ON HIGH SUBGROUPS
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One of the purposes of this paper is to answer the following three questions:

1. What groups $G$ with $G^1 = 0$ are direct summands of all groups containing them as high subgroups?

2. If $G$ is a $\Sigma$-group, are all high subgroups of $G$ endomorphic images of $G$ (see [3] and [4])?

3. If $G$ is a torsion $\Sigma$-group, is every subgroup of $G$ a $\Sigma$-group (see [3])?

The answer is affirmative to (2) and negative to (3). However an affirmative answer can be given to (3) when $|G^1| \leq \aleph_0$.

All groups in this paper will be assumed to be additively written abelian groups. For the most part, the notation and terminology of [2] will be followed. If $G$ is a group, $G_\pi$ will denote the torsion subgroup of $G$ and $G^1$ the subgroup of elements of infinite height, that is, $G_\pi = \bigcap_{n=1}^\infty nG$. A torsion group is said to be closed if each $p$-primary component is a closed $p$-group (see [2], pp. 114-117). A mixed abelian group is said to split if it decomposes into a direct sum of a torsion and torsion free group. By the $n$-adic topology on the group $G$, we shall mean the topology defined by taking as neighborhoods of 0 the subgroups $nG$ for each positive integer $n$. A subgroup $H$ of $G$ is said to be a high subgroup if $H$ is maximal in $G$ with respect to $H \cap G^1 = 0$. If $H$ is a high subgroup of $G$, then $H$ is pure in $G$ and $G/H$ is divisible (see [3]). If all high subgroups of $G$ are direct sums of cyclic groups, then $G$ is said to be a $\Sigma$-group. If one high subgroup of $G$ is a direct sum of cyclic groups, then all high subgroups of $G$ are isomorphic and $G$ is a $\Sigma$-group (see [4]).

1. High subgroups. Let $G$ be an arbitrary abelian group and let $D$ be a minimal divisible group containing $G^1$. Then let $K$ be the amalgamated sum of $G$ and $D$ over $G^1$, that is, $K$ is the abelian group generated by the elements of $G$ and $D$ subject only to $G \cap D = G^1$. ($K$ can be realized as $(G + D)/L$ where $L$ is the subgroup of $G + D$ consisting of all elements of the form $(x, -x)$ with $x \in G^1$.) It then follows that $K/G = \{G, D\}/G \cong D/(G \cap D) = D/G^1$, and similarly that $K/D \cong G/G^1$.

**Lemma 1.** If $D$ is minimal divisible containing $G^1$ and if $K$ is the amalgamated sum of $G$ and $D$ over $G^1$, then

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(i) $G$ is a pure subgroup of $K$;
(ii) $K = H + D$ with $H \cong G/G^1$;
(iii) $H \cap G$ is a high subgroup of $G$;
(iv) $K = \{H, G\}$; and
(v) $G$ is a subdirect sum of $H$ and $D$.

Proof. If $g \in G$ and $nk = g$ for some $k \in K$, then we write $k = g_1 + d$ with $g_1 \in G$ and $d \in D$. Then $nd = g - ng_1$ is an element of $G \cap D = G^1$ and hence there is a $g_2 \in G$ such that $ng_2 = g - ng_1$, that is, $g = n(g_1 + g_2)$ and we conclude that $G$ is pure in $K$.

(ii) is immediate since divisible subgroups are always direct summands and, as observed above, $K/D \cong G/G^1$.

In order to show that $H \cap G$ is high in $G$ we need only prove for every $g \in G \setminus H$ that $\{H \cap G, g\} \cap G^1 \neq 0$. If $g \in G \setminus H$, we write $g = h + d$ with $h \in H$, $d \in D$ and $d \neq 0$. Since $D$ is minimal divisible containing $G^1$, for some integer $n$, $nd$ is a nonzero element in $G^1$. Then $nh = n(g - d) \in H \cap G$ and $nd = ng - nh$ is a nonzero element of $\{H \cap G, g\} \cap G^1$.

Let $p$ be an arbitrary prime. Since $G^1[p] = D[p]$, $K[p] \subseteq \{H, G\}$. Assume that we have established that $K[p^n]$ is contained in $\{H, G\}$. In order to show that $K[p^{n+1}]$ is contained in $\{H, G\}$, we need only consider elements in $D[p^{n+1}]$. If $d \in D[p^{n+1}]$, then $p^nd \in G^1$ and therefore there is a $g \in G$ such that $d - g \in K[p^n] \subseteq \{H, G\}$, that is, $d \in \{H, G\}$ and we conclude that $K[p^{n+1}] \subseteq \{H, G\}$. Clearly then the torsion subgroup of $K$ is contained in $\{H, G\}$. To complete the proof that $K = \{H, G\}$, we show that $D \subseteq \{H, G\}$. Indeed if $d \in D$ with $d \neq 0$, then $nd$ is a nonzero element of $G^1$ for some integer $n$. Then there is a $g \in G$ such that $d - g \in K[n] \subseteq \{H, G\}$ and therefore $d \in \{H, G\}$.

