FAITHFUL \*-REPRESENTATIONS OF NORMED ALGEBRAS. II

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1. Introduction. Let $A$ be a complex Banach algebra with an involution $x \rightarrow x^*$. By the positive cone $P$ of $A$ is meant the closure, in the set $H$ of self-adjoint elements of $A$, of the set of all finite sums of elements of the form $x^*x$. Kelley and Vaught [5] have shown that, if $A$ has an identity, $A$ has a faithful *-representation (as bounded linear operators on a Hilbert space) if and only if (1) $x \rightarrow x^*$ is continuous and (2) $P \cap (-P) = (0)$. Consider the (incomplete) normed algebra case. Examples exist with a faithful *-representation and both conditions false, with (1) true and (2) false, and with (1) false and (2) true. Moreover, even if (1) holds so that $x \rightarrow x^*$ extends to the completion $A_\epsilon$ of $A$, one can have a continuous faithful *-representation for $A$ when none exists for $A_\epsilon$. It follows that the results which we now describe, even for the normed algebra case, can not be deduced from the theory of Banach algebras.

These facts led us to consider the development of a theory of *-representations of a complex algebra $A$ with involution (with or without an identity) under minimal assumptions on $A$ but with results sufficiently definitive to illuminate the counter-examples mentioned above. We suppose that the real linear space $H$ has a norm in terms of which it is a real normed linear space such that

(a) the real subalgebra generated by each $h \in H$ is a normed algebra and

(b) the Jordan product $x \cdot h = xh + hx$ is a continuous function on $H$ for each fixed $h \in H$.

It is shown that $A$ has a faithful *-representation continuous on $H$ if and only if $A$ is semi-simple and $P \cap (-P) = (0)$. If $A$ is a normed *-$Q$-algebra, any *-representation is automatically continuous on $H$ so that these conditions are necessary and sufficient there for a faithful *-representation. As already noted, this can fail if the $Q$-algebra hypothesis is dropped.

For previous work on *-representations we refer to [5], [7], [8], and [10].

2. Preliminaries. Let $A$ be an algebra over the complex field
with an involution \( x \rightarrow x^* \). The set of self-adjoint (s.a.) elements of \( A \) is denoted by \( H \). By a \(*\)-representation of \( A \) we mean a homomorphism \( x \rightarrow T_x \) of \( A \) into the algebra of bounded linear operators on a Hilbert space where, for each \( x \), \( T_x^* \) is the adjoint of \( T_x \). A \(*\)-representation which is one to one is called faithful. A general representation procedure of Gelfand and Naimark [7] which we adapt to our needs leads to \(*\)-representations via positive linear functionals.

A complex linear functional \( f \) on \( A \) is called positive if \( f(x^*x) \geq 0 \) for all \( x \in A \). We call \( f \) hermitian if \( f(x^*) = f(x) \) for all \( x \in A \) or equivalently if \( f \) is a real linear functional when restricted to the real linear space \( H \). As in [8, p. 200] we define \( L_f = \{ x : f(zx) = 0 \} \) for all \( z \in A \) = \{ x : f(x^*x) = 0 \}; \( L_f \) is a left ideal of \( A \). Let \( X_f \) be the linear space \( A - L_f \) and \( \pi \) be the natural homomorphism of \( A \) onto \( X_f \). Then, [8, p. 212], \( (\pi(x), \pi(y)) = f(y^*x) \) defines an inner product on \( X_f \) in terms of which \( X_f \) is a pre-Hilbert space. Let \( H_f \) be the completion of \( X_f \) in the pre-Hilbert space norm. As in [7, p. 120] we associate with \( y \in A \) a linear operator \( T'_y \) defined on \( X_f \) by the rule \( T'_y[\pi(x)] = \pi(yx) \). In order that every \( T'_y \), \( y \in A \), be extendable to a bounded linear operator \( U'_y \) on \( H_f \) it is necessary and sufficient [8, p. 213] that \( f \) be admissible, that is, to each \( x \in A \) there corresponds a number \( K(x) < \infty \) such that \( f(y^*x^*xy) \leq K(x)f(y^*y) \) for all \( y \in A \). If \( f \) is admissible, the mapping \( x \rightarrow U'_y \) is a \(*\)-representation of \( A \).

For any positive linear functional \( f \) and any \( y \in A \) we define the positive linear functional \( f_y(x) = f(y^*x) \).

