A GENERALIZATION OF POWER-ASSOCIATIVITY

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Probably the most promising new identity to arise in a recent study of identities on commutative algebras [3] is

\begin{equation}
2((x^2 \cdot x)x + (x^3 \cdot x)x^2) = 3(x^2 \cdot x^3)x.
\end{equation}

This identity generalizes not only the power-associative identity, \( x^2 \cdot x^2 = (x^2 \cdot x)x \), but also the generalization of the Jordan identity considered in [4]. In the present paper, we study the structure of commutative rings of characteristic relatively prime to 2, 3, 5, or 7 satisfying (1). This restriction on the characteristic will be assumed throughout the paper without further mention.

There are two obvious ways in which the structure theory of the class of rings studied here is noticeably weaker than the structure theory of power-associative rings. First of all, given a ring \( A \) satisfying (1) containing an idempotent \( e \), there can exist elements of \( A \) which are annihilated by the operator \((2R_e - I)^2\) but not by \((2R_e - I)\). Secondly, defining the additive subgroups \( A_\lambda = A_\lambda(e) = \{x \mid x \in A, xe = \lambda x\} \) for \( \lambda = 0, 1/2, \) and \( 1 \), the relations \( A_0A_0 = 0 \) and \( A_1/2A_1/2 \subset A_1 + A_0 \) are not valid in general. Despite these impediments, we see in §1 that \( A \) may be decomposed simultaneously with respect to a set of mutually orthogonal idempotents in much the usual fashion. In §2 we prove that, if \( A \) is simple of degree \( \geq 3 \) satisfying the condition that \( x(2R_e - I)^2 = 0 \) if and only if \( x(2R_e - I) = 0 \) for all \( x \) in \( A \), then \( A \) is a Jordan ring.

1. We begin our investigation by partially linearizing (1) to obtain

\begin{equation}
4((yx \cdot x)x)x + 2(yx^3 \cdot x)x + 2yx^3 \cdot x + 2y(x^3 \cdot x) + 2(yx \cdot x)x^3 \\
+ yx^3 \cdot x^3 + 2yx \cdot x^3 = 12(yx \cdot x^3)x + 3y(x^2 \cdot x^5).
\end{equation}

Then, setting \( x = e \) in (2) immediately yields

\begin{equation}
4yR_e^4 - 8yR_e^3 + 5yR_e^2 - yR_e = 0, \quad \text{or}
\end{equation}

\begin{equation}
y[(R_e - I)(2R_e - I)^2R_e] = 0.
\end{equation}

Defining \( B_{1/2} = B_{1/2}(1/2) = \{x \mid x \in A, \ x(2R_e - I)^2 = 0\} \), it follows from (3) that \( A \) may be decomposed into the additive direct sum

\begin{equation}
A = A_1 + B_{1/2} + A_0.
\end{equation}

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Another additive subgroup of $A$ which will be useful is $C_{1/2} = C_e(1/2) = \{ x \mid \exists y \in B_{1/2} \in y(2R_e - I) = x \}$. It is easy to see that $C_{1/2} \subset A_{1/2} \subset B_{1/2}$.

**Theorem 1** Let $A$ be a ring satisfying (1) with an idempotent $e$, and let $A_1, A_0, A_{1/2}, B_{1/2}, C_{1/2}$ be defined as above. Then $A_1$ and $A_0$ are subrings, $A_1A_0 \subset A_{1/2}, A_1B_{1/2} \subset B_{1/2} + A_0, A_0B_{1/2} \subset B_{1/2} + A_1, A_1C_{1/2} \subset A_{1/2}, A_0C_{1/2} \subset A_{1/2}$, $A_{1/2}A_{1/2} \subset A_1 + A_{1/2} + A_0, A_{1/2}C_{1/2} \subset A_1 + C_{1/2} + A_0$, and $C_{1/2}C_{1/2} \subset C_{1/2}$.

To establish this theorem, we first partially linearize (2) and assume that the new variable of degree 3 is idempotent. This gives

\begin{align*}
4(yx)R_x^3 + 4(ye \cdot x)R_x^3 + 4(yR_x^3 \cdot x)R_e + 4yR_x^3 \cdot x + 4(y \cdot xe)R_e^3 \\
+ 2(ye \cdot x)R_e + 2yR_x^3 \cdot x + 4(y \cdot xR_x^3)R_e + 2(y \cdot xe)R_e + 2ye \cdot x \\
+ 4y(xR_x^3) + 2y(xR_x^3) + 2y \cdot xe + 2(yx)R_x^3 + 2(ye \cdot x)R_e \\
+ 4(yR_x^3)(xe) + 2(y \cdot xe)R_e + 2(ye)(xe) + 2(yx)R_e \\
+ 4(yx)(xR_x^3) + 2(ye)(xe) \\
= 12(yx)R_x^3 + 24(ye \cdot xe)R_e + 12yR_x^3 \cdot x + 12y(xR_x^3),
\end{align*}

which simplifies to

\begin{align*}
(yx)[2R_x^3 - 5R_e^2 + R_e] + (ye \cdot x + y \cdot xe)[2R_x^3 + 2R_e + I] \\
+ (yR_x^3 \cdot x + y \cdot xR_x^3)[2R_e - 5I] + 2(yR_x^3 \cdot x + y \cdot xR_x^3) \\
+ (ye \cdot xe)[-12R_e + 2I] + 2(yR_x^3 \cdot x + ye \cdot xR_x^3) = 0. 
\end{align*}

(5)

First, letting $x, y \in A_1$, this reduces to $(yx)[2R_x^3 - 5R_e^2 + R_e] = 0$. Since 1 is a root of this operator but 0 and 1/2 are not for any characteristic, we have $ye \in A_1$, or $A_1A_1 \subset A_1$. Similarly, if $x, y \in A_0$, (5) reduces to $(yx)[2R_x^3 - 5R_e^2 + R_e] = 0$, which gives $A_0A_0 \subset A_0$. And, choosing $y \in A_1, x \in A_0$ in (5) yields $(yx)[2R_x^3 - 5R_e^2 + R_e] = 0$, or $A_1A_0 \subset A_{1/2}$.