Since $K = \{H, G\} = \{D, G\}$, for each $h \in H$ there is a $g \in G$ and a $d \in D$ such that $g = h + d$; and similarly, for each $d \in D$ there is a $g \in G$ and an $h \in H$ such that $g = h + d$. Thus, $G$ is a subdirect sum of $H$ and $D$ and the kernels are obviously $H \cap G$ and $D \cap G = G^1$.

Lemma 1 suggests a useful method for constructing groups with certain properties. Indeed, it can be shown without difficulty that if $D$ is minimal divisible containing the group $A$ and if $H$ is a group without elements of infinite height having a pure subgroup $B$ such that $H/B \cong D/A$, then any subdirect sum $G$ of $H$ and $D$ with kernels $B$ and $A$ is a pure subgroup of $H + D$ such that

(i) $G^1 = A$,
(ii) $G/G^1 \cong H$, and
(iii) $B$ is a high group of $G$.

**Theorem 1.** Let $M$ be an abelian group without elements of
infinite height. Then \( M \) is a direct summand of every group containing it as a high subgroup if and only if \( M \) is closed.

**Proof.** Suppose that \( M \) is closed and that \( G \) contains \( M \) as a high subgroup with \( G/M = D \). The \( G \) can be represented as a subdirect sum of \( H \) and \( D \) where \( H \cong G/G^1 \) and \( M = H \cap G \). Therefore \( D/G^1 \cong H/M \cong (H/M_t)/(M/M_t) \) and since \( M/M_t \) is a torsion free pure subgroup of \( H/M \) and \( D/G^1 \) is torsion, \( H/M_t = J/M_t + M/M_t \). But since \( M \) is closed and is pure in the torsion group \( J, J = L + M_t \) with \( L \cong D/G^1 \). Thus \( H = L + M \) and since \( M \) is a direct summand of \( H, M \) is necessarily a direct summand of \( G \).

Suppose now that \( M \) is not closed. Let \( M^* \) be the \( n \)-adic completion of \( M \). Then \( M \) is a pure subgroup of \( M^* \) with \( M^*/M \) divisible. Since \( M_t^* \) is closed, \( \{M, M_t^*\}/M \) is nonzero and is moreover the torsion subgroup of \( M^*/M \). Let \( H = \{M, M_t^*\} \) and choose a direct sum \( A \) of cyclic groups such that if \( D \) is minimal divisible containing \( A \), then \( D/A \cong H/M \). If \( G \) is a subdirect sum of \( H \) and \( D \) with kernels \( M \) and \( A \), then \( G \) will be a reduced group having \( M \) as a high subgroup.

**Theorem 2.** If some high subgroup of \( G \) splits, then \( G/G^1 \) splits.

**Proof.** Let \( T + F \) be a high subgroup of \( G \) where \( T \) is torsion and \( F \) is torsion free. Let \( D \) be minimal divisible containing \( G^2 \) and let \( K \) be the amalgamated sum of \( G \) and \( D \) over \( G^1 \). Then if \( K = H + D \), we may assume that \( G \cap H = T + F \). Then \( H/(T + F) = H/(H \cap G) \cong (H, G)/G = K/G \cong D/G^1 \). Therefore since \( D/G^1 \) is torsion and \( (T + F)/T \) is a torsion free pure subgroup of \( H/T, H/T = M/T + (T + F)/T \) for some subgroup \( M \) of \( H \). Hence \( H = M + F \) where \( M \) is necessarily a torsion group since \( M/T \) is torsion, that is, \( H \) splits and \( H \cong G/G^1 \).

**Corollary 1.** If some high subgroup of \( G \) is torsion, then \( G/G^1 \) is torsion and therefore all high subgroups of \( G \) are torsion.

**Corollary 2.** (Irwin, Peercy and Walker [4]) If \( A \) is a high subgroup of \( G \) and \( A = T + F \) where \( T \) is torsion and \( F \) is torsion free, then \( G = L + F \) with \( L/T \) divisible.

**Proof.** Let \( D \) and \( K \) be as in the proof of Theorem 2 and suppose that \( K = H + D \) with \( H \cap G = A \). Then \( K = (M + F) + D \) and therefore \( G = L + F \) where \( L = G \cap (M + D) \). Finally, we observe that \( L/T \cong G/(T + F) = G/(H \cap G) \cong D \).

**Remark.** From an example in [4], \( G \) need not split if \( G/G^1 \) splits.
2. $\Sigma$-groups.

**Theorem 3.** If $G$ is a $\Sigma$-group, then every high subgroup of $G$ is an endomorhpic image of $G$. More generally, if the high subgroup $H$ of $G$ splits and the torsion subgroup of $H$ is a direct sum of cyclic groups, then $H$ is an endomorphic image of $G$.