2.1. Lemma. Let \( f \) be a positive linear functional on \( A \). Then \( f \) is admissible if and only if

\[
\text{sup}_{n} [f_y(h^{2^n})]^{1-\alpha} < \infty ,
\]

for each \( y \in A \), \( h \in H \), where the sup is taken over the set of positive integers.

Suppose that \( f \) is admissible. Then, for \( h \in H \), \( U'_h \) is a bounded s.a. operator on the Hilbert space \( H_f \). For convenience, let \( U'_h \) where \( z = h^{2^n} \) be denoted by \( V_n \). For each \( y \in A \),

\[
f_y(h^{2n+1}) = \| V_n \pi(y) \|^2 \leq \| U'_y \|^{2n+1} f(y^*y)
\]

for \( n = 0, 1, 2, \ldots \). This implies (2.1).

For the converse we make use of an inequality due to Kaplansky [4, p. 55] concerning a positive linear functional \( f \) which asserts that

\[
f_y(x^*x) \leq f(y^*y)^{1-\alpha} [f_y(x^*x)^{\alpha}]^{1-\alpha}
\]
for all $x, y \in A$ and all positive integers $n$. Assume (2.1). It is clearly sufficient to show that $T'_h$ is a bounded operator on $X_f$ for each $h$ s.a. Using (2.2) we have

$$
|| T'_h \pi(y) ||^n = f_h(h^2) \leq f(y^*y)^{1-\delta_k} [f_h(h^{2n+1})]^{1-n}
$$

so that $|| T'_h ||$ cannot exceed the sup of (2.1).

2.2. **Lemma.** Suppose $H$ is given a topology in which it is a real linear topological space. Then the mapping $p \rightarrow (1 + h) x(1 + h)$ is continuous on $H$, for each $h \in H$, if and only if the Jordan product $x \cdot h = hx + xh$ is continuous on $H$ for each $h \in H$.

Let $a(x, h) = x + xh + hx + hxx$. Then $x \cdot h = [a(x, h) - a(x, -h)]/2$ and $a(x, h) = x + xh + [(x \cdot h) - h - x \cdot h]/2$ from which the lemma is immediate.

We now state metric requirements which we put on the algebra $A$ with involution. We suppose given a norm $|| h ||$ on $H$ in terms of which $H$ is a real normed linear space and, for each $h \in H$, the real subalgebra generated by $h$ is a normed algebra. No assumptions are made about the elements not in $H$ nor are there any requirements of completeness or identity element. We assume that the Jordan product $x \cdot h$ is continuous on $H$ for each $h \in H$. We call $A$ a normed *-algebra if, $A$ is a normed algebra. Following [3] we say that the normed *-algebra $A$ is a normed $Q$*-algebra if the set of quasi-regular elements of $A$ is open. If $A$ is a Banach algebra it has this property [3, p. 155].

For $h \in H$, $\lim || h^n ||^{1/n} = \nu(h)$ exists. Clearly $\nu(h) \leq || h ||$ and $\nu(h^2) = [\nu(h)]^2$ (see [8, p. 10]).

2.3. **Lemma.** Let $f$ be a positive linear functional on $A$. The following statements are equivalent.

(a) Each $f_h$ is continuous on $H$.
(b) $f_h(x^*x) \leq \nu(x^*x)f(y^*y)$ for all $x, y \in A$.
(c) $f$ is admissable and the mapping $x \rightarrow T'_x$ is continuous on $H$.

Suppose (a) holds. From the inequality (2.2) we obtain

$$
f_h(x^*x) \leq f(y^*y)^{1-\delta_k} (|| f_h || || (x^*x)^{2n} ||)^{1-n}.
$$

If we let $n \rightarrow \infty$ we obtain (b).

Suppose (b) holds. Clearly $f$ is admissable. For $h \in H$ we have $f_h(h^2) \leq \nu(h)f(y^*y)$ so that $|| T'_h \pi(y) || \leq || h || || \pi(y) ||$ and $|| T'_h || \leq || h ||$.

Suppose (c) with $|| T'_h || \leq k || h ||$, $h \in H$. Then, by the Cauchy-Schwarz inequality,
so that \( f_y \) is continuous on \( H \).