Suppose next that $y \in A_1$ and $x \in B_{1/2}$. Letting $w = x(R_e - 1/2I)$, we have $xe = 1/2x + w, we = 1/2w, xe \cdot e = 1/4x + w, (xe \cdot e)e = 1/8x + 3/4w$, and (5) becomes $(yx)[(2R_x^3 - 5R_e^2 + R_e) + (3R_x^3 + 3R_e + 3/2I) + (5/2R_e - 25/4I) + 9/4I + (-6R_e + I) + 3/2I] + (yw)[2R_x^3 + 2R_e + I + 2R_e - 5I + 3/2I - 12R_e + 2I + 4I] = 0$, or

\begin{align*}
(6) \quad (yx)[2R_x^3 - 2R_e^2 + R_e] + (yw)[2R_x^3 - 8R_e + \frac{7}{2}I] = 0. 
\end{align*}

Taking $w = 0$ in (6), we see that $A_1A_{1/2} \subset B_{1/2} + A_0$. But then $(yw) \in B_{1/2} + A_0$ in general and the component of (6) in $A_1$ is $\frac{1}{2}(yx), = 0$, giving $A_1B_{1/2} \subset B_{1/2} + A_0$. This shows that the first term in (6) is zero, which implies that $(yw)(2R_e - I)(R_e - 7/2I) = 0$, or $A_1C_{1/2} \subset A_{1/2}$. Similarly, letting $y \in A_0, x \in B_{1/2}$ in (5) yields
(y) \left[ 2R_e^3 - 4R_e^2 + \frac{5}{2} R_e - \frac{1}{2} I \right] + (yw) \left[ 2R_e^2 + 4R_e - \frac{5}{2} I \right] = 0 ,

from which one gets $A_0B_{1/3} \subset B_{1/2} + A_1$ and $A_0C_{1/3} \subset A_{1/3}$.

Finally, let $x, y \in B_{1/3}$, $x(R_e - 1/2I) = w$, $y(R_e - 1/2I) = z$ in (5) to get

$$(yx) \left[ (2R_e^3 - 5R_e^2 + R_e) + (2R_e^2 + 2R_e + I) \right] + \left( R_e - \frac{5}{2} I \right) + \frac{1}{2} I$$

$$+ \left( -3R_e + \frac{1}{2} I \right) + \frac{1}{2} I$$

$$+ (yw + zx) \left[ (2R_e^2 + 2R_e + I) \right]$$

$$+ (2R_e - 5I) + \frac{3}{2} I + (-6R_e + I) + \frac{3}{2} I$$

$$+ (zw) \left[ (-12R_e + 2I) + 4I \right] = 0 ,$$

or

(7) $$(yx) \left[ 2R_e^3 - 3R_e^2 + R_e \right] + (yw + zx) \left[ 2R_e^2 - 2R_e \right]$$

$$+ (zw) [-12R_e + 6I] = 0 .$$

Taking $w = z = 0$ in (7), we obtain first the relation $A_{1/3}A_{1/3} \subset A_1 + A_{1/3} + A_0$. If only $z$ is zero, then the component of (7) in $B_{1/3}$ is $(yx)_{1/3}[2R_e - I] + 2(yw)_{1/3}(R_e - I) = 0$, showing that $A_{1/3}C_{1/3} \subset A_1 + C_{1/2} + A_0$. If neither $w$ nor $z$ is zero, we may apply the operator $(2R_e - I)^3$ to (7) to get $(zw)(2R_e - I)^3 = 0$, or $C_{1/2}C_{1/2} \subset B_{1/3}$. But since $C_{1/2}C_{1/2} \subset A_{1/3}C_{1/2} \subset A_1 + C_{1/2} + A_0$, we have $C_{1/3}C_{1/3} \subset C_{1/2}$ to finish the proof of Theorem 1.

By constructing examples, it is not difficult to show that the relations given in Theorem 1 cannot be improved. To illustrate this procedure, we shall show that the relation $A_0A_0 \subset A_{1/3}$ cannot be improved. Consider the commutative algebra spanned by the four elements $e, a_{1/3}, a_0$ over any field $F$, and let multiplication be defined by $e^2 = e$, $a_{1/3}a_{1/3} = a_{1/3}$, $ea_{1/3} = i\bar{a}_{1/3}$ ($i = 0, 1/2, 1$), where all other products of basis elements are assumed to be zero. To show that this algebra satisfies (1), it is sufficient to show that the complete linearization of (1) is satisfied for all ways of replacing the variables by basis elements. If either four or five of these variables are replaced by $e$, the equation is satisfied by (3). If exactly three of the variables are replaced by $e$ and the other two variables by $a_i$ and $a_0$ respectively, then the equation reduces to $(a_ia_0)[2R_e^3 - 3R_e^2 + 5R_e - 2I] = 0$ as in the proof of Theorem 1, and hence is satisfied. If any other combination of basis elements is substituted into the linearized form of (1), it is clear that every term will vanish, and the identity will be trivially satisfied.

Suppose now that a ring $A$ satisfying (1) contains two orthogonal
idempotents $u$ and $v$. Although the elements of of $A_u(1)$ are not in general orthogonal to the elements of $A_u(1)$, we can prove that $v$ is orthogonal to $A_u(1)$.

**Lemma 1** If $u$ and $v$ are orthogonal idempotents, then $A_u(1) \subset A_v(0)$.

For the proof of this lemma we linearize (2) so that two of the $x$'s in each term become $u$'s and the other two become $v$'s. This gives

$$
4((yu\cdot u)v)v + 4((yu\cdot v)u)v + 4((yu\cdot v)v)u + 4((yv\cdot u)u)v \\
+ 4((yv\cdot u)v)u + 4((yv\cdot v)u)u + 2(yu\cdot v)v + 2(yv\cdot u)u \\
+ 2(yu\cdot u)v + 2(yv\cdot v)u + yu\cdot v + yv\cdot u \\
= 12(yu\cdot v)v + 12(yv\cdot u)v .
$$

Taking $y \in A_u(1)$ and using the relation $yv\cdot u = 1/2yv$ which follows from Theorem 1, this becomes $(yv\cdot v)[4R_u^2 + 8R_u + 3I] = 2yv$, or

$$
(9) \quad (yv\cdot v)[2R_u + I)(2R_v + 3I)] = 2yv .
$$

Since $yv \in A_u(1/2)$, we see from (9) that $(yv\cdot v) \in A_u(1/2)$ also. But then (9) reduces to $8yv\cdot v = 2yv$, or $(yv)[4R_v - I] = 0$. Thus, $yv = 0$ and $A_u(1) \subset A_v(0)$ as desired.

