

Pacific Journal of Mathematics

ON COVERINGS

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1. Introduction. Recently [2, 3, 4, 5] renewed interest has been aroused in the notion of covering and related problems, originally posed by Steiner [8] and later reformulated by Moore [6] as problems of the existence of *tactical configurations*.

A tactical configuration $C(k, l, \lambda, n)$ ($n \geq k \geq l$) is a set of unordered k -tuples of n different elements, such that each l -tuple of these elements appears exactly λ times.

In view of the importance of the special cases $\lambda = 1$ and $l = 2$ the notions of *tactical systems* $S(k, l, n)$ for $C(k, l, 1, n)$ and *balanced incomplete block designs (BIBD)* $B(k, \lambda, n)$ for $C(k, 2, \lambda, n)$ have also been used.

A necessary condition [6] for the existence of a tactical configuration $C(k, l, \lambda, n)$ is known to be

$$(1) \quad \lambda \binom{n-h}{l-h} / \binom{k-h}{l-h} = \text{integer}, \quad h = 0, 1, \dots, l-1.$$

For $h = 0$ this integer, namely

$$(2) \quad \lambda \binom{n}{l} / \binom{k}{l}$$

is clearly the number of elements in $C(k, l, \lambda, n)$.

Condition (1) has been proved to be sufficient for $l = 2$, $k = 3$, $\lambda = 1$ by Moore [6] and Reiss [7], for $l = 2$, $k = 3$, $\lambda = 2$ by Bose [1], for $l = 2$, $k = 3$ and $k = 4$ and every λ , for $l = 2$, $k = 5$ $\lambda = 1, 4$ and 20, and for $l = 3$, $k = 4$ and every λ by Hanani [3, 4, 5].

These results for $\lambda = 1$ show—and we note this here for future references—that necessary and sufficient conditions for the existence of tactical systems $S(4, 2, n)$, $S(5, 2, n)$ and $S(4, 3, n)$ are, respectively

$$(3) \quad n \equiv 1 \text{ or } 4 \pmod{12}$$

$$(4) \quad n \equiv 1 \text{ or } 5 \pmod{20}$$

$$(5) \quad n \equiv 2 \text{ or } 4 \pmod{6}$$

More general *coverings* $R(k, l, \lambda, n)$ existing for every n may be defined.

Received May 15, 1963. This paper will form part of the author's D. Sci. thesis in preparation at the Technion, Israel Institute of Technology, Haifa.

A covering $R(k, l, \lambda, n)$ ($n \geq k \geq l$) is a set of unordered k -tuples of n different elements, such that each l -tuple of these n elements appears at least λ times.

Coverings $R(3, 2, 1, n)$ have been studied by Fort Jr. and Hedlund [2]. These authors have proved that:

(i) every covering $R(3, 2, 1, n)$ contains at least

$$\varphi(n) = \begin{cases} n^2/6 & \text{if } n \equiv 0 \\ n(n-1)/6 & \text{if } n \equiv 1 \text{ or } 3 \\ n^2 + 2/6 & \text{if } n \equiv 2 \text{ or } 4 \\ n^2 - n + 4/6 & \text{if } n \equiv 5 \end{cases} \pmod{6}$$

triples;

(ii) for each n there exists a covering $R(3, 2, 1, n)$ containing exactly $\varphi(n)$ triples.

In this paper we define the function

$$\psi(k, l, \lambda, n) = \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{k-1} \left\lceil \dots \left\lceil \frac{n-l+2}{k-l+2} \left\lceil \frac{\lambda(n-l+1)}{k-l+1} \right\rceil \right\rceil \dots \right\rceil \right\rceil \right\rceil$$

where $\lceil x \rceil$ denotes the smallest integer $y, y \geq x$. This is a generalization of the function $\varphi(n)$. Indeed, $\varphi(n)$ equals $\psi(3, 2, 1, n)$.

We shall then prove (Theorem I) that every covering $R(k, l, \lambda, n)$ contains at least $\psi(k, l, \lambda, n)$ k -tuples.

Further, we denote coverings $R(k, l, \lambda, n)$ containing exactly $\psi(k, l, \lambda, n)$ k -tuples as *admissible coverings* $M(k, l, \lambda, n)$. Tactical configurations are such admissible coverings, because the number (2) of k -tuples in a tactical configuration $C(k, l, \lambda, n)$ equals $\psi(k, l, \lambda, n)$ as a consequence of conditions (1).