We note that, under these conditions, the norm of the mapping \( x \to T'_x \) on \( H \) does not exceed one.

\[ |f'_x(h)|^2 \leq f(y^*y)f'_x(h^2) = f(y^*y)\|T'_x\| \leq k^2 \|h\|^2[f(y^*y)]^2 \]

2.4. **Lemma.** Any \(*\)-representation of a normed \(*\)-algebra \( A \) is continuous on \( H \).

Let \( x \to T_x \) be a \(*\)-representation of \( A \). Let \( \rho(u) \) denote the spectral radius of \( u \) \([8, p. 30]\). For \( h \in H \) we have \( \|T_h\| = \rho(T_h) \leq \rho(h) \leq \|h\| \) by \([9, p. 373]\). Thus in the Q-algebra case the admissible positive linear functionals are those satisfying (b) of Lemma 2.3; if also \( A \) has an identity the admissible positive linear functionals are those continuous on \( H \).

2.5. **Lemma.** Suppose \( f \) is positive linear functional on \( A \) which is continuous on \( H \). Then \( f_y \) is continuous on \( H \) for each \( y \in A \).

It follows from Lemma 2.2 that the mapping \( x \to hxh \) is continuous on \( H \) for each \( h \in H \). Therefore the functional \( f_h \) is continuous on \( H \) for each \( h \in H \). Now, if \( y = u + iv, u, v \in H \) we have \( f_y(x) = f_u(x) + f_v(x) + if(uxv - vxu) \). But, by the Cauchy-Schwarz inequality, for any \( x \in H, |f(uxv)|^2 \leq f(u^2)f_v(x^2) \leq f(u^2)\|f_x\|\|x\|^2 \) where \( \|f_x\| \) is the norm of \( f_x \) considered as a linear functional on \( H \). This makes \( \|f_y(x)\| \leq K\|x\|, x \in H, \) where

\[ K = \|f_u\| + \|f_v\| + 2[f(u^2)\|f_x\| + f(v^2)\|f_x\|]\]^{1/2}.

In view of Lemma 2.3, \( f \) is admissible.

We give an example of a normed \(*\)-algebra \( A \) whose involution is continuous with the following properties.

1. \( A \) has a faithful \(*\)-representation.
2. Every \(*\)-representation of \( A \) other than the zero representation is discontinuous on \( H \).
3. The completion \( A_c \) of \( A \) has only the zero \(*\)-representation.

Let \( A \) be the set of all polynomials in the complex variable \( z \) which vanish at the origin. For \( p(z) = \Sigma \alpha_kz^k \) we define \( p^*(z) = \Sigma \alpha_kz^k \) and \( \|p(z)\| = \Sigma |\alpha_k|/k! \). Then (see \([3, p. 158]\)) \( A \) is a normed \(*\)-algebra. That (1) holds will be pointed out in \( \S 4 \). Let \( p \to T_p \) be a \(*\)-representation of \( A \) continuous on \( H \). The polynomial \( z \) is s.a. For each real scalar \( \lambda, ||\lambda z^*|| \to 0 \). Therefore \( ||\lambda^* T_p || = ||\lambda || T_p ||^* \to 0 \). This makes \( T_z = 0 \) so that \( T_p = 0 \) on \( A \). Now the involution on \( A \), being bicontinuous, extends to an involution on \( A_c \). Any \(*\)-representation \( x \to V_x \) of the Banach algebra \( A_c \) must be continuous by \([8, Theorem\].
Therefore, by the above, \( V_x = 0 \) for all \( x \in A \).

Let \( F \) be a set of admissible positive linear functionals on \( A \). We call \( F \) a compatible set if for each \( x \in A \) there exists a real number \( K(x) \) such that \( \| U_f x \| \leq K(x) \) for all \( f \in F \). This is equivalent to requiring that, for each \( x \in A \), there exists \( C(x) < \infty \), such that \( f(x^*x) \leq C(x) f(y^*y) \) for all \( y \in A \) and all \( f \in F \). By Lemmas 2.3 and 2.4 the set of all admissible positive linear functionals on a normed *-Q-algebra is a compatible set.

