.

COROLLARY 1. *If some high subgroup of G is torsion, then G/G^1 is torsion and therefore all high subgroups of G are torsion.*

COROLLARY 2. (Irwin, Percy and Walker [4]) *If A is a high subgroup of G and $A = T + F$ where T is torsion and F is torsion free, then $G = L + F$ with L/T divisible.*

Proof. Let D and K be as in the proof of Theorem 2 and suppose that $K = H + D$ with $H \cap G = A$. Then $K = (M + F) + D$ and therefore $G = L + F$ where $L = G \cap (M + D)$. Finally, we observe that $L/T \cong G/(T + F) = G/(H \cap G) \cong D$.

REMARK. From an example in [4], G need not split if G/G^1 splits.

2. Σ -groups.

THEOREM 3. *If G is a Σ -group, then every high subgroup of G is an endomorphic image of G . More generally, if the high subgroup H of G splits and the torsion subgroup of H is a direct sum of cyclic groups, then H is an endomorphic image of G .*

Proof. If $H = T + F$ where T is torsion and F is torsion free, then $G/G^1 \cong M + F$ where M is torsion and contains T as a pure subgroup. If T is a direct sum of cyclic groups, then T is a basic subgroup of M and therefore an endomorphic image of M . Clearly then if T is a direct sum of cyclic groups, H is an endomorphic image of G .

Requiring a Σ -group to have at most countably many elements of infinite height imposes severe restrictions on the structure of the group.

THEOREM 4. *If G is a Σ -group such that $|G^1| \leq \aleph_0$, then G/G^1 is a direct sum of cyclic groups.*

Proof. If $H = T + F$ is a high subgroup of G and if F is free and T is a torsion direct sum of cyclic groups, then $G/G^1 \cong M + F$ where M is a torsion direct sum of cyclic groups provided $|G^1| \leq \aleph_0$. Indeed, $M/T \cong D/G^1$, where D is minimal divisible containing G^1 , and since D/G^1 is necessarily at most countable and M is without elements of infinite height, Theorem 33.4 in [2] implies that M is a direct sum of cyclic groups.

REMARK. A glance at the proof of Theorem 4 should suggest an extremely simple proof of the fact that if one high subgroup of a group is a direct sum of cyclic groups then all high subgroups of the group are isomorphic.

THEOREM 5. *If G is a torsion Σ -group and G^1 has an at most countable basic subgroup, then G/G^1 is a direct sum of cyclic groups.*

Proof. We need only observe that if G^1 has an at most countable basic subgroup and if D is minimal divisible containing G^1 , then $|D/G^1| \leq \aleph_0$ (see [2], p. 110).

EXAMPLE 1. The restrictions in Theorems 3 and 4 are necessary. Indeed, let $B = \sum_{n=1}^{\infty} C(p^n)$ where p is a prime and let \bar{B} be the torsion subgroup of $\sum_{n=1}^{\infty} {}^*C(p^n)$. Then B is pure in \bar{B} and \bar{B}/B is isomorphic to 2^{\aleph_0} copies of $C(p^\infty)$. Next set $A = \sum_{\lambda \in \Lambda} \{a_\lambda\}$, where $\{a_\lambda\} \cong C(p)$ for each λ and $|\Lambda| = 2^{\aleph_0}$. Then if D is minimal divisible containing

A , $D/A \cong \bar{B}/B$. If G is a subdirect sum of \bar{B} and D with kernels \bar{B} and A , then G is a Σ -group such that $G/G^1 \cong \bar{B}$.

The proof of the following lemma follows immediately from results in [1].

LEMMA 2. *If A is an at most countable subgroup of the torsion group G such that G/A is a direct sum of cyclic groups, then G is the direct sum of an at most countable group and a direct sum of cyclic groups.*

THEOREM 6. *If G is a torsion group such that $|G^1| \leq \aleph_0$, then the following three conditions are equivalent:*

- (i) G is a Σ -group.
- (ii) G/G^1 is a direct sum of cyclic groups.
- (iii) $G = H + C$ where $|H| \leq \aleph_0$ and C is a direct sum of cyclic groups.

Proof. (i) implies (ii) by Theorem 4. (ii) implies (iii) by Lemma 2. And, finally, it is easy to see that (iii) always implies (i).

THEOREM 7. *If G is a torsion Σ -group such that $|G^1| \leq \aleph_0$, then every subgroup of G is a Σ -group.*

Proof. Let G be a torsion Σ -group such that $|G^1| \leq \aleph_0$ and let H be a subgroup of G . In order to show that H is a Σ -group, it suffices to show that H/H^1 is a direct sum of cyclic groups. Since $H/H \cap G^1 \cong \{H, G^1\}/G^1$, $H/H \cap G^1$ is a direct sum of cyclic groups. But this group is isomorphic to $(H/H^1)/(H \cap G^1/H^1)$ and since $H \cap G^1/H^1$ is at most countable and H/H^1 is without elements of infinite height, we conclude from Lemma 2 that H/H^1 is a direct sum of cyclic groups.

COROLLARY 3. *If the torsion group G is the direct sum of a countable group and a direct sum of cyclic groups, then every subgroup of G has a similar direct decomposition.*

EXAMPLE 2. The restriction that $|G^1| \leq \aleph_0$ in Theorem 7 is necessary. Indeed, if \bar{B} is as in Example 1, it is then easy to construct by methods we have used above a primary Σ -group G with $G^1 = \bar{B}$. Then G^1 is itself a subgroup of G which is not a Σ -group.

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