Finally, we shall prove (Theorem II) the existence of other admissible coverings, establishing that the existence of a tactical system $S(k, l, n)$ implies the existence of an admissible covering $M(k, l, 1, n+1)$. Thus, particularly (Corollaries 1, 2, 3) from conditions (3), (4), (5), derives the existence of admissible coverings $M(k, l, 1, n)$ for

$$\begin{aligned} k = 4, l = 2 & \text{ if } n \equiv 2 \text{ or } 5 \pmod{12} \\ k = 5, l = 2 & \text{ if } n \equiv 2 \text{ or } 6 \pmod{20} \\ k = 4, l = 3 & \text{ if } n \equiv 3 \text{ or } 5 \pmod{6}. \end{aligned}$$

Our last result means in terms of *minimal coverings* (coverings containing the least possible number of k -tuples), that a minimal covering $R(k, l, \lambda, n)$ contains exactly $\psi(k, l, \lambda, n)$ k -tuples if a tactical system $S(k, l, n-1)$ exists.

2. The lower bound for the number of k -tuples in a covering.

THEOREM I. Every covering $R(k, l, \lambda, n)$ contains at least

$$(6) \quad \left[\frac{n}{k} \left[\frac{n-1}{k-1} \left[\dots \left[\frac{n-l+2}{k-l+2} \left[\frac{\lambda(n-l+1)}{k-l+1} \right] \right] \dots \right] \right] \right]$$

k -tuples.

Proof. We denote by $q(R, k, l, \lambda, n)$ the number of k -tuples contained in $R(k, l, \lambda, n)$ and by $\psi(k, l, \lambda, n)$ the expression (6). Under this notation, the statement of Theorem I is

$$(7) \quad q(R, k, l, \lambda, n) \geq \psi(k, l, \lambda, n).$$

We prove this inequality by induction on l . Let $l = 1$. Obviously $q(R, k, 1, \lambda, n) \geq \lceil \lambda n/k \rceil = \psi(k, 1, \lambda, n)$. Suppose that inequality (7) is established for each $n \geq k > l$ and $l \leq l_0$. Now let $l = l_0 + 1$. Consider a $R(k, l_0 + 1, \lambda, n)$. It will contain $q(R, k, l_0 + 1, \lambda, n)$ k -tuples and therefore $k \cdot q(R, k, l_0 + 1, \lambda, n)$ elements. But each element must appear at least $q(R_1, k - 1, l_0, \lambda, n - 1)$ times, for otherwise $R(k, l_0 + 1, \lambda, n)$ could not contain λ times the l_0 -tuples of n elements containing a given element. According to the hypothesis of the induction

$$q(R_1, k - 1, l_0, \lambda, n - 1) \geq \psi(k - 1, l_0, \lambda, n - 1).$$

It follows that

$$\begin{aligned} k \cdot q(R, k, l_0 + 1, \lambda, n) &\geq nq(R_1, k - 1, l_0, \lambda, n - 1) \\ &\geq n\psi(k - 1, l_0, \lambda, n - 1) \end{aligned}$$

and, since q must be an integer, and as a consequence of the definition of $\psi(k - 1, l_0, \lambda, n - 1)$, we have

$$\begin{aligned} q(R, k, l_0 + 1, \lambda, n) &\geq \left\lceil \frac{n}{k} \cdot \psi(k - 1, l_0, \lambda, n - 1) \right\rceil \\ &= \psi(k, l_0 + 1, \lambda, n). \end{aligned}$$

This proves the validity of inequality (7) for each l , and the theorem is proved.

Theorem I justifies the following definition:

A covering $R(k, l, \lambda, n)$ may be called an admissible covering $M(k, l, \lambda, n)$ if it contains exactly $\psi(k, l, \lambda, n)$ k -tuples.

3. The existence of admissible coverings which are not tactical configurations. The fact that there exist admissible coverings which are not tactical configurations will be shown in Corollaries 1, 2

and 3 to Lemma 3, but for the purpose of obtaining the more general Theorem II, we shall prove the following four lemmas:

LEMMA 1. *If the expression*

$$\binom{n-h-1}{l-h} / \binom{k-h}{l-h}$$

is an integer for $h = 0, 1, \dots, l-1$, and if we denote it by α_{l-h} and 1 by α_0 , we have, for $i = 1, \dots, l$

$$(8) \quad \sum_{j=0}^i \alpha_j = \left[\frac{n-l+i}{k-l+i} \left[\frac{n-l+i-1}{k-l+i-1} \left[\dots \left[\frac{n-l+1}{k-l+1} \right] \dots \right] \right] \right].$$

Proof. We proceed by induction on i . Let $i = 1$. Then

$$\left[\frac{n-l+1}{k-l+1} \right] = \left[\alpha_1 + \frac{1}{k-l+1} \right] = \alpha_1 + 1 = \alpha_1 + \alpha_0 = \sum_{j=0}^1 \alpha_j.$$