For each \( f \) in the compatible set \( F \) consider the Hilbert space \( H_f \) and the corresponding *-representation \( x \mapsto U_f x \). Let \( H \) be the Hilbert space direct sum of the Hilbert spaces \( H_f \). Since \( \| U_f \| \leq K(x) \) for all \( f \in F \) we can take [7, p. 113] the direct sum \( x \mapsto U_f x \) of the *-representations \( x \mapsto U_f x \), \( f \in F \) where \( U_f \) is a bounded operator on \( H \) and \( \| U_f \| \leq K(x) \). We call this *-representation the canonical *-representation of \( A \) induced by \( F \). For a left ideal \( L \) of \( A \) we use the notation \((L : A)\) as in [8, p. 53] to denote the set of all \( x \in A \) such that \( xA \subset L \). The kernel of the canonical *-representation induced by \( F \) is given by \( \bigcap (L_f : A) \) where the intersection is taken over all \( f \in F \).

3. On *-representations. For our purposes we wish to define the *-radical \( \mathfrak{R}^* \) of \( A \) as the intersection of the kernels of all *-representations of \( A \) which are continuous on \( H \). Let \( A^* \) denote the set of all positive linear functionals on \( A \). At the outset we consider three subsets of \( A^* \). Let \( \mathfrak{B} = \{ f \in A^* : f(x^*x) \leq \nu(x^*x)f(y^*y), \text{ for all } x, y \in A \} \). Let \( \mathfrak{D} \) be the set of dual functionals by which we mean \( \{ f \in A^* : f \text{ is hermitian and } f \text{ is continuous on } H \} \). Let \( \mathfrak{G} = \{ f \in D : |f(x)|^2 \leq f(x^*x) \text{ for all } x \in A \} \). By Lemmas 2.3 and 2.5 we see that \( \mathfrak{B} \supset \mathfrak{D} \supset \mathfrak{G} \) and that these are compatible sets. Let \( \mathfrak{B}_0, \mathfrak{D}_0, \text{ and } \mathfrak{G}_0 \) be the kernels of the canonical *-representations of \( A \) induced by \( \mathfrak{B}, \mathfrak{D}, \) and \( \mathfrak{G} \) respectively. Then \( \mathfrak{G}_0 \supset \mathfrak{D}_0 \supset \mathfrak{B}_0 \).

3.1. Lemma. \( \mathfrak{R}^* = \mathfrak{G}_0 = \mathfrak{D}_0 = \mathfrak{B}_0 \). \( A/\mathfrak{R}^* \) is semi-simple.

For any \( f \in \mathfrak{B}_0 \), and \( x, y \in A \), \( |T_x^* \pi(y)|^2 \leq \nu(x^*x) |\pi(y)|^2 \) so that \( |T_x^* \| \leq \sqrt{\nu(x^*x)} \). Consequently \( |T_x^* \| \leq \nu(h) \leq \| h \|, \ h \in H \). Therefore if \( x \mapsto T_x^* \) is any of the canonical *-representations in question, \( \| T_x^* \| \leq \sqrt{\nu(x^*x)} \), \( h \in H \), and the *-representation is continuous on \( H \). This proves that \( \mathfrak{R}^* \subset \mathfrak{B}_0 \subset \mathfrak{D}_0 \subset \mathfrak{G}_0 \). We show that \( \mathfrak{G}_0 \subset \mathfrak{R}^* \).

Let \( x \mapsto V_x \) be any *-representation of \( A \) continuous on \( H \), say as operators on the Hilbert space \( M \). For each \( \alpha \in M \) the functional \( g^*(x) = (V_x(\alpha), \alpha) \) is continuous on \( H \) and is a dual functional. For \( \alpha \) in the unit ball \( \Sigma \) of \( M \), \( |g^*(x)|^2 \leq \|V_x(\alpha)\|^2 = g^*(x^*x) \) so that \( g^* \in \mathfrak{G} \). We have
\( \mathcal{E}_0 = \bigcap_{f \in \mathcal{E}} (L_f : A) \subset \bigcap_{a \in \mathcal{E}} (L_{a^*} : A) \\
= \bigcap_{a \in \mathcal{E}} \{ z \in A : g^*((zy)^*(zy)) = 0 \text{ for all } y \in A \} \\
= \bigcap_{a \in \mathcal{E}} \{ z \in A : V_{zy}(x) = 0 \text{ for all } y \in A \} \\
= \{ z \in A : V_{zy} = 0 \text{ for all } y \in A \} \\
\subset \{ z \in A : ||V_{zy}|| = ||V_{y}||^2 = 0 \} . \\

Therefore \( \mathcal{E}_0 \subset \mathcal{R}^* \).