We are now ready to consider how the decomposition of $A$ with respect to the idempotent $u+v$ is related to the decompositions with respect to $u$ and $v$ separately. We shall prove.

**Theorem 2** Let $u$ and $v$ be orthogonal idempotents in a ring $A$ satisfying (1). Then $R_u R_v = R_v R_u$ and

$$
A_{u+v}(1) = A_u(1) + B_u \left( \frac{1}{2} \right) \cap B_v \left( \frac{1}{2} \right) + A_v(1) ,
$$

$$
B_{u+v} \left( \frac{1}{2} \right) = B_u \left( \frac{1}{2} \right) \cap A_v(0) + A_u(0) \cap B_v \left( \frac{1}{2} \right) ,
$$

$$
A_{u+v} \left( \frac{1}{2} \right) = A_u \left( \frac{1}{2} \right) \cap A_v(0) + A_u(0) \cap A_v \left( \frac{1}{2} \right) ,
$$

$$
C_{u+v} \left( \frac{1}{2} \right) - C_u \left( \frac{1}{2} \right) \cap A_v(0) + A_u(0) \cap C_v \left( \frac{1}{2} \right) ,
$$

$$
A_{u+v}(0) = A_u(0) \cap A_v(0) .
$$

For the proof of Theorem 2 we shall need

**Lemma 2** If $u$ and $v$ are orthogonal idempotents and if $y \in B_u(1/2) \cap B_v(1/2)$, then $yv \in B_u(1/2) \cap B_v(1/2)$, $yu\cdot v = yv\cdot u = 1/4y$, and $y \in A_{u+v}(1)$. Hence, $A_u(1/2) \cap B_v(1/2) = A_v(1/2) \cap A_u(1/2)$.
By Theorem 1, we have \( yv \in B_u(1/2) + A_u(1) \) and hence \((yv)(2R_u - I)^2 \in A_u(1) \subset A_u(0)\). On the other hand, \( yv \in B_u(1/2) \), giving \((yv)(2R_u - I)^2 \in B_u(1/2) + A_u(1)\). Thus, \((yv)(2R_u - I)^2 = 0\), or \((yv) \in B_u(1/2)\), to give the first assertion of the lemma.

From Theorem 1 we also get the relation \( y(2R_u - I)R_u(2R_u - I) = 0\), or \( 4(uy\cdot v)u = 2yu\cdot v + 2yv\cdot u - yv \). Using this relation and \( 4yu\cdot u = 4yu - y \), equation (8) with \( y \in B_u(1/2) \cap B_u(1/2) \) becomes

\[
4(uy\cdot v)v - yv\cdot v + 2(uy\cdot v)v + 2(uy\cdot u)v - yv\cdot v + 4(uy\cdot u)v \\
- yu\cdot u + 4(uy\cdot u)v - yv\cdot v + 2(uy\cdot u)v + 2(yv\cdot v)v + 2(uy\cdot u)v + 2(yu\cdot u)v \\
+ 4(yv\cdot u) - yv\cdot v + 2(yv\cdot v)v + 2(yv\cdot u)v + 2(yu\cdot u)v \\
+ 2(yv\cdot v)u + yu\cdot v + yv\cdot u - 12(yu\cdot v)u - 12(yv\cdot u)v = 0 ,
\]

or

\[
8(uy\cdot v)v - 4(uy\cdot u)v + 8(yv\cdot v)u + 2(uy\cdot u)v - 8(uy\cdot v)u \\
+ 2(yu\cdot u)v + yu\cdot v + yv\cdot u - 5yv\cdot v - yu\cdot u = 0 .
\]

Reducing this equation again given

\[
8yu\cdot v - 2yu - 2yu\cdot v - 2yv\cdot u + yu + 8yv\cdot u - 2yu + 2yv\cdot u \\
- \frac{1}{2}yv - 4yu\cdot v - 4yv\cdot u + 2yv + 2yu\cdot v - \frac{1}{2}yv + yu\cdot v \\
+ yv\cdot u - 5yv + \frac{5}{4}y - yu + \frac{1}{4}y = 0 ,
\]

or \( 5yu\cdot v + 5yv\cdot u - 4yu - 4yv + 3/2y = 0\), which may be put in the form

\[
y\left[ (R_u - \frac{1}{2}I)(5R_u - \frac{3}{2}I) + (R_u - \frac{1}{2}I)(5R_u - \frac{3}{2}I) \right] = 0 .
\]

If \( y \in A_u(1/2) \cap B_u(1/2) \), then (10) reduces to \( y(R_u - 1/2I)(5R_u - 3/2I) = 0\), or \( y \in A_u(1/2) \). Thus \( A_u(1/2) \cap B_u(1/2) = A_u(1/2) \cap A_u(1/2) \). But then \( y \in B_u(1/2) \cap B_u(1/2) \) implies that \( y(R_u - 1/2I) \in A_u(1/2) \cap A_u(1/2) \) and \( y(R_u - 1/2I)(5R_u - 3/2I) = y(R_u - 1/2I) \). Using this relation, (10) reduces to \( y[R_u - 1/2I + R_u - 1/2I] = 0 \), or \( y \in A_u(1/2) \). Since \( y(R_u - 1/2I)R_u = 1/2y(R_u - 1/2I) \), we also have \( yu\cdot v = 1/2yu\cdot v + 1/2yu\cdot v - 1/4y = 1/2y(u + v) - 1/4y = 1/4y \). And finally, \( yv\cdot u = 1/4y \) by symmetry.

Returning to the proof of the theorem, let \( y \) be an arbitrary element of \( A_u(1) \) and let \( y = y_1 + y_{1/2} + y_0 \) be its decomposition with respect to \( u \). Then the equation \( y(u + v) = y \) gives \( y_1 + y_{1/2}(u + v) + y_0v = y_1 + y_{1/2} + y_0 \), which breaks into the two equations \( y_{1/2}(u + v) = y_{1/2} \)
and $y_0v = y_0$ since $y_{1/2}(u + v) \in B_u(1/2) + A_u(1)$ and $y_0v \in A_u(0)$. Thus, $y_0 \in A_u(1)$ and $y_{1/2}(2R_u - I) = -y_{1/2}(2R_u - I) \in A_u(1/2) \cap A_{u+0}(1)$, leading to $y_{1/2}(2R_u - I)^2 = y_{1/2}(2R_u - I)[2(R_u + R_v) - I - 2R_u] = y_{1/2}(2R_u - I)(I - 2R_u) = 0$ and $y_{1/2} \in B_u(1/2)$. We have shown that $A_{u+0}(1)$ is contained in $A_u(1) + B_u(1/2) \cap B_v(1/2) + A_v(1)$. Conversely, $A_u(1)$ and $A_v(1)$ are clearly in $A_{u+0}(1)$, while $B_u(1/2) \cap B_v(1/2)$ is in by Lemma.

