

# Pacific Journal of Mathematics

**ON COVERINGS**

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# ON COVERINGS

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1. **Introduction.** Recently [2, 3, 4, 5] renewed interest has been aroused in the notion of covering and related problems, originally posed by Steiner [8] and later reformulated by Moore [6] as problems of the existence of *tactical configurations*.

A tactical configuration  $C(k, l, \lambda, n)$  ( $n \geq k \geq l$ ) is a set of unordered  $k$ -tuples of  $n$  different elements, such that each  $l$ -tuple of these elements appears exactly  $\lambda$  times.

In view of the importance of the special cases  $\lambda = 1$  and  $l = 2$  the notions of *tactical systems*  $S(k, l, n)$  for  $C(k, l, 1, n)$  and *balanced incomplete block designs (BIBD)*  $B(k, \lambda, n)$  for  $C(k, 2, \lambda, n)$  have also been used.

A necessary condition [6] for the existence of a tactical configuration  $C(k, l, \lambda, n)$  is known to be

$$(1) \quad \lambda \binom{n-h}{l-h} / \binom{k-h}{l-h} = \text{integer}, \quad h = 0, 1, \dots, l-1.$$

For  $h = 0$  this integer, namely

$$(2) \quad \lambda \binom{n}{l} / \binom{k}{l}$$

is clearly the number of elements in  $C(k, l, \lambda, n)$ .

Condition (1) has been proved to be sufficient for  $l = 2$ ,  $k = 3$ ,  $\lambda = 1$  by Moore [6] and Reiss [7], for  $l = 2$ ,  $k = 3$ ,  $\lambda = 2$  by Bose [1], for  $l = 2$ ,  $k = 3$  and  $k = 4$  and every  $\lambda$ , for  $l = 2$ ,  $k = 5$   $\lambda = 1, 4$  and  $20$ , and for  $l = 3$ ,  $k = 4$  and every  $\lambda$  by Hanani [3, 4, 5].

These results for  $\lambda = 1$  show—and we note this here for future references—that necessary and sufficient conditions for the existence of tactical systems  $S(4, 2, n)$ ,  $S(5, 2, n)$  and  $S(4, 3, n)$  are, respectively

$$(3) \quad n \equiv 1 \text{ or } 4 \pmod{12}$$

$$(4) \quad n \equiv 1 \text{ or } 5 \pmod{20}$$

$$(5) \quad n \equiv 2 \text{ or } 4 \pmod{6}$$

More general *coverings*  $R(k, l, \lambda, n)$  existing for every  $n$  may be defined.

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A covering  $R(k, l, \lambda, n)$  ( $n \geq k \geq l$ ) is a set of unordered  $k$ -tuples of  $n$  different elements, such that each  $l$ -tuple of these  $n$  elements appears at least  $\lambda$  times.

Coverings  $R(3, 2, 1, n)$  have been studied by Fort Jr. and Hedlund [2]. These authors have proved that:

(i) every covering  $R(3, 2, 1, n)$  contains at least

$$\varphi(n) = \begin{cases} n^2/6 & \text{if } n \equiv 0 \\ n(n-1)/6 & \text{if } n \equiv 1 \text{ or } 3 \\ n^2 + 2/6 & \text{if } n \equiv 2 \text{ or } 4 \\ n^2 - n + 4/6 & \text{if } n \equiv 5 \end{cases} \pmod{6}$$

triples;

(ii) for each  $n$  there exists a covering  $R(3, 2, 1, n)$  containing exactly  $\varphi(n)$  triples.

In this paper we define the function

$$\psi(k, l, \lambda, n) = \left[ \frac{n}{k} \left[ \frac{n-1}{k-1} \left[ \dots \left[ \frac{n-l+2}{k-l+2} \left[ \frac{\lambda(n-l+1)}{k-l+1} \right] \right] \dots \right] \right] \right]$$

where  $[x]$  denotes the smallest integer  $y$ ,  $y \geq x$ . This is a generalization of the function  $\varphi(n)$ . Indeed,  $\varphi(n)$  equals  $\psi(3, 2, 1, n)$ .