Let equality (8) be valid for $i = m < l$. This implies

$$\begin{aligned} & \left[\frac{n-l+m+1}{k-l+m+1} \left[\frac{n-l+m}{k-l+m} \left[\dots \left[\frac{n-l+1}{k-l+1} \right] \dots \right] \right] \right] \\ &= \left[\frac{n-l+m+1}{k-l+m+1} \left(\sum_{j=0}^m \alpha_j \right) \right] \\ &= \left[\frac{\sum_{j=0}^m (n-l+j)\alpha_j + \sum_{j=0}^m (m-j+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[\frac{\sum_{j=0}^m (k-l+j+1)\alpha_{j+1} + \sum_{j=0}^m (m-j+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[\frac{\sum_{j=1}^{m+1} (k-l+j)\alpha_j + \sum_{j=0}^m (m-j+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[\frac{(k-l+m+1)\alpha_{m+1} + m+1 + \sum_{j=1}^m (k-l+m+1)\alpha_j}{k-l+m+1} \right] \\ &= \left[\alpha_{m+1} + \sum_{j=1}^m \alpha_j + \frac{m+1}{k-l+m+1} \right] = \sum_{j=0}^{m+1} \alpha_j. \end{aligned}$$

And the lemma is proved.

LEMMA 2. *If the expression*

$$(9) \quad \binom{n-h-1}{l-h} / \binom{k-h}{l-h}$$

is an integer for $h = 0, 1, \dots, l-1$ we have

$$(10) \quad \frac{(n-1)(n-2) \cdots (n-l)}{k(k-1) \cdots (k-l+1)} + \left[\frac{n-1}{k-1} \left[\frac{n-2}{k-2} \left[\cdots \left[\frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right] = \left[\frac{n}{k} \left[\frac{n-1}{k-1} \left[\cdots \left[\frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right].$$

Proof. Denote the integer (9) by α_{l-h} and $\alpha_0 = 1$. According to Lemma 1 and under this notation, the left hand side of equality (10) becomes

$$\alpha_l + \sum_{j=0}^{l-1} \alpha_j = \sum_{j=0}^l \alpha_j = \left[\frac{n}{k} \left[\frac{n-1}{k-1} \left[\cdots \left[\frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right].$$

LEMMA 3. *If there exists a tactical system $S(k, l, n-1)$ and an admissible covering $M(k-1, l-1, 1, n-1)$, then there also exists an admissible covering $M(k, l, 1, n)$.*

Proof. Let N be a fixed element. Let $V = \{(x, N) : x \in M(k-1, l-1, 1, n-1)\}$ and $T = S(k, l, n-1) \cup V$. It will then, be shown that T is an admissible covering.

Indeed, it is a covering $R(k, l, 1, n)$, as all the l -tuples of n elements not containing the element N appear in one of the k -tuples in $S(k, l, n-1)$, while the l -tuples containing the element N appear in at least one of the k -tuples in V . Moreover, the covering $R(k, l, 1, n)$ is an admissible covering $M(k, l, 1, n)$. In fact, it contains

$$(11) \quad \frac{(n-1)(n-2) \cdots (n-l)}{k(k-1) \cdots (k-l+1)} + \left[\frac{n-1}{k-1} \left[\frac{n-2}{k-2} \left[\cdots \left[\frac{n-l+1}{k-l+1} \right] \cdots \right] \right] \right]$$

k -tuples, which is the sum of the number of k -tuples in $S(k, l, n-1)$ and of $(k-1)$ -tuples in $M(k-1, l-1, 1, n-1)$.

The conditions of Lemma 2 are satisfied, and accordingly, (11) equals $\psi(k, l, 1, n)$, which proves the lemma.

COROLLARY 1. *If*

$$(12) \quad n \equiv 3 \text{ or } 5 \pmod{6}$$

then there exists an admissible covering $M(4, 3, 1, n)$.

Proof. For M satisfying (12), according to (5), there exists a tactical system $S(k, 3, n - 1)$, and according to (ii) there also exists an admissible covering $M(3, 2, 1, n - 1)$. Lemma 3 then implies the existence of an admissible covering $M(4, 3, 1, n)$

COROLLARY 2. *If*

$$(13) \quad n \equiv 2 \text{ or } 5 \pmod{12}$$

then there exists an admissible covering $M(4, 2, 1, n)$.

Proof. For n satisfying (13), according to (3), there exists a BIBD $B(4, 1, n - 1)$. The existence of an admissible covering $M(3, 1, 1, n - 1)$ being obvious, Lemma 3 implies the existence of an admissible covering $M(4, 2, 1, n)$.