Next, suppose that $y \in B_{u+0}(1/2)$ and let $y = y_1 + y_{1/2} + y_0$ again be the decomposition of $y$ with respect to $u$. Then,

$$0 = (y_1 + y_{1/2} + y_0) \left[ (R_u + R_v)^2 - (R_u + R_v) + \frac{1}{4} I \right] = \frac{1}{4} y_1$$

and breaking this equation into components gives $1/4 y_1 + y_{1/2} [R_u R_v + R_v R_u + R^2_v - R_u] = 0$, and $y_0 [R^2_v - R_v + 1/4 I] = 0$ or $y_0 \in B_v(1/2)$. Letting $y_{1/2} = w_1 + w_{1/2} + w_0$ be the decomposition of $y_{1/2}$ with respect to $v$, the former equation becomes $1/4 y_1 + w_{1/2} [R_u R_v + R_v R_u + R^2_v - R_u] = 0$. But $1/4 y_1$ is the only term in the last equation with a component in $4(0)$, so that $y_1 = 0$ and $y_{1/2} \in B_{u+0}(1/2)$. By symmetry, $w_1 = 0$ and $w_0 \in B_v(1/2)$, giving $w_{1/2} = (y_{1/2} - w_0) \in B_u(1/2) \cap B_v(1/2)$. Then Lemma

$$0 = w_{1/2} [R_u R_v + R_v R_u + R^2_v - R_v] = w_{1/2} \left[ \frac{1}{4} I + \frac{1}{4} I - \frac{1}{4} I \right] \frac{1}{4} w_{1/2},$$

showing that $y_{1/2} = w_0 \in A_v(0)$. This proves that $B_{u+0}(1/2)$ is contained in $B_u(1/2) \cap A_u(0) + A_v(0) \cap B_v(1/2)$, and the converse is immediate.

If $y \in A_{u+0}(1/2)$, the argument above shows that $y = y_{1/2} + y_0$ where $y_{1/2} \in B_u(1/2) \cap A_u(0)$ and $y_0 \in A_v(0) \cap B_v(1/2)$. Then, $0 = (y_{1/2} + y_0) [R_u + R_v - 1/2 I] = y_{1/2} (R_u - 1/2 I) + y_0 (R_v - 1/2 I)$, and breaking into components gives $y_{1/2} \in A_u(1/2)$ and $y_0 \in A_v(1/2)$. Hence $A_{u+0}(1/2)$ is contained in $A_u(1/2) \cap A_u(0) + A_u(0) \cap A_v(1/2)$, and the converse is obvious. If $z \in C_{u+0}(1/2)$, then there exists an element $y \in B_{u+0}(1/2)$ such that $z = y [R_u + R_v - 1/2 I]$. Then $z = (y_{1/2} + y_0) [R_u + R_v - 1/2 I] = y_{1/2} (R_u - 1/2 I) + y_0 (R_v - 1/2 I) \in C_u(1/2) \cap A_u(0) + A_u(0) \cap C_v(1/2)$, and the converse is again obvious.

Finally, let $y \in A_{u+0}(0)$ and let $y = y_1 + y_{1/2} + y_0$ be the decomposition of $y$ with respect to $u$. Then $0 = y(u + v) = y_1 + y_{1/2} (u + v) + y_0 v = 0$, giving $y_0 \in A_v(0)$ and $y_1 + y_{1/2} (u + v) = 0$. If $y_{1/2} = w_1 + w_{1/2} + w_0$ is the decomposition of $y_{1/2}$ with respect to $v$, the latter equation gives $y_1 + w_1 + w_{1/2} (u + v) + w_0 u = 0$, and the component of this equation in $A_v(0)$ is $y_1 + w_0 u = 0$. But then $w_0 \in A_u(1) + A_u(0)$, so that
0 = y_{1/2}(4R_u^2 - 4R_u + I) = w_1 + w_{1/2}(4R_u^2 - 4R_u + I) + w_0. The component of the last equation in \( A_u(0) \) is \( w_0 = 0 \), implying that \( y_1 = 0 \) and that \( y_{1/2} \in A_{u+\alpha}(0) \). By symmetry, we also have \( w_1 = 0 \), so that \( y_{1/2} = w_{1/2} \in B_u(1/2) \cap B_\alpha(1/2) \subset A_{u+\alpha}(1) \). Thus, \( y_{1/2} = 0 \), and \( A_{u+\alpha}(0) \subset A_u(0) \cap A_\alpha(0) \). The converse of this inclusion is trivial.

The relation \( R_u R_\alpha = R_\alpha R_u \) was shown to hold on elements of \( B_u(1/2) \cap B_\alpha(1/2) \) in Lemma 2, and it is easy to check that it also holds for elements of each of the other additive subgroups into which we have decomposed \( A \).

Now that we have established Theorem 2, it is an easy matter to decompose \( A \) simultaneously with respect to any number of mutually orthogonal idempotents.

**Theorem 3** Let \( e_1, e_2, \cdots, e_n \) be a set of orthogonal idempotents in a ring \( A \) satisfying (1) whose sum is the unity element of \( A \), and define \( A_i = A_{e_i}(1) \), \( A_{ij} = A_{e_i}(1/2) \cap A_{e_j}(1/2) \), \( B_{ij} = B_{e_i}(1/2) \cap B_{e_j}(1/2) \), and \( C_{ij} = C_{e_i}(1/2) \cap C_{e_j}(1/2) \) for \( 1 \leq i, j \leq n \) and \( i \neq j \). Then \( A \) is the additive direct sum of the \( A_i \)'s and the \( B_{ij} \)'s, and \( A_i A_j \subset A_{ij} \), \( A_i B_{ij} \subset B_{ij} + A_j \), \( A_i C_{ij} \subset A_{ij} \), \( B_{ij} B_{jk} \subset B_{ik} \), \( A_i A_j k \subset A_{ij} \), \( B_{ij} C_{jk} \subset C_{ik} \), and \( C_{ij} C_{jk} = A_i B_{jk} = B_{ij} B_{kl} = 0 \) for \( 1 \leq i, j, k, l \leq n \) and \( i, j, k, l \) distinct.

