

Pacific Journal of Mathematics

ON THE DEFINING RELATIONS OF A FREE PRODUCT

ELVIRA RAPAPORT STRASSER

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If $F_n = F(x_1, \dots, x_n)$ is the free group generated by the symbols x_1, \dots, x_n , and $R_i = R_i(x_1, \dots, x_n)$, $i: 1, \dots, k$ are element in it, let

$$R = \{R_1, \dots, R_k\}$$

be their normal closure in F_n , and let

$$F/R = (x_1, \dots, x_n; R_1, \dots, R_k)$$

be the factor group of F_n by R , with n and k assumed finite.

My object is to prove the following theorem and corollaries.

THEOREM. *If*

$$F/R = (x_1, \dots, x_n; R_1, \dots, R_k)$$

and

$$F^*/R^* = (x_1^*, \dots, x_n^*; R_1^*, \dots, R_k^*)$$

are isomorphic groups and $n^ + k < n + k^*$, then in the free group F_{n+n^*} generated by $x_1^*, \dots, x_n^*, z_1, \dots, z_n$ the normal subgroup $\{R_1^*(x_1^*, \dots, x_n^*), \dots, R_k^*(x_1^*, \dots, x_n^*), z_1, \dots, z_n\}$ is normal closure of $n^* + k < n + k^*$ elements.*

The two corollaries concern the cases $k \leq 1$ and $k^* \leq 1$ — that is, free groups and groups possessing presentations on a single defining relation. (Deficiency is defined in Remark 1. below.)

COROLLARY 1. *If a group possesses the two presentations of the theorem, then the one with the lesser deficiency has at least two defining relations ($k^* > 1$).*

COROLLARY 2. *If $G = F/R$ and $k \leq 1$, then $n - k = d$ is maximal for all presentations of G .*

The theorem is trivial if for example certain of the k^* defining relations are redundant. It becomes interesting when the number k^* is least possible for the subgroup R^* . For suppose that k^* is minimal for R^* :

$$R^* = \{R_1^*, \dots, R_{k^*}^*\} \neq \{S_1, \dots, S_{k^*-1}\}$$

for any elements S_1, \dots, S_{k^*-1} of F_{n^*} . Then if z_1, \dots, z_n are new symbols and F_{n+n^*} the free group on the symbols $x_1^*, \dots, x_{n^*}^*, z_1, \dots, z_n$, the normal closure of $R_1^*, \dots, R_{k^*}^*$ in F_{n+n^*} still requires k^* elements to generate it. Consider however the normal subgroup

$$T = \{R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n\}$$

of F_{n+n^*} . The R_j^* do not involve the symbols z_i ; k^* is least possible for R^* , and, of course, n is least possible for the group

$$(z_1, \dots, z_n; z_1, \dots, z_n) = 1.$$

Now the group

$$(x_1^*, \dots, x_{n^*}^*, z_1, \dots, z_n; R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n)$$

is the free product $G_1 * G_2$ of $G_1 = F^*/R^*$ and G_2 the trivial group $(z_1, \dots, z_n; z_1, \dots, z_n)$. The theorem claims that the sum of the (minimal) numbers of defining relations for the G_i is not always minimal for $G_1 * G_2$.

Compare this with Grushko's theorem, which implies that the number of generators of a free product is the sum of those for the factors.

REMARK 1. If one takes the number k in the presentation F/R (of any group) to be least possible for R in F , then this presentation is said to have the deficiency

$$d = n - k.$$

Thus, setting $d^* = n^* - k^*$, the inequality $n^* + k < n + k^*$ is the same as

$$d^* < d,$$

provided that k is minimal for R and k^* is minimal for R^* in their respective free groups. The deficiency of a presentation is not a group invariant [1].

REMARK 2. The group with noninvariant deficiency given in [1] happens to be non-Hopfian (it is isomorphic to a proper factorgroup of itself: $G \simeq G/N$, $N \neq 1$). It is not known whether such groups must be non-Hopfian; however, two presentations, G and G/N of a non-Hopfian group may have identical deficiencies even if N is not trivial. That is, if a group is given by the presentation

$$G_1 = F/R = (x_1, \dots, x_n; R_1, \dots, R_k)$$

and is isomorphic to its own proper factorgroup given by

$$G_2 = F/R^* = (G_1; R_{k+1}) ,$$

it need not follow that $k + 1$ is minimal for R^* whenever k is minimal for R . For example, the first known pair of presentations of a non-Hopfian group, due to Higman and quoted in [2], is

$$\begin{aligned} G_1 &= (a, b, c; a^{-1}bab^{-2}, bcb^{-1}c^{-1}) \\ G_2 &= (G_1; ab^{-1}a^{-1}c^{-1}aba^{-1}c) , \end{aligned}$$

but $R^* = \{R_1, R_2, R_3\} = \{R_1, R_3\}$ (the R_i being the three defining words above in the order given).