We shall then prove (Theorem I) that every covering  $R(k, l, \lambda, n)$  contains at least  $\psi(k, l, \lambda, n)$   $k$ -tuples.

Further, we denote coverings  $R(k, l, \lambda, n)$  containing exactly  $\psi(k, l, \lambda, n)$   $k$ -tuples as *admissible coverings*  $M(k, l, \lambda, n)$ . Tactical configurations are such admissible coverings, because the number (2) of  $k$ -tuples in a tactical configuration  $C(k, l, \lambda, n)$  equals  $\psi(k, l, \lambda, n)$  as a consequence of conditions (1).

Finally, we shall prove (Theorem II) the existence of other admissible coverings, establishing that the existence of a tactical system  $S(k, l, n)$  implies the existence of an admissible covering  $M(k, l, 1, n+1)$ . Thus, particularly (Corollaries 1, 2, 3) from conditions (3), (4), (5), derives the existence of admissible coverings  $M(k, l, 1, n)$  for

$$\begin{aligned} k = 4, l = 2 & \text{ if } n \equiv 2 \text{ or } 5 \pmod{12} \\ k = 5, l = 2 & \text{ if } n \equiv 2 \text{ or } 6 \pmod{20} \\ k = 4, l = 3 & \text{ if } n \equiv 3 \text{ or } 5 \pmod{6} . \end{aligned}$$

Our last result means in terms of *minimal coverings* (coverings containing the least possible number of  $k$ -tuples), that a minimal covering  $R(k, l, \lambda, n)$  contains exactly  $\psi(k, l, \lambda, n)$   $k$ -tuples if a tactical system  $S(k, l, n-1)$  exists.

2. The lower bound for the number of  $k$ -tuples in a covering.

THEOREM I. Every covering  $R(k, l, \lambda, n)$  contains at least

$$(6) \quad \left[ \frac{n}{k} \left[ \frac{n-1}{k-1} \left[ \dots \left[ \frac{n-l+2}{k-l+2} \left[ \frac{\lambda(n-l+1)}{k-l+1} \right] \right] \dots \right] \right] \right]$$

$k$ -tuples.

*Proof.* We denote by  $q(R, k, l, \lambda, n)$  the number of  $k$ -tuples contained in  $R(k, l, \lambda, n)$  and by  $\psi(k, l, \lambda, n)$  the expression (6). Under this notation, the statement of Theorem I is

$$(7) \quad q(R, k, l, \lambda, n) \geq \psi(k, l, \lambda, n).$$

We prove this inequality by induction on  $l$ . Let  $l = 1$ . Obviously  $q(R, k, 1, \lambda, n) \geq \lceil \lambda n/k \rceil = \psi(k, 1, \lambda, n)$ . Suppose that inequality (7) is established for each  $n \geq k > l$  and  $l \leq l_0$ . Now let  $l = l_0 + 1$ . Consider a  $R(k, l_0 + 1, \lambda, n)$ . It will contain  $q(R, k, l_0 + 1, \lambda, n)$   $k$ -tuples and therefore  $k \cdot q(R, k, l_0 + 1, \lambda, n)$  elements. But each element must appear at least  $q(R_1, k - 1, l_0, \lambda, n - 1)$  times, for otherwise  $R(k, l_0 + 1, \lambda, n)$  could not contain  $\lambda$  times the  $l_0$ -tuples of  $n$  elements containing a given element. According to the hypothesis of the induction

$$q(R_1, k - 1, l_0, \lambda, n - 1) \geq \psi(k - 1, l_0, \lambda, n - 1).$$

It follows that

$$\begin{aligned} k \cdot q(R, k, l_0 + 1, \lambda, n) &\geq nq(R_1, k - 1, l_0, \lambda, n - 1) \\ &\geq n\psi(k - 1, l_0, \lambda, n - 1) \end{aligned}$$

and, since  $q$  must be an integer, and as a consequence of the definition of  $\psi(k - 1, l_0, \lambda, n - 1)$ , we have

$$\begin{aligned} q(R, k, l_0 + 1, \lambda, n) &\geq \left\lceil \frac{n}{k} \cdot \psi(k - 1, l_0, \lambda, n - 1) \right\rceil \\ &= \psi(k, l_0 + 1, \lambda, n). \end{aligned}$$

This proves the validity of inequality (7) for each  $l$ , and the theorem is proved.