The first eight inclusion relations listed in this theorem follow immediately from Theorem 1. To show \( B_{ij} B_{jk} \subset B_{ik} \), we let \( u = e_i + e_j \) and \( w = e_i + e_j + e_k \) and observe that \( B_{ij} B_{jk} \subset B_{u}(1/2) + A_{u}(0) \) and \( B_{ij} B_{jk} \subset A_{u}(1) \), leading to \( B_{ij} B_{jk} \subset B_{ik} + B_{jk} + A_k \). But, by symmetry, we also have \( B_{ij} B_{jk} \subset B_{ik} + B_{ij} + A_k \). But, by symmetry, we also have \( B_{ij} B_{jk} \subset B_{ik} + B_{ij} + A_k \), giving \( B_{ij} B_{jk} \subset B_{ik} \). This same calculation also shows that \( C_{ij} C_{jk} \subset B_{ik} \). However, \( C_{ij} C_{jk} = C_{ij} + C_{jk} \), giving \( C_{ij} C_{jk} = 0 \). Looking at the product \( A_i B_{jk} \) with respect to the three idempotents \( e_i, e_j, e_k \), we get that this product is contained respectively in \( A_{ij} + A_{ik} \), \( B_{jk} + B_{ij} + A_j \), and \( B_{jk} + B_{ik} + A_k \). Since the mutual intersection of three is zero, \( A_i B_{jk} = 0 \). Observing that \( B_{ij} B_{kl} \subset A_{u}(1)B_{kl} \) for \( u = e_i + e_j \), we also have \( B_{ij} B_{kl} = 0 \).

For the two remaining inclusion relations given in Theorem 3, we must make a little longer calculation. Linearing (2) completely and setting two of the variables equal to \( e_i \), and the other three equal to \( e_j, x, y \) respectively where \( x \in B_{ij} \) and \( y \in B_{jk} \), we get

\[
4((ye_j \cdot x)e_i)e_i + 4((y \cdot xe_j)e_i)e_i + 4(y(xe_i \cdot e_j))e_i + 4(y(xe_j \cdot e_i))e_i
\]
\[
4y((xe_i \cdot e_j)e_i) + 4y((xe_i \cdot e_j)e_i) + 4y((xe_j \cdot e_i)e_i + 2y(xe_i \cdot e_j)
\]
\[
2(ye_j \cdot xe_i)e_i + 2(y \cdot xe_i)e_i + 4(ye_j)(xe_i \cdot e_i) + 2(ye_j)(xe_i)
\]
\[
= 24(ye_j \cdot xe_i)e_i + 12y(xe_j \cdot e_i).
\]
Using the relation \( xe_i \cdot e_j = xe_j \cdot e_i = 1/4 \) from Lemma 2, this reduces to
\[
(ye_j \cdot x + xe_j \cdot y)[4R_{ei}^2 + 2R_{ei}] + 3xe_i \cdot y + (yx)[2R_{ei} - \frac{5}{2}I] \\
+ 4(ye_j)(xe_i \cdot e_i) + 2(ye_x)(xe_i) - 24(ye_j \cdot xe)e_i = 0.
\]
Letting \( xe_j = 1/2x + w \) and \( ye_j = 1/2y + z \), and noting that \( xe_i = x - xe_j = 1/2x - w \) and that \( zw = 0 \), our equation becomes
\[
(yx)[4R_{ei}^2 - 2R_{ei}] + (zx)[4R_{ei}^3 - 10R_{ei} + 2I] \\
+ (yw)[4R_{ei}^2 + 14R_{ei} - 6I] = 0.
\]
Since \( yx, zx, \) and \( yw \) are all in \( B_{ik} \), we may replace \( 4R_{ei}^2 \) by \( 4R_{ei} - I \) here, giving
\[
(11) \quad (yx)[2R_{ei} - I] + (zx)[- 6R_{ei} + I] + (yw)[18R_{ei} - 7I] = 0,
\]
Applying the operator \( (2R_{ei} - I) \) to (11), we get
\[
(zx)[- 12R_{ei}^2 + 8R_{ei} - I] + (yw)[36R_{ei}^2 - 32R_{ei} + 7I] = 0,
\]
which reduces to \( (zx)[- 4R_{ei} + 2I] + (yw)[4R_{ei} - 2I] = 0 \), or \( (yw - zx) \in A_{ik} \). On the other hand, we may set \( e = e_i \) in (7) to obtain
\[
(yx)[2R_{ei} - I] - 2zx + 2yw = 0,
\]
which simplifies to \( (yx)[1/2R_{ei} - 1/4I] + (yw + zx)[- 1/2I] = 0 \), or \( (yw + zx) \in C_{ik} \). Thus, \( yw \) and \( zx \) are both in \( A_{ik} \), and (11) reduces to \( (yx)[2R_{ei} - I] - 2zx + 2yw = 0 \), or \( (yw - zx) \in C_{ik} \). We finally have \( zx \in C_{ik} \), giving the relation \( B_{ij}C_{jk} \subset C_{ik} \). The remaining relation \(- A_{ij}A_{ik} \subset A_{ik} \) may be derived by taking \( z = w = 0 \) in (11).

2. This section will be devoted to the proof of

**Theorem 4.** Let \( A \) be a simple ring satisfying (1) and containing two orthogonal idempotents \( u \) and \( v \) such that \( u + v \) is not the unity element of \( A \) and such that \( B_u(1/2) = A_u(1/2) \) and \( B_v(1/2) = A_v(1/2) \). Then \( A \) is a Jordan ring.

If \( A \) doesn’t contain a unity element, then we may adjoin one and the resulting ring will still satisfy the same identity [3, Theorem 1]. It is therefore sufficient to prove the theorem for a ring \( R \) which contains a unity element and which is either simple or is the result of adjoining a unity element to a simple ring. In the latter case, every ideal of the augmented ring contains the original ring [1, Lem. 2, p. 506], and in either case, the idempotents \( e_i = u \), \( e_i = v \), and \( e_i = 1 - u - v \) are mutually orthogonal idempotents of \( R \) which add to the unity element. Adopting the terminology of Theorem 3, we see from the
last sentence of Lemma 2 that the remaining hypotheses of Theorem 4 are equivalent to the relations $B_{ij} = A_{ij}$ for $1 \leq i, j \leq 3$ and $i \neq j$.