Since \( A/\mathcal{R}^* = A/\mathcal{E}_0 \) is algebraically \(-\)-isomorphic to a \(-\)-subalgebra of the algebra of all bounded linear operators on a Hilbert space, we see from [8, Theorem 4.1.19] that \( A/\mathcal{R}^* \) is semi-simple. From this we see also that the radical of \( A \) is contained in \( \mathcal{R}^* \).

3.2. Lemma. A normed \(-\)-algebra \( A \) has a faithful \(-\)-representation continuous on \( H \) if and only if \( \mathcal{R}^* = (0) \).

Suppose \( \mathcal{R}^* = (0) \). The preceding Lemma 3.1 then asserts the canonical \(-\)-representations induced by \( \mathfrak{B} \), \( \mathfrak{D} \), or \( \mathcal{E} \) are faithful. As noted above, these \(-\)-representations are continuous on \( H \). We naturally seek conditions on \( A \) which force \( \mathcal{R}^* = (0) \).

We set forth notation which will be used below. Let \( R_0 \) be the collection of all finite sums of elements of \( A \) of the form \( x^*x \) and let \( P \) be the closure of \( R_0 \) in \( H \). The set \( P \) will be considered as a closed cone in the real normed linear space \( H \). Let \( A_1 \) be the algebra obtained by adjoining an identity \( e \) to \( A \). As usual the involution on \( A \) is extended to \( A_1 \) by \((\lambda e + x)^* = \overline{\lambda} e + x^* \) where \( \lambda \) is a scalar and \( x \in A \). We shall have occasion to consider the sets \( H_1, \mathfrak{R}^*_1, \mathfrak{D}_1, R_0, \) and \( P \) in \( A_1 \) simultaneously with the corresponding sets defined for \( A \). When we do so, we denote the latter sets by \( H_1, \mathfrak{R}_1, \mathfrak{D}_1, R_0, \) and \( P \) respectively. The given norm on \( H \) leads to a norm on \( H_1 \) via \( ||\lambda e + h|| = |\lambda| + ||h||, \lambda \text{ real}, h \in H \). \( A_1 \) satisfies the requirements of our theory.

We set \( Z(\mathfrak{D}) = \bigcap f^{-1}(0), f \in \mathfrak{D} \) and \( Z(\mathcal{E}) = \bigcap f^{-1}(0), f \in E \). We define two versions of the reducing ideal [7, p. 130] suitable for this setting. Let \( \bigcap L_f \) where \( f \) runs over \( \mathfrak{D}(\mathcal{E}) \) be denoted by \( RI(\mathfrak{D}) \) and \( RI(\mathcal{E}) \) respectively.

Let \( g \) be a continuous real linear functional on \( H, g(P) \geq 0 \). If we extend \( g \) to \( A \) by the rule \( g(x) = g(h_1 + \overline{h_2})(x) = h_1 + ih_2, h_1, h_2 \text{ s.a.}, \) we obtain an element of \( \mathfrak{D} \). Conversely the restriction to \( H \) of any \( f \in D \) has the property that \( g(P) \geq 0 \). From the theory of closed cones in a normed linear space [5, Lemma 1.2] it follows that \( P \cap (-P) \) is the s.a. part of \( Z(\mathfrak{D}) \) so that \( Z(\mathfrak{D}) = P \cap (-P) + iP \cap (-P) \).

In a more restrictive context, this was pointed out and used in [1].
It will appear that $Z(\mathfrak{D})$ can differ from $Z(\mathfrak{C})$; $Z(\mathfrak{C})$ does not seem to have as neat an interpretation as $Z(\mathfrak{D})$. For that reason the results of § 4 involving $\mathfrak{D}$ are more interesting than the theory for $\mathfrak{C}$.

3.3. LEMMA. $\mathcal{R}^* = (RI(\mathfrak{D}) : A) = (RI(\mathfrak{C}) : A)$

We see, by Lemma 3.1, that $\mathcal{R}^*$ is the kernel of the canonical $*$-representation induced by $\mathfrak{D}$. Thus
\[
\mathcal{R}^* = \bigcap_{f \in \mathfrak{D}} (L_f : A) = \bigcap_{f \in \mathfrak{D}} \{ x : xy \in L_f, \text{ for all } y \in A \}
\]

Let $x \rightarrow V_x$ be the canonical $*$-representation induced by $\mathfrak{C}$. We show, by direct computation, (see [7, p. 132]) that
\[
(3.1) \quad ||V_x||^2 = \sup_{f \in \mathfrak{C}} f(x^*x), \quad x \in A.
\]

