A NOTE ON ORTHOGONAL LATIN SQUARES

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1. Introduction. The purpose of this note is to give an improved estimate for $N(n)$, the maximal number of pairwise orthogonal Latin squares, by following the method of Chowla, Erdős and Straus [2]. The difference is that we use a result of Buchstab [1] rather than that of Rademacher in the sieve argument. Our result is that if $c$ is any number less than $1/42$, then for all large $n$ we have $N(n) > n^c$.

In the notation of Buchstab, write $P_\omega(x; x^{1/\omega})$ for the number of positive integers not exceeding $x$ which do not lie in any of the progressions $a_0 \mod p_0$, $a_i \mod p_i$, or $b_i \mod p_i$, where $p_0 = 2$, and $p_i$ runs over the primes from 3 to $x^{1/\omega}$. The subscript $\omega$ refers to the fact that $P$ depends on the $a_i, b_i$. Buchstab proves that

$$P_\omega(x; x^{1/\omega}) > \frac{c'x}{(\log x)^\omega} + O\left(\frac{x}{(\log x)^\omega}\right),$$

where $c'$ is a constant 0.4161 and $\lambda(5) \geq 0.96$.

The properties of $N(n)$ used for the proof are those of [2]:

A. $N(ab) \geq \min\{N(a), N(b)\}$.

B. $N(n) \leq n - 1$, with equality when $n$ is a prime-power.

C. If $k \leq 1 + N(m)$ and $1 < u < m$, then

$$N(u + km) \geq \min\{N(k), N(k + 1), 1 + N(m), 1 + N(u)\} - 1.$$

We note that A and B are due to H. F. MacNeish, while C was found by Bose and Shrikhande.

2. Lower estimation of $N(n)$. We must deal separately with odd $n$ and even $n$, and we use a fact proven in [1], called there "Lemma D":

D. The number of integers no greater than $x$, which have a prime factor in common with $n$ and greater than $n^\omega$, is no greater than $x/n^\omega$.

Estimate for even $n$. We pick $k$ so that

$$k \equiv -1 \pmod{2 \log_{2^{1/(\log x)}}},$$

$$k \equiv 0 \text{ or } -1 \pmod{p} \text{ for } 3 \leq p \leq n^{1/\beta},$$

$$k \leq n^{1/\gamma}.$$
Since \( k = -1 + h2^{\log_2 a} \), say, we know the number of such \( k \) is 
\[ P_{\omega}\left(\frac{1 + n^{1/\gamma}}{2^{1/\log_2 a}}; n^{1/\beta}\right). \]
In view of Buchstab's theorem, we take \( 1/\gamma - 1/\alpha = 5/\beta \) and then have, for some positive constant \( c \) and all large \( n \),
\[ P_{\omega} > c \cdot \frac{n^{6/\beta}}{\log^2 n}, \]

Our \( k \) have no prime factor below \( n^{1/\beta} \), so to choose \( k \) also prime to \( n \) we must deal with the primes in \( n \) which are greater than \( n^{1/\beta} \).

By \( D \), the number of integers below \( n^{1/\gamma} \), which have a prime factor which exceeds \( n^{1/\beta} \) and divides \( n \), is at most \( n^{1/\gamma}/(1/\beta)n^{1/\beta} \). Since we want this to be less than the number of \( k \), we take \( 1/\gamma = (6-\varepsilon)/\beta \), where \( 0 < \varepsilon < 1 \). Then, for all large \( n \) we can choose \( k \) as above so as to be prime to \( n \). Note that we now have \( 1/\alpha = (1-\varepsilon)/\beta \). Since all prime factors of \( k \) exceed \( n^{1/\beta} \), and due to the restrictions on \( k+1 \), we deduce from \( A \) and \( B \) that:

\[ N(k) > n^{1/\beta} - 1 \]
\[ N(k + 1) > \min \left( \frac{1}{2}n^{1/\alpha}, n^{1/\beta} \right) - 1, \]

and we note that for all large \( n \) both these estimates exceed \( n^{1/\alpha}/3 \).

Now, since we want to have \( n = u + mk \), write
\[ n = n_1 + n_2k, \quad 0 < n_1 < k, \quad (n_1, k) = 1, \]

and
\[ u = n_1 + u_2k. \]

Now choose \( u_1 \) so that:

\[ \begin{align*}
\text{If} & \quad (3) \quad \begin{cases} u_1 \equiv n_1 \pmod{2}, \\
u_1 \equiv -n_3/k \pmod{p}, \quad p \nmid k \\
u_2 \equiv n_2 \pmod{p} \\
u_1 < n^{1/\beta}.
\end{cases}
\end{align*} \]

By Buchstab, this is all right as long as \( k \leq n^{1/\beta} \), so we choose \( 1/\delta = 5/\gamma = 5(6-\varepsilon)/\beta \). No prime less than or equal to \( k \) can divide \( u \); for \( u \) is prime to \( k \), and those primes below \( k \) which don't divide \( k \) do not divide \( u \), by (3). Hence

\[ N(u) \geq k > N(k) > \frac{1}{3}n^{1/\alpha}. \]

Finally, \( m = (n - u)/k \), of course; so \( m - u = \{n - (1 + k)u\}/k \), which
we want to make positive. Since \((1 + k)u < n^{3/7 + 1/8}\), choose \(\beta\) so that \(7 \cdot (6 - \varepsilon)/\beta < 1\), or equivalently \(1/\alpha < (1 - \varepsilon)/7(6 - \varepsilon)\). Thus we can achieve the conditions so far expressed for all large \(n\), as long as \(\alpha\) is any chosen number exceeding 42. As to \(N(m)\), note that \(m = n_2 - u_1 \equiv 0 \pmod{p}\) for \(3 \leq p \leq k\). Also \(u\) is odd, by (3), and \(n\) is even; hence \(m\) is odd. Thus

\[
N(m) \geq k > N(k) > \frac{1}{3} n^{1/\alpha}.
\]

The conditions of \(C\) apply now, and the above estimates and \(C\) imply that for any constant \(c\) less than \(1/42\) we have:

\[N(n) > n^c, \quad \text{for all large even } n.\]

**Estimate for odd } n.** This time \(k\) is chosen even, the conditions being:

\[
\begin{align*}
k + 1 &\equiv 1 \pmod{2^{\log n/\alpha}}, \\
k + 1 &\not\equiv 0 \text{ or } 1 \pmod{p} \quad \text{for } 3 \leq p \leq n^{1/\beta}, \\
k + 1 &\leq n^{1/\gamma}.
\end{align*}
\]

With obvious changes in detail from the previous case, we still get \(\operatorname{Min}\{N(k), N(k + 1)\} > 1/3(n)^{1/\alpha}\), and \((n, k) = 1\). This time, the relation \(n - u = (n_2 - u_1)k\) ensures that \(u\) is odd, but we must adjust the parity condition on \(u_1\) to ensure that \(m\) is odd:

\[
\begin{align*}
u_1 &\not\equiv n_2 \pmod{2} \\
u_1 &\not\equiv -n_3/k \pmod{p}, \quad \text{for } p \nmid k, \\
u_1 &\equiv n_3 \pmod{p} \\
u_1 &< n^{1/3}.
\end{align*}
\]

Thus \(m = n_2 - u_1\) is odd, and now the details are as before, giving finally the following result.

**Theorem.** To each number \(c\) which is less than \(1/42\), there corresponds an integer \(n_0 = n_0(c)\), such that for all \(n > n_0\) we have

\[N(n) > n^c.\]

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