We must next deduce more information from our identity about the products of elements from different components of $R$. Linearizing (1) completely, replacing two of the variables by the idempotent $e = e_i (i = 1, 2, 3)$, and assuming that the other three variables satisfy $xe = \lambda x, ye = \mu y$, and $ze = \nu z$, we obtain

\begin{align*}
[yz \cdot x + yx \cdot z + xz \cdot y]R_e^2 + [(yz \cdot e)x + (yx \cdot e)z + (xz \cdot e)y]R_e \\
+ [(yz)R_e^2 \cdot x + (yx)R_e^2 \cdot z + (xz)[R_e^2 \cdot y] \\
(12) + \left[(\mu + \nu - 6\lambda + \frac{1}{2})(yz \cdot x) + (\lambda + \mu - 6\nu + \frac{1}{2})(yx \cdot z) \\
+ (\nu - \lambda - 6\mu + \frac{1}{2})(x \cdot y)R_e \\
+ \left(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \lambda\nu - 6\mu\nu + \frac{1}{2}\lambda \\
+ \frac{1}{2}\mu + \frac{1}{2}\nu\right)yz \cdot x + (\lambda^2 + m^2 + \nu^2 \\
+ \lambda\mu + \mu\nu - 6\lambda\mu + \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu)yx \cdot z \\
+ \left(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \mu\nu - 6\lambda\nu + \frac{1}{2}\lambda + \frac{1}{2}\mu \\
+ \frac{1}{2}\nu\right)xz \cdot y = 0 \, .
\end{align*}

We first set $\lambda = \mu = 0, \nu = 1$ in this equation to get

\begin{align*}
(yz \cdot x + xz \cdot y)\left[R_e^2 + \frac{3}{2}R_e + \frac{3}{2}I\right] + [(yz \cdot e)x + (xz \cdot e)y][R_e - 2I] \\
+ (yz)R_e^2 \cdot x + (xz)R_e^2 \cdot y + (yx \cdot z)\left[R_e^2 - \frac{11}{2}R_e + \frac{3}{2}I\right] = 0 \, ,
\end{align*}

which reduces to

\begin{align*}
(yz \cdot x + xz \cdot y)\left[R_e^2 + 2R_e + \frac{3}{4}I\right] + (yx \cdot z)\left[R_e^2 - \frac{11}{2}R_e + \frac{3}{2}I\right] = 0 \, .
\end{align*}

Separating this equation into components and using the convention that the subscript $1, 1/2, \text{ or } 0$ indicates the component in $A_4(1), A_4(1/2), \text{ or } A_4(0)$ respectively, the last equation yields

\begin{align*}
(13) \quad 2[yz \cdot x + xz \cdot y]_{1/2} = yx \cdot z \, .
\end{align*}

Next, setting $\lambda = \mu = 0, \nu = 1/2$ in (12) gives
\[(yz \cdot x + xz \cdot y) \left[ R_e^3 + R_e + \frac{1}{2} I \right] + (yx \cdot z) \left[ R_e^3 - \frac{5}{2} R_e + \frac{1}{2} I \right] + [(yz) x + (xz) y][R_e - 5/2 I] + [(yz) R_e^2 \cdot x + (xz) R_e^2 \cdot y] = 0 , \]

which becomes
\[
[(yz)_1 \cdot x + (xz)_1 \cdot y][R_e^2 + 2R_e - I] + [(yz)_{1/2} \cdot x + (xz)_{1/2} \cdot y][R_e^2 + 3/2R_e - 1/2 I] + (yx \cdot z)[R_e^2 - 5/2R_e + 1/2 I] = 0 .
\]

This separates into the two equations
\[
2[(yz)_{1/2} \cdot x + (xz)_{1/2} \cdot y] = [yx \cdot z]_1 , 
\]
\[
[(yz)_1 \cdot x + (xz)_1 \cdot y] + 2[(yz)_{1/2} \cdot x + (xz)_{1/2} \cdot y]_{1/2} = 2[yx \cdot z]_{1/2} .
\]

The equations that we have just derived may be put in operator form by defining for each \(x \in A_e(0)\) the mappings \(S_x : A_e(1) \rightarrow A_e(1/2)\), \(T_x : A_e(1/2) \rightarrow A_e(1)\), and \(U_x : A_e(1/2) \rightarrow A_e(1)\) by the equations \((z_1) S_x = xx\), \((z_{1/2}) T_x = (xx)_1\), and \((z_{1/2}) U_x = (xx)_{1/2}\) respectively. In this notation, equations (13) — (15) become
\[
\frac{1}{2} S_{yx} = S_y U_x + S_x U_y ,
\]
\[
\frac{1}{2} T_{yx} = U_x T_x + U_x U_y ,
\]
\[
U_{yx} = U_y U_x + U_x U_y + \frac{1}{2} (T_y S_x + T_x S_y) .
\]

We shall make use of these relations to prove.

**Lemma 3** In the ring \(R\), \(A_{ij}A_{ij} \subset A_i + A_j\) for \(1 \leq i, j \leq 3\) and \(i \neq j\).

Choosing \(e\) to be that one of \(e_i, e_j, e_k\) which is neither \(e_i\) nor \(e_j\), we see that \(A_e(0) = A_i + A_{ij} + A_j\). Consider the subalgebra of \(A_e(0)\) defined by \(D = \{x | x \in A_o, S_x = T_x = 0\}\). By (18), the mapping \(x \rightarrow U_x\) defines a homomorphism of \(D\) into the Jordan ring of all endomorphisms of \(A_e(1/2)\) with kernel \(C = \{x | x \in A_o, S_x = T_x = U_x = 0\}\). If \(x \in C\) and \(y \in A_o\), then \(S_{yx} = T_{yx} = U_{yx} = 0\) by (16), (17), and (18), so that \(C\) is an ideal of \(A_o\). Furthermore, \(CA_e(1) = CA_e(1/2) = 0\) by the definition of \(S, T,\) and \(U,\) showing that \(C\) is an ideal of \(R\). But \(R\) contains no nonzero ideals lying within \(A_e(0)\), implying that \(C = 0\) and that \(D\) is a Jordan ring.