Theorem I justifies the following definition:

A covering  $R(k, l, \lambda, n)$  may be called an admissible covering  $M(k, l, \lambda, n)$  if it contains exactly  $\psi(k, l, \lambda, n)$   $k$ -tuples.

3. The existence of admissible coverings which are not tactical configurations. The fact that there exist admissible coverings which are not tactical configurations will be shown in Corollaries 1, 2

and 3 to Lemma 3, but for the purpose of obtaining the more general Theorem II, we shall prove the following four lemmas:

LEMMA 1. *If the expression*

$$\binom{n-h-1}{l-h} / \binom{k-h}{l-h}$$

is an integer for  $h = 0, 1, \dots, l - 1$ , and if we denote it by  $\alpha_{l-h}$  and 1 by  $\alpha_0$ , we have, for  $i = 1, \dots, l$

$$(8) \quad \sum_{j=0}^i \alpha_j = \left[ \frac{n-l+i}{k-l+i} \left[ \frac{n-l+i-1}{k-l+i-1} \left[ \dots \left[ \frac{n-l+1}{k-l+1} \right] \dots \right] \right] \right].$$

*Proof.* We proceed by induction on  $i$ . Let  $i = 1$ . Then

$$\left[ \frac{n-l+1}{k-l+1} \right] = \left[ \alpha_1 + \frac{1}{k-l+1} \right] = \alpha_1 + 1 = \alpha_1 + \alpha_0 = \sum_{j=0}^1 \alpha_j.$$

Let equality (8) be valid for  $i = m < l$ . This implies

$$\begin{aligned} & \left[ \frac{n-l+m+1}{k-l+m+1} \left[ \frac{n-l+m}{k-l+m} \left[ \dots \left[ \frac{n-l+1}{k-l+1} \right] \dots \right] \right] \dots \right] \\ &= \left[ \frac{n-l+m+1}{k-l+m+1} \left( \sum_{j=0}^m \alpha_j \right) \right] \\ &= \left[ \frac{\sum_{j=0}^m (n-l+j)\alpha_j + \sum_{j=0}^m (m-j+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[ \frac{\sum_{j=0}^m (k-l+j+1)\alpha_{j+1} + \sum_{j=0}^m (m-j+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[ \frac{\sum_{j=1}^{m+1} (k-l+j)\alpha_j + \sum_{j=0}^m (m-j+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[ \frac{(k-l+m+1)\alpha_{m+1} + m+1 + \sum_{j=1}^m (k-l+m+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[ \alpha_{m+1} + \sum_{j=1}^m \alpha_j + \frac{m+1}{k-l+m+1} \right] = \sum_{j=0}^{m+1} \alpha_j. \end{aligned}$$

And the lemma is proved.

LEMMA 2. *If the expression*

$$(9) \quad \binom{n-h-1}{l-h} / \binom{k-h}{l-h}$$

is an integer for  $h = 0, 1, \dots, l-1$  we have

$$(10) \quad \frac{(n-1)(n-2) \cdots (n-l)}{k(k-1) \cdots (k-l+1)} + \left[ \frac{n-1}{k-1} \left[ \frac{n-2}{k-2} \left[ \cdots \left[ \frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right] = \left[ \frac{n}{k} \left[ \frac{n-1}{k-1} \left[ \cdots \left[ \frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right].$$