Proof of the theorem. Set

$$x = (x_1, \dots, x_n), x^* = (x_1^*, \dots, x_n^*), f(x^*) = (f_1(x^*), \dots, f_n(x^*)) ,$$

etc., and let

$$\begin{aligned} G_1 &= F/R , \\ G_2 &= IG_1 = F^*/R^* , \end{aligned}$$

with the isomorphism I given by

$$\begin{aligned} I(x_i) &= f_i(x^*), \quad i : 1, \dots, n , \\ I(g_j(x)) &= x_j^*, \quad j : 1, \dots, n^* . \end{aligned}$$

Form the groups

$$H_1 = \overset{\circ}{F}/\overset{\circ}{R} = (x_1, \dots, x_n, y_1, \dots, y_n; R_1, \dots, R_k, y_j g_j^{-1}(x), j : 1, \dots, n^*)$$

and

$$\begin{aligned} H_2 = \overset{\circ}{F}^*/\overset{\circ}{R}^* &= (x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*; R_1^*, \dots, R_k^*, y_i^* f_i^{-1}(x^*), \\ &\quad i : 1, \dots, n) . \end{aligned}$$

These are new presentations of the same group and the following mapping, J_1 , defines, and isomorphism between H_1 and H_2 [cf. 4]:

$$\begin{aligned} J_1(x_i) &= y_i^*, \quad i : 1, \dots, n \\ J_1(y_j) &= x_j^*, \quad j : 1, \dots, n^* . \end{aligned}$$

Since

$$x_j^* = I(g_j(x)) = g_j(I(x)) = g_j(f(x^*)) \text{ modulo } R^*$$

and

$$R(f(x^*)) = 1 \text{ modulo } R^*$$

in G_2 , and hence in H_2 , one gets the following identities in H_2 :

$$J_1(R) = R(y^*) = R(f(x^*)) = 1 ,$$

$$J_1(y_j g_j^{-1}(x)) = x_j^* g_j^{-1}(y^*) = x_j^* g_j^{-1}(f(x^*)) = 1 .$$

Clearly, J_1 maps not only H_1 on H_2 , but also $\overset{\circ}{F}$ on $\overset{\circ}{F}^*$, and $\overset{\circ}{R}$ on $\overset{\circ}{R}^*$, isomorphically.

Finally let

$$H_3 = F'/R' = (x_1^*, \dots, x_n^*, z_1, \dots, z_n ; R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n) .$$

Under the transformation J_2 given by

$$J_2(y_i^*) = z_i f_i(x^*), \quad i : 1, \dots, n ,$$

$$J_2(x_j^*) = x_j^*, \quad j : 1, \dots, n^* ,$$

H_2 is mapped isomorphically on H_3 and J_2 is also an isomorphism between the free groups involved :

$$J_2(\overset{\circ}{F}^*) = F' \quad J_2(\overset{\circ}{R}^*) = R' .$$

Writing $J = J_2 J_1$ gives $JH_1 = H_3$, $J\overset{\circ}{F} = F'$, $J\overset{\circ}{R} = R'$, since J maps two free groups of equal rank (namely $n + n^*$) onto each other. Hence

$$\overset{\circ}{JR} = \{JR_1, \dots, JR_k, J(y_1 g_1^{-1}(x)), \dots, J(y_n^* g_n^{-1}(x))\}$$

$$= R' = \{R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n\} ,$$

showing that R' is the normal closure of $n^* + k$ elements in spite of the possibility that k^* is minimal for R^* and that the R_i^* contain no z -symbols.

This concludes the proof.

Proof of Corollary 1. If $k^* = 0$ then R' becomes $\{z_1, \dots, z_n\}$, the normal closure of a free factor of F' , and so requires $n = n + k^* > n^* + k$ defining elements. This contradicts the theorem, so $k^* \neq 0$.

If $k^* = 1$, then $R' = \{R_1^*, z_1, \dots, z_n\} = \{s_1, \dots, s_n\}$ obtains. Now R' contains the free factor $F(z_1, \dots, z_n)$ of F' and is the closure of n elements in F' . Therefore, by [3], $R' = \{z_1, \dots, z_n\}$; and since the R_j^* contain no z -symbols, R_1^* is empty, so $k^* = 0$. Hence $k^* > 1$.

Proof of Corollary 2. Suppose, on the contrary, that there is a presentation with deficiency $d^{**} > d$. Then k can play the role of k^* above and Corollary 1 is contradicted.

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50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$3.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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