Since \(e_i\) and \(e_j\) are contained in \(D\), we have \(D = D_i + D_{ij} + D_j,\)
where $D_i \subset A_i$, $D_{ij} \subset A_{ij}$, and $D_j \subset A_j$. The fact that $D$ is Jordan implies that $D_{ij}D_{ij} \subset D_i + D_j \subset A_i + A_j$. But since $A_{ij}A_i(1) = 0$ and $A_{ij}A_i(1/2) \subset A_i(1/2)$, we see that $A_{ij} \subset D_i$ giving $A_{ij} \subset D_{ij}$ and $A_{ij}A_{ij} \subset D_{ij}D_{ij} \subset A_i + A_j$.

In order to prove our next lemma, we need to compute two more special cases of (12). Using Lemma 2, we may now assume that $A_i(1/2)A_i(0) \subset A_i(1) + A_i(0)$. First, taking $\lambda = 0$, $\mu = 1/2$, $\nu = 1$ and saving just the component in $A_i(0)$ gives

$$
- \frac{3}{2} \left[ \frac{1}{2} yx \cdot z + \frac{1}{2} xz \cdot y \right]_0 + \left[ \frac{1}{4} yx \cdot z + \frac{1}{4} xz \cdot y \right]_0
+ \left[ -yz \cdot x + \frac{5}{2} yx \cdot z + \frac{5}{2} xz \cdot y \right]_0 = 0,
$$

or

(19)

$$
2[y_{1/2}x_0 \cdot z_1 + x_0z_1 \cdot y_{1/2}]_0 = (y_{1/2}z_0)_0 \cdot x_0.
$$

Secondly, setting $\lambda = 0$, $\mu = \nu = 1/2$ in (12) and keeping just the component in $A_i(0)$, we get

$$
- \frac{2}{2} \left[ \frac{1}{2} (yx)_{1/2} \cdot z + (yx)_{1/2} \cdot z + \frac{1}{2} (xz)_{1/2} \cdot y + (xz)_{1/2} \cdot y \right]_0
+ \left[ \frac{1}{4} (yx)_{1/2} \cdot z + (yx)_{1/2} \cdot z + \frac{1}{4} (xz)_{1/2} \cdot y + (xz)_{1/2} \cdot y \right]_0
+ \left[ -\frac{1}{2} yz \cdot x + \frac{5}{4} yx \cdot z + \frac{5}{4} xz \cdot y \right]_0 = 0,
$$

which simplifies to

(20)

$$
2[(y_{1/2}x_0)_{1/2} \cdot z_{1/2} + (z_{1/2}x_0)_{1/2} \cdot y_{1/2}]_0
+ [(y_{1/2}z_0)_{1/2} \cdot x_{1/2} + (z_{1/2}y_0)_{1/2}]_0 = 2(y_{1/2}z_{1/2})_0 \cdot x_0.
$$

**Lemma 4.** Let $G_0$ be the additive subgroup of $A_0 = A_i(0)$ generated by all elements of the form $(y_{1/2}z_1)_0$ and $(y_{1/2}z_{1/2})_0$. Then either $G_0 = A_0$ or we may adjoin $e_3$ to $G_0$ to obtain $i_2$. In either case the lemma holds.

If $x_0$ is any element of $A_0$, we see from (19) that $(y_{1/2}z_1)_0 \cdot x_0$ is in $G_0$ and from (20) that $(y_{1/2}z_{1/2})_0 \cdot x_0$ is in $G_0$. Thus, $G_0$ is an ideal of $A_0$. Defining the ideal $G_1$ of $A_1$ analogously, we now consider $G = G_1 + A_{1/2} + G_0$. But $GA_1 = G_1A_1 + (A_{1/2}A_1)_{1/2} + (A_{1/2}A_1)_0 + A_0A_1 \subset G_1 + A_{1/2} + G_0 + A_{1/2} = G$, and similarly $GA_{1/2} \subset G$ and $GA_0 \subset G$. Thus, $G$ is an ideal of $R$ and is nonzero since $A_{1/2} \subset G$. It follows from the definition of $R$ that either $G = R$ or we may adjoin $e_3$ to obtain $R$. In either case the lemma holds.

We shall assume hereafter that $i, j, k$ form a permutation of $1, 2, 3,$
and we shall indicate in which part of the decomposition of $R$ an element lies by attaching the appropriate subscripts. Then, taking $e = e_j$ in Lemma 4 yields the following.

**COROLLARY.** $A_{ik}$ is generated by the elements of the form $(y_{ij}z_{jk})$ and $A_i$ is generated by $e_i$ and by the elements of the form $(y_{ij}z_{ij})$.

**LEMMA 5.** The following relations hold in $R$: $x_{ik}(y_{ij}z_{jk}) = A_{ik}z_{jk} = A_{ik}z_{jk} = A_{ik} + A_k$. On the other hand, using (19) with $e = e_j$ gives $x_{ik}(y_{ij}z_{jk}) = 2[y_{ij}x_{ik}z_j + x_{ik}z_j](y_{ij}) = 2y_{ij}x_{ik}z_j \in A_{jk}A_j \subset A_{jk} + A_k$, and combining the two relations gives $x_{ik}(y_{ij}z_{jk}) \in A_k$. Secondly, taking $e = e_k$ in (13) gives $x_{ik}(y_{ij}z_{jk}) = 2[x_{ik}y_{ij}z_{ij} + x_{ik}z_{ij}z_{ij}] = 0$. And finally, setting $e = e_j$ in (20) yields $x_{ik}(y_{ij}z_{ij}) = y_{ij}x_{ik}z_{ij} + x_{ik}z_{ij}y_{ij} \in A_{ik}$.

**LEMMA 6.** If $x_i$ is an element of $A_i$ such that $x_iA_{ik} \subset A_{ik}$, then $x_iA_{ik} \subset A_{ij}$. Similarly, $x_iA_{ik} \subset A_k$ implies $x_iA_{ij} = 0$.

Suppose that $x_iA_{ik} \subset A_{ik}$. Then (20) gives $x_i(y_{ik}z_{jk}) = (x_iy_{ik})z_{jk} \in A_{ik}z_{jk} \subset A_{ij}$ to show that $x_iA_{ij} \subset A_{ij}$. On the other hand, if $x_iA_{ik} \subset A_k$, then (20) yields $(x_i(y_{ik}z_{jk}) = (x_iy_{ik})z_{jk} \in A_{ik}z_{jk} \subset A_{jk} + A_k$. However, we also have $x_i(y_{ik}z_{jk}) \in x_iA_{ij} \subset A_{ij} + A_j$, and thus $x_iA_{ij} = 0$.