COROLLARY 3. *If*

$$n \equiv 2 \text{ or } 6 \pmod{20}$$

then there exists an admissible covering $M(5, 2, 1, n)$.

Proof. Similar to that of the preceding corollary, but using (1) instead of (3).

LEMMA 4. *The existence of a tactical system $S(k, l, n)$ implies that of an admissible covering $M(k - 1, l - 1, 1, n)$.*

Proof. By induction on l . Let $l = 2$. The existence of an admissible covering $M(k - 1, 1, 1, n)$ is obvious. Suppose now that the lemma is proved for $l = l_0$ and let $l = l_0 + 1$. The existence of a $S(k, l_0 + 1, n)$ implies that of a $S(k - 1, l_0, n - 1)$ which, according to the hypothesis of the induction, implies the existence of a $M(k - 2, l_0 - 1, 1, n - 1)$. The existence of a $S(k - 1, l_0, n - 1)$ and of a $M(k - 2, l_0 - 1, 1, n - 1)$ implies, according to Lemma 3, that of a $M(k - 1, l_0, 1, n)$.

THEOREM II. *If a tactical system $S(k, l, n)$ exists, then there also exists an admissible covering $M(k, l, 1, n + 1)$.*

Proof. According to Lemma 4, the second hypothesis of Lemma 3 is automatically satisfied if the first hypothesis holds.

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$3.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Homer Franklin Bechtell, Jr., <i>Pseudo-Frattini subgroups</i>	1129
Thomas Kelman Boehme and Andrew Michael Bruckner, <i>Functions with convex means</i>	1137
Lutz Bungart, <i>Boundary kernel functions for domains on complex manifolds</i>	1151
L. Carlitz, <i>Rings of arithmetic functions</i>	1165
D. S. Carter, <i>Uniqueness of a class of steady plane gravity flows</i>	1173
Richard Albert Dean and Robert Harvey Oehmke, <i>Idempotent semigroups with distributive right congruence lattices</i>	1187
Lester Eli Dubins and David Amiel Freedman, <i>Measurable sets of measures</i>	1211
Robert Pertsch Gilbert, <i>On class of elliptic partial differential equations in four variables</i>	1223
Harry Gonshor, <i>On abstract affine near-rings</i>	1237
Edward Everett Grace, <i>Cut points in totally non-semi-locally-connected continua</i>	1241
Edward Everett Grace, <i>On local properties and G_δ sets</i>	1245
Keith A. Hardie, <i>A proof of the Nakaoka-Toda formula</i>	1249
Lowell A. Hinrichs, <i>Open ideals in $C(X)$</i>	1255
John Rolfe Isbell, <i>Natural sums and abelianizing</i>	1265
G. W. Kimble, <i>A characterization of extremals for general multiple integral problems</i>	1283
Nand Kishore, <i>A representation of the Bernoulli number B_n</i>	1297
Melven Robert Krom, <i>A decision procedure for a class of formulas of first order predicate calculus</i>	1305
Peter A. Lappan, <i>Identity and uniqueness theorems for automorphic functions</i>	1321
Lorraine Doris Lavallee, <i>Mosaics of metric continua and of quasi-Peano spaces</i>	1327
Mark Mahowald, <i>On the normal bundle of a manifold</i>	1335
J. D. McKnight, <i>Kleene quotient theorems</i>	1343
Charles Kimbrough Megibben, III, <i>On high subgroups</i>	1353
Philip Miles, <i>Derivations on B^* algebras</i>	1359
J. Marshall Osborn, <i>A generalization of power-associativity</i>	1367
Theodore G. Ostrom, <i>Nets with critical deficiency</i>	1381
Elvira Rapaport Strasser, <i>On the defining relations of a free product</i>	1389
K. Rogers, <i>A note on orthogonal Latin squares</i>	1395
P. P. Saworotnow, <i>On continuity of multiplication in a complemented algebra</i>	1399
Johan Schoneim, <i>On coverings</i>	1405
Victor Lenard Shapiro, <i>Bounded generalized analytic functions on the torus</i>	1413
James D. Stafney, <i>Arens multiplication and convolution</i>	1423
Daniel Sterling, <i>Coverings of algebraic groups and Lie algebras of classical type</i>	1449
Alfred B. Willcox, <i>Šilov type C algebras over a connected locally compact abelian group. II</i>	1463
Bertram Yood, <i>Faithful $*$-representations of normed algebras. II</i>	1475
Alexander Zabrodsky, <i>Covering spaces of paracompact spaces</i>	1489