*Proof.* Denote the integer (9) by  $\alpha_{l-h}$  and  $\alpha_0 = 1$ . According to Lemma 1 and under this notation, the left hand side of equality (10) becomes

$$\alpha_l + \sum_{j=0}^{l-1} \alpha_j = \sum_{j=0}^l \alpha_j = \left[ \frac{n}{k} \left[ \frac{n-1}{k-1} \left[ \cdots \left[ \frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right].$$

**LEMMA 3.** *If there exists a tactical system  $S(k, l, n-1)$  and an admissible covering  $M(k-1, l-1, 1, n-1)$ , then there also exists an admissible covering  $M(k, l, 1, n)$ .*

*Proof.* Let  $N$  be a fixed element. Let  $V = \{(x, N) : x \in M(k-1, l-1, 1, n-1)\}$  and  $T = S(k, l, n-1) \cup V$ . It will then, be shown that  $T$  is an admissible covering.

Indeed, it is a covering  $R(k, l, 1, n)$ , as all the  $l$ -tuples of  $n$  elements not containing the element  $N$  appear in one of the  $k$ -tuples in  $S(k, l, n-1)$ , while the  $l$ -tuples containing the element  $N$  appear in at least one of the  $k$ -tuples in  $V$ . Moreover, the covering  $R(k, l, 1, n)$  is an admissible covering  $M(k, l, 1, n)$ . In fact, it contains

$$(11) \quad \frac{(n-1)(n-2) \cdots (n-l)}{k(k-1) \cdots (k-l+1)} + \left[ \frac{n-1}{k-1} \left[ \frac{n-2}{k-2} \left[ \cdots \left[ \frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right]$$

$k$ -tuples, which is the sum of the number of  $k$ -tuples in  $S(k, l, n-1)$  and of  $(k-1)$ -tuples in  $M(k-1, l-1, 1, n-1)$ .

The conditions of Lemma 2 are satisfied, and accordingly, (11) equals  $\psi(k, l, 1, n)$ , which proves the lemma.

**COROLLARY 1.** *If*

$$(12) \quad n \equiv 3 \text{ or } 5 \pmod{6}$$

then there exists an admissible covering  $M(4, 3, 1, n)$ .

*Proof.* For  $M$  satisfying (12), according to (5), there exists a tactical system  $S(k, 3, n - 1)$ , and according to (ii) there also exists an admissible covering  $M(3, 2, 1, n - 1)$ . Lemma 3 then implies the existence of an admissible covering  $M(4, 3, 1, n)$ .

COROLLARY 2. *If*

$$(13) \quad n \equiv 2 \text{ or } 5 \pmod{12}$$

then there exists an admissible covering  $M(4, 2, 1, n)$ .

*Proof.* For  $n$  satisfying (13), according to (3), there exists a BIBD  $B(4, 1, n - 1)$ . The existence of an admissible covering  $M(3, 1, 1, n - 1)$  being obvious, Lemma 3 implies the existence of an admissible covering  $M(4, 2, 1, n)$ .

COROLLARY 3. *If*

$$n \equiv 2 \text{ or } 6 \pmod{20}$$

then there exists an admissible covering  $M(5, 2, 1, n)$ .

*Proof.* Similar to that of the preceding corollary, but using (1) instead of (3).

LEMMA 4. *The existence of a tactical system  $S(k, l, n)$  implies that of an admissible covering  $M(k - 1, l - 1, 1, n)$ .*

*Proof.* By induction on  $l$ . Let  $l = 2$ . The existence of an admissible covering  $M(k - 1, 1, 1, n)$  is obvious. Suppose now that the lemma is proved for  $l = l_0$  and let  $l = l_0 + 1$ . The existence of a  $S(k, l_0 + 1, n)$  implies that of a  $S(k - 1, l_0, n - 1)$  which, according to the hypothesis of the induction, implies the existence of a  $M(k - 2, l_0 - 1, 1, n - 1)$ . The existence of a  $S(k - 1, l_0, n - 1)$  and of a  $M(k - 2, l_0 - 1, 1, n - 1)$  implies, according to Lemma 3, that of a  $M(k - 1, l_0, 1, n)$ .