We are now in a position to prove.

**LEMMA 7.** In the ring $R$, $A_iA_{ij} \subset A_{ij}$ and $A_iA_j = 0$. Hence $A_i + A_{ij} + A_j$ is a Jordan ring.

By the corollary to Lemma 4, $A_i$ is generated by $e_i$ and elements of the form $(y_{ij}z_{ij})$ and $(y_{ij}z_{ij})$. Then $(y_{ij}z_{ij})A_{ik} \subset A_k$ by Lemma 5 and so $(y_{ij}z_{ij})A_{ij} = 0$ by Lemma 6. On the other hand, Lemma 5 also gives $(y_{ij}z_{ij})A_{ik} \subset A_{ik}$, which implies $(y_{ij}z_{ij})A_{ij} \subset A_{ij}$ by Lemma 6. Hence, $A_iA_{ij} \subset A_{ij}$, $(y_{ij}z_{ij})i = 0$, and $A_i$ is generated by $e_i$ and elements of the form $(y_{ij}z_{ij})$. But then $A_iA_k = 0$ by the second relation of Lemma 5. The relations which we have just established show that the Jordan ring $D = \{x \mid x \in A_{e_k}(0), xA_e(1) = 0, xA_e(1/2) \subset A_{e_k}(1/2)\}$ used in the proof of Lemma 3 is all of $A_{e_k}(0) = A_i + A_{ij} + A_j$.

Now that Lemma 7 has been proved, equation (15) yields the three special cases.
while (20) yields

\[ [y_{jk}x_{ij}z_{ik}]_i = [y_{jk}z_{ik}x_{ij}]_i. \]

Since \( A_i + A_{ij} + A_j \) is a Jordan ring, we also have

\[ z_{ij}x_i y_i = z_{ij}x_i y_i + z_{ij}y_i x_i, \]
\[ y_{ij}z_{ij}x_i = [y_{ij}z_{ij}x_i + z_{ij}x_i y_{ij}]_i, \]
\[ [y_{ij}x_i z_{ij}]_j = [z_{ij}x_i y_{ij}]_j. \]

Theorem 4 may now be established by verifying that the linearized Jordan identity is satisfied for all possible ways of choosing the arguments in the various components of \( R \). These calculations all proceed easily using Theorem 3, Lemma 3, Lemma 7, and equations (21)–(23). However, this computation may be avoided by appealing to [2, Theorem 5], which states that a certain set of hypotheses implied by the properties that we have established for \( R \) implies power-associativity. It should be remarked that Mrs. Losey's theorem is stated only for simple algebras in which the decomposition is well behaved with respect to any idempotent in the algebra. However, her proof actually establishes the theorem for simple rings containing a unity element or with unity element adjoined in which properties about the decomposition with respect to an idempotent are only assumed for three particular idempotents which add to the unity element.

REFERENCES

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Homer Franklin Bechtell, Jr., *Pseudo-Frattini subgroups* ........................................ 1129
Thomas Kelman Boehme and Andrew Michael Bruckner, *Functions with convex means* ................................................................. 1137
Lutz Bungart, *Boundary kernel functions for domains on complex manifolds* ........ 1151
L. Carlitz, *Rings of arithmetic functions* ................................................................. 1165
D. S. Carter, *Uniqueness of a class of steady plane gravity flows* ......................... 1173
Richard Albert Dean and Robert Harvey Oehmke, *Idempotent semigroups with distributive right congruence lattices* ......................................... 1187
Lester Eli Dubins and David Amiel Freedman, *Measurable sets of measures* .... 1211
Robert Pertsch Gilbert, *On class of elliptic partial differential equations in four variables* ................................................................. 1223
Harry Gonshor, *On abstract affine near-rings* ......................................................... 1237
Edward Everett Grace, *Cut points in totally non-semi-locally-connected continua* .... 1241
Edward Everett Grace, *On local properties and $G_δ$ sets* ....................................... 1245
Keith A. Hardie, *A proof of the Nakaoka-Toda formula* ........................................... 1249
Lowell A. Hinrichs, *Open ideals in $C(X)$* .............................................................. 1255
John Rolfe Isbell, *Natural sums and abelianizing* .................................................. 1265
G. W. Kimble, *A characterization of extremals for general multiple integral problems* ................................................................. 1283
Nand Kishore, *A representation of the Bernoulli number $B_n$* ................................. 1297
Melven Robert Krom, *A decision procedure for a class of formulas of first order predicate calculus* ................................................................. 1305
Peter A. Lappan, *Identity and uniqueness theorems for automorphic functions* ....... 1321
Lorraine Doris Lavallee, *Mosaics of metric continua and of quasi-Peano spaces* ..... 1327
Mark Mahowald, *On the normal bundle of a manifold* ............................................ 1335
J. D. McKnight, *Kleene quotient theorems* ............................................................. 1343
Charles Kimbrough Megibben, III, *On high subgroups* ............................................ 1353
Philip Miles, *Derivations on $B^*$ algebras* ............................................................ 1359
J. Marshall Osborn, *A generalization of power-associativity* ................................... 1367
Theodore G. Ostrom, *Nets with critical deficiency* .................................................... 1381
Elvira Rapaport Strasser, *On the defining relations of a free product* ....................... 1389
K. Rogers, *A note on orthogonal Latin squares* ....................................................... 1395
P. P. Saworotnow, *On continuity of multiplication in a complemented algebra* .... 1399
Johanan Schonheim, *On coverings* ........................................................................ 1405
Victor Lenard Shapiro, *Bounded generalized analytic functions on the torus* ....... 1413
James D. Stafney, *Arens multiplication and convolution* ......................................... 1423
Daniel Sterling, *Coverings of algebraic groups and Lie algebras of classical type* ................................................................. 1449
Alfred B. Willcox, *Šilov type $C$ algebras over a connected locally compact abelian group. II* ................................................................. 1463
Bertram Yood, *Faithful $^*$-representations of normed algebras. II* ........................ 1475
Alexander Zabrodsky, *Covering spaces of paracompact spaces* ............................... 1489