**Proof.** If $H = T + F$ where $T$ is torsion and $F$ is torsion free, then $G/G^1 \cong M + F$ where $M$ is torsion and contains $T$ as a pure subgroup. If $T$ is a direct sum of cyclic groups, then $T$ is a basic subgroup of $M$ and therefore an endomorphic image of $M$. Clearly then if $T$ is a direct sum of cyclic groups, $H$ is an endomorphic image of $G$.

Requiring a $\Sigma$-group to have at most countably many elements of infinite height imposes severe restrictions on the structure of the group.

**Theorem 4.** If $G$ is a $\Sigma$-group such that $|G^1| \leq \aleph_0$, then $G/G^1$ is a direct sum of cyclic groups.

**Proof.** If $H = T + F$ is a high subgroup of $G$ and if $F$ is free and $T$ is a torsion direct sum of cyclic groups, then $G/G^1 \cong M + F$ where $M$ is a torsion direct sum of cyclic groups provided $|G^1| \leq \aleph_0$. Indeed, $M/T \cong D/G^1$, where $D$ is minimal divisible containing $G^1$, and since $D/G^1$ is necessarily at most countable and $M$ is without elements of infinite height, Theorem 33.4 in [2] implies that $M$ is a direct sum of cyclic groups.

**Remark.** A glance at the proof of Theorem 4 should suggest an extremely simple proof of the fact that if one high subgroup of a group is a direct sum of cyclic groups then all high subgroups of the group are isomorphic.

**Theorem 5.** If $G$ is a torsion $\Sigma$-group and $G^1$ has an at most countable basic subgroup, then $G/G^1$ is a direct sum of cyclic groups.

**Proof.** We need only observe that if $G^1$ has an at most countable basic subgroup and if $D$ is minimal divisible containing $G^1$, then $|D/G^1| \leq \aleph_0$ (see [2], p. 110).

**Example 1.** The restrictions in Theorems 3 and 4 are necessary. Indeed, let $B = \sum_{n=1}^{\infty} C(p^n)$ where $p$ is a prime and let $\bar{B}$ be the torsion subgroup of $\sum_{n=1}^{\infty} C(p^n)$. Then $B$ is pure in $\bar{B}$ and $\bar{B}/B$ is isomorphic to $2^{\aleph_0}$ copies of $C(p^n)$. Next set $A = \sum_{\lambda \in \Lambda} \{a_\lambda\}$, where $\{a_\lambda\} \cong C(p)$ for each $\lambda$ and $|A| = 2^{\aleph_0}$. Then if $D$ is minimal divisible containing
A, \(D/A \cong \bar{B}/B\). If \(G\) is a subdirect sum of \(\bar{B}\) and \(D\) with kernels \(\bar{B}\) and \(A\), then \(G\) is a \(\Sigma\)-group such that \(G/G^1 \cong \bar{B}\).

The proof of the following lemma follows immediately from results in [1].

**Lemma 2.** If \(A\) is an at most countable subgroup of the torsion group \(G\) such that \(G/A\) is a direct sum of cyclic groups, then \(G\) is the direct sum of an at most countable group and a direct sum of cyclic groups.

**Theorem 6.** If \(G\) is a torsion group such that \(|G^1| \leq \aleph_0\), then the following three conditions are equivalent:

(i) \(G\) is a \(\Sigma\)-group.

(ii) \(G/G^1\) is a direct sum of cyclic groups.

(iii) \(G = H + C\) where \(|H| \leq \aleph_0\) and \(C\) is a direct sum of cyclic groups.

**Proof.** (i) implies (ii) by Theorem 4. (ii) implies (iii) by Lemma 2. And, finally, it is easy to see that (iii) always implies (i).

**Theorem 7.** If \(G\) is a torsion \(\Sigma\)-group such that \(|G^1| \leq \aleph_0\), then every subgroup of \(G\) is a \(\Sigma\)-group.

**Proof.** Let \(G\) be a torsion \(\Sigma\)-group such that \(|G^1| \leq \aleph_0\) and let \(H\) be a subgroup of \(G\). In order to show that \(H\) is a \(\Sigma\)-group, it suffices to show that \(H/H^1\) is a direct sum of cyclic groups. Since \(H/H \cap G^1 \cong (H, G^1)/G^1\), \(H/H \cap G^1\) is a direct sum of cyclic groups. But this group is isomorphic to \((H/H^1)/(H \cap G^1/H^1)\) and since \(H \cap G^1/H^1\) is at most countable and \(H/H^1\) is without elements of infinite height, we conclude from Lemma 2 that \(H/H^1\) is a direct sum of cyclic groups.

**Corollary 3.** If the torsion group \(G\) is the direct sum of a countable group and a direct sum of cyclic groups, then every subgroup of \(G\) has a similar direct decomposition.

**Example 2.** The restriction that \(|G^1| \leq \aleph_0\) in Theorem 7 is necessary. Indeed, if \(\bar{B}\) is as in Example 1, it is then easy to construct by methods we have used above a primary \(\Sigma\)-group \(G\) with \(G^1 = \bar{B}\). Then \(G^1\) is itself a subgroup of \(G\) which is not a \(\Sigma\)-group.

**References**


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