THEOREM II. *If a tactical system  $S(k, l, n)$  exists, then there also exists an admissible covering  $M(k, l, 1, n + 1)$ .*

*Proof.* According to Lemma 4, the second hypothesis of Lemma 3 is automatically satisfied if the first hypothesis holds.

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# Pacific Journal of Mathematics

Vol. 14, No. 4

August, 1964

Homer Franklin Bechtell, Jr., <i>Pseudo-Frattini subgroups</i> .....	1129
Thomas Kelman Boehme and Andrew Michael Bruckner, <i>Functions with convex means</i> .....	1137
Lutz Bungart, <i>Boundary kernel functions for domains on complex manifolds</i> .....	1151
L. Carlitz, <i>Rings of arithmetic functions</i> .....	1165
D. S. Carter, <i>Uniqueness of a class of steady plane gravity flows</i> .....	1173
Richard Albert Dean and Robert Harvey Oehmke, <i>Idempotent semigroups with distributive right congruence lattices</i> .....	1187
Lester Eli Dubins and David Amiel Freedman, <i>Measurable sets of measures</i> .....	1211
Robert Pertsch Gilbert, <i>On class of elliptic partial differential equations in four variables</i> .....	1223
Harry Gonshor, <i>On abstract affine near-rings</i> .....	1237
Edward Everett Grace, <i>Cut points in totally non-semi-locally-connected continua</i> .....	1241
Edward Everett Grace, <i>On local properties and <math>G_\delta</math> sets</i> .....	1245
Keith A. Hardie, <i>A proof of the Nakaoka-Toda formula</i> .....	1249
Lowell A. Hinrichs, <i>Open ideals in <math>C(X)</math></i> .....	1255
John Rolfe Isbell, <i>Natural sums and abelianizing</i> .....	1265
G. W. Kimble, <i>A characterization of extremals for general multiple integral problems</i> .....	1283
Nand Kishore, <i>A representation of the Bernoulli number <math>B_n</math></i> .....	1297
Melven Robert Krom, <i>A decision procedure for a class of formulas of first order predicate calculus</i> .....	1305
Peter A. Lappan, <i>Identity and uniqueness theorems for automorphic functions</i> .....	1321
Lorraine Doris Lavalley, <i>Mosaics of metric continua and of quasi-Peano spaces</i> .....	1327
Mark Mahowald, <i>On the normal bundle of a manifold</i> .....	1335
J. D. McKnight, <i>Kleene quotient theorems</i> .....	1343
Charles Kimbrough Megibben, III, <i>On high subgroups</i> .....	1353
Philip Miles, <i>Derivations on <math>B^*</math> algebras</i> .....	1359
J. Marshall Osborn, <i>A generalization of power-associativity</i> .....	1367
Theodore G. Ostrom, <i>Nets with critical deficiency</i> .....	1381
Elvira Rapaport Strasser, <i>On the defining relations of a free product</i> .....	1389
K. Rogers, <i>A note on orthogonal Latin squares</i> .....	1395
P. P. Saworotnow, <i>On continuity of multiplication in a complemented algebra</i> .....	1399
Johanan Schonheim, <i>On coverings</i> .....	1405
Victor Lenard Shapiro, <i>Bounded generalized analytic functions on the torus</i> .....	1413
James D. Stafney, <i>Arens multiplication and convolution</i> .....	1423
Daniel Sterling, <i>Coverings of algebraic groups and Lie algebras of classical type</i> .....	1449
Alfred B. Willcox, <i>Šilov type <math>C</math> algebras over a connected locally compact abelian group. II</i> .....	1463
Bertram Yood, <i>Faithful <math>*</math>-representations of normed algebras. II</i> .....	1475
Alexander Zabrodsky, <i>Covering spaces of paracompact spaces</i> .....	1489