Let $\beta(x)$ denote the right hand side of (3.1). Take $f \in \mathfrak{C}$. Then $|f_x(x)|^2 = |f(y^*xy)|^2 \leq f(y^*y)f_v(x^*x)$ by the Cauchy-Schwarz inequality. Therefore $f_x \in \mathfrak{C}$ whenever $f(y^*y) \leq 1$. Now $||T'_y\pi(y)||^2 = f_y(x^*x)$ so that $||T'_y||^2 \leq \beta(x)$ from which we see that $||V_x||^2 \leq \beta(x)$. On the other hand, for $f \in \mathfrak{C}$,
\[
[f(x^*x)]^2 \leq f(x^*xx^*x) = ||T'_y\pi(x)||^2 \leq ||T'_y||^2 f(x^*x)
\]
which shows that $\beta(x) \leq ||V_x||^2$.

From Lemma 3.1 we observe that $\mathcal{R}^* = \{ x : f(x^*x) = 0, \text{ for all } f \in \mathfrak{C} \} = RI(\mathfrak{C})$. This formula, as we shall see in § 4, can be invalid if $\mathfrak{C}$ is replaced by $\mathfrak{D}$.

We consider next a version of Kelley and Vaught's result [5, Theorem 4.4].

3.4. THEOREM. Let $x \rightarrow V_x$ be the conical $*$-representation of $A$ induced by $\mathfrak{C}$. Then $||V_x||^2 = \text{dist} (-x^*x, P_1)$.

Let $h \in H$, $||h|| \leq 1$. In the algebra $A_i$ let $B$ be the real subalgebra generated by $e$ and $h$ and let $B_e$ be its completion. For $m = 1, 2, \cdots$ let
\[
w_m = \sum_{k=1}^{m} \left( \begin{array}{c} 1/2 \\ k \end{array} \right) (-1)^{k} h^k.
\]
Clearly $w_m \in H$. In $B_e$ we have $(e - h) = [\lim (e + w_m)]^2$ so that, in $A_i$, we get
\[
(3.3) \quad e - h = \lim_{m} (e + w_m)^2.
\]
This shows that, in \( H \), \( e \) is an interior point of the cone \( P \).

The discussion in \([6, \text{p. 96}]\) shows that any \( f \in \mathbb{C} \) is extendable to \( A_1 \) so as to belong to \( \mathcal{D}_1 \) where \( f(e) \leq 1 \). On the other hand if \( g \in \mathcal{D}_1 \), \( g(e) \leq 1 \) then its restriction to \( A \) lies in \( \mathbb{C} \) by the Cauchy-Schwarz inequality. Since \( \| e \| = 1 \) and \( e \) is an interior point of \( P_1 \) we see, from Lemma 1.3 of \([5]\), that, for each \( x \in A \),

\[
\text{dist}(-x^*x, P_1) = \sup_{f \in \mathcal{D}_1 \setminus \{0\}} f(x^*x) = \sup_{f \in \mathcal{D}_1 \setminus \{0\}} f(x^*x).
\]

An application of formula (3.1) completes the proof.

4. Faithful \(^*\)-representations.

4.1. **Theorem.**

(a) \( Z(\mathcal{D}) \) is a two-sided ideal of \( A \).

(b) \( Z(\mathcal{D}) \subset R(\mathcal{D}) \subset \mathbb{R}^* \) and the inclusions can be proper.

(c) If \( A \) has an identity, \( Z(\mathcal{D}) = R(\mathcal{D}) = \mathbb{R}^* \).

(d) If \( x \in \mathbb{R}^* \) then \( x^* \in Z(\mathcal{D}) \).

(e) \( \mathbb{R}^* \) is the complete inverse image of the radical of \( A/Z(\mathcal{D}) \) under the natural homomorphism of \( A \) onto \( A/Z(\mathcal{D}) \).

We refer to formula (3.2) for notation. For each \( m = 1, 2, \ldots \) we define the operator \( \alpha_m \) on \( H \) by the rule \( \alpha_m(x) = (e + w_m)x(e + w_m) \). Since \( \alpha_m(x^*x) = (x + xw_m)^*(x + xw_m) \) we see that \( \alpha_m(R_\mathbb{C}) \subset R_\mathbb{C} \). Because \( \alpha_m \) is continuous on \( H \) by Lemma 2.2, we also get \( \alpha_m(P) \subset P \).

Suppose next that also \( h \in P \). Then \( (e + w_m)h(e + w_m) = h(e + w_m)^2 \in P \). Passing to the limit as \( m \to \infty \) we see from (3.3) that \( h - h^2 \in P \). We have established that, for any \( h \in P \) whatever its norm, \( f(h) \| h \| \geq f(h^2) \geq 0 \), \( f \in \mathcal{D} \). By the Cauchy-Schwarz inequality \( |f(hx)|^2 \leq f(h^2)f(x^*x) \) and \( |f(xh)|^2 \leq f(xx^*)f(h^2) \), \( f \in \mathcal{D} \). Now \( P \cap (-P) = \{ y \in H : f(y) = 0, f \in \mathcal{D} \} \), so that \( f(yx) = f(xy) = 0 \) for all \( x \in A, f \in \mathcal{D} \). Next let \( w \in Z(\mathcal{D}) \). We can write \( w = y_1 + iy_2 \) where each \( y_s \in P \cap (-P) \). We then see that \( f(wxw) = f(wx) \) for all \( f \in \mathcal{D}, x \in A \), so that \( wxw \) and \( xw \) lie in \( Z(\mathcal{D}) \). This establishes (a).

Let \( x \in Z(\mathcal{D}) \). By (a) we see that \( x^*x \in Z(\mathcal{D}) \) so that \( f(x^*x) = 0 \) for all \( f \in \mathcal{D} \). Thus \( Z(\mathcal{D}) \subset R(\mathcal{D}) \). Next let \( x \in R(\mathcal{D}), y \in A \). Then \( xy \in R(\mathcal{D}) \) so that \( x \in (R(\mathcal{D}) : A) = \mathbb{R}^* \) by Lemma 3.3.

We now produce an example for which \( Z(\mathcal{D}) \neq R(\mathcal{D}) \). Let \( A = C([0,1]) \) with the usual norm and involution but considered as a zero algebra. Then all linear functionals on \( A \) are positive. This implies that \( R(\mathcal{D}) = A \). On the other hand it is trivial that \( Z(\mathcal{D}) = \{0\} \).

We now provide an instance where \( R(\mathcal{D}) \neq \mathbb{R}^* \). Let \( q(w) \) be the function \( q(w) = w \) on \([0,1]\). Again we take \( A = C([0,1]) \) with the
usual norm and involution but define the product by the rule \( xy = x(0)y(0)q \). Under these definitions \( A \) is a Banach algebra and \( A^* = 0 \). Since the radical of \( A \) is contained in \( \mathcal{R}^* \) by Lemma 3.1, we see that \( \mathcal{R}^* \subset A \). Now for any linear functional \( f \) on \( A \), \( f(x^*x) = |x(0)|^2f(q) \).

By the Hahn-Banach theorem, there exists a continuous real linear functional \( g \) on \( H \) such that \( g(q) = 1 \). We extend \( g \) to \( A \) by the rule \( g(h_1 + ih_2) = g(h_1) + ig(h_2) \) where \( h_1, h_2 \in H \). Then \( g \in \mathcal{D} \). If \( x \in \mathcal{R}I(\mathcal{D}) \), \( g(x^*x) = |x(0)|^2 = 0 \). Thus \( \mathcal{R}I(\mathcal{D}) = \{ x \in A : x(0) = 0 \} \). This completes the proof of (b).

Suppose that \( A \) has an identity \( e \). For any \( x \in \mathcal{R}^* \), \( x = xe \subset \mathcal{R}I(\mathcal{D}) \) by Lemma 3.3. Next take \( x \in \mathcal{R}I(\mathcal{D}) \). Since \( |f(x)|^2 \leq f(e)f(x^*x) = 0 \), for all \( f \in \mathcal{D} \), we see that \( x \in \mathcal{Z}(\mathcal{D}) \). Combining this information with the set inequalities of (b) we obtain (c).

By Lemma 3.1 there exists a \(*\)-representation of \( A \), continuous on \( H \), with kernel \( \mathcal{R}^* \). By restricting this \(*\)-representation to \( A \) we see that

\[
\mathcal{R}^* \subset A \cap \mathcal{R}^*_1.
\]

Let \( \lambda \) be a scalar and \( x, y \in A \). Then \( y^*(\lambda e + x)^*(\lambda e + x)y = (\lambda y + xy)^