

Pacific Journal of Mathematics

A NOTE ON ORTHOGONAL LATIN SQUARES

K. ROGERS

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KENNETH ROGERS

1. **Introduction.** The purpose of this note is to give an improved estimate for $N(n)$, the maximal number of pairwise orthogonal Latin squares, by following the method of Chowla, Erdős and Straus [2]. The difference is that we use a result of Buchstab [1] rather than that of Rademacher in the sieve argument. Our result is that if c is any number less than $1/42$, then for all large n we have $N(n) > n^c$.

In the notation of Buchstab, write $P_\omega(x; x^{1/a})$ for the number of positive integers not exceeding x which do not lie in any of the progressions $a_0 \pmod{p_0}$, $a_i \pmod{p_i}$, or $b_i \pmod{p_i}$, where $p_0 = 2$, and p_i runs over the primes from 3 to $x^{1/a}$. The subscript ω refers to the fact that P depends on the a_i, b_i . Buchstab proves that

$$(1) \quad P_\omega(x; x^{1/a}) > \lambda(a) \frac{c'x}{(\log x)^2} + o\left(\frac{x}{(\log x)^3}\right),$$

where c' is a constant 0.4161 and $\lambda(5) \geq 0.96$.

The properties of $N(n)$ used for the proof are those of [2]:

- A. $N(ab) \geq \text{Min} \{N(a), N(b)\}$.
- B. $N(n) \leq n - 1$, with equality when n is a prime-power.
- C. If $k \leq 1 + N(m)$ and $1 < u < m$, then

$$N(u + km) \geq \text{Min} \{N(k), N(k + 1), 1 + N(m), 1 + N(u)\} - 1.$$

We note that A and B are due to H. F. MacNeish, while C was found by Bose and Shrikhande.

2. **Lower estimation of $N(n)$.** We must deal separately with odd n and even n , and we use a fact proven in [1], called there "Lemma D ":

D. The number of integers no greater than x , which have a prime factor in common with n and greater than n^α , is no greater than x/gn^α .

Estimate for even n . We pick k so that

$$(2) \quad \begin{cases} k \equiv -1 \pmod{2^{\lceil \log_2 n / \alpha \rceil}}, \\ k \not\equiv 0 \text{ or } -1 \pmod{p} \text{ for } 3 \leq p \leq n^{1/\beta}, \\ k \leq n^{1/\gamma}. \end{cases}$$

Since $k = -1 + h2^{\lceil \log_2 n / \alpha \rceil}$, say, we know the number of such k is $P_\omega((1 + n^{1/\gamma})/2^{\lceil \log_2 n / \alpha \rceil}; n^{1/\beta})$. In view of Buchstab's theorem, we take $1/\gamma - 1/\alpha = 5/\beta$ and then have, for some positive constant c and all large n ,

$$P_\omega > c \cdot \frac{n^{5/\beta}}{\log^2 n},$$

Our k have no prime factor below $n^{1/\beta}$, so to choose k also prime to n we must deal with the primes in n which are greater than $n^{1/\beta}$. By D , the number of integers below $n^{1/\gamma}$, which have a prime factor which exceeds $n^{1/\beta}$ and divides n , is at most $n^{1/\gamma}/(1/\beta)n^{1/\beta}$. Since we want this to be less than the number of k , we take $1/\gamma = (6-\varepsilon)/\beta$, where $0 < \varepsilon < 1$. Then, for all large n we can choose k as above so as to be prime to n . Note that we now have $1/\alpha = (1-\varepsilon)/\beta$. Since all prime factors of k exceed $n^{1/\beta}$, and due to the restrictions on $k+1$, we deduce from A and B that:

$$\begin{aligned} N(k) &> n^{1/\beta} - 1 \\ N(k + 1) &> \text{Min} \left(\frac{1}{2}n^{1/\alpha}, n^{1/\beta} \right) - 1, \end{aligned}$$

and we note that for all large n both these estimates exceed $n^{1/\alpha}/3$. Now, since we want to have $n = u + mk$, write

$$n = n_1 + n_2k, \quad 0 < n_1 < k, \quad (n_1, k) = 1,$$

and

$$u = n_1 + u_1k.$$

Now choose u_1 so that:

$$(3) \quad \left. \begin{aligned} u_1 &\not\equiv n_1 \pmod{2}, \\ u_1 &\not\equiv -n_1/k \pmod{p}, \quad p \nmid k \\ u_1 &\not\equiv n_2 \pmod{p} \\ u_1 &< n^{1/\delta}. \end{aligned} \right\} 3 \leq p \leq k,$$

By Buchstab, this is all right as long as $k \leq n^{1/\delta}$, so we choose $1/\delta = 5/\gamma = 5(6-\varepsilon)/\beta$. No prime less than or equal to k can divide u : for u is prime to k , and those primes below k which don't divide k do not divide u , by (3). Hence

$$(4) \quad N(u) \geq k > N(k) > \frac{1}{3} n^{1/\alpha}.$$

Finally, $m = (n - u)/k$, of course; so $m - u = \{n - (1 + k)u\}/k$, which

we want to make positive. Since $(1 + k)u \ll n^{2/\gamma+1/\delta}$, choose β so that $7 \cdot (6 - \varepsilon)/\beta < 1$, or equivalently $1/\alpha < (1 - \varepsilon)/7(6 - \varepsilon)$. Thus we can achieve the conditions so far expressed for all large n , as long as α is any chosen number exceeding 42. As to $N(m)$, note that $m = n_2 - u_1 \not\equiv 0 \pmod{p}$ for $3 \leq p \leq k$. Also u is odd, by (3), and n is even; hence m is odd. Thus

$$(5) \quad N(m) \geq k > N(k) > \frac{1}{3} n^{1/\alpha}.$$

The conditions of C apply now, and the above estimates and C imply that for any constant c less than $1/42$ we have:

$$N(n) > n^c, \text{ for all large even } n.$$

Estimate for odd n . This time k is chosen even, the conditions being:

$$\begin{aligned} k + 1 &\equiv 1 \pmod{2^{\lceil \log_2 n / \alpha \rceil}}, \\ k + 1 &\not\equiv 0 \text{ or } 1 \pmod{p} \text{ for } 3 \leq p \leq n^{1/\beta}, \\ k + 1 &\leq n^{1/\gamma}. \end{aligned}$$

With obvious changes in detail from the previous case, we still get $\text{Min}\{N(k), N(k + 1)\} > 1/3(n)^{1/\alpha}$, and $(n, k) = 1$. This time, the relation $n - u = (n_2 - u_1)k$ ensures that u is odd, but we must adjust the parity condition on u_1 to ensure that m is odd:

$$\left. \begin{aligned} u_1 &\not\equiv n_2 \pmod{2} \\ u_1 &\not\equiv -n_1/k \pmod{p}, \text{ for } p \nmid k, \\ u_1 &\not\equiv n_2 \pmod{p} \end{aligned} \right\} 3 \leq p \leq k,$$

$$u_1 < n^{1/\delta}.$$

Thus $m = n_2 - u_1$ is odd, and now the details are as before, giving finally the following result.

THEOREM. *To each number c which is less than $1/42$, there corresponds an integer $n_0 = n_0(c)$, such that for all $n > n_0$ we have*

$$N(n) > n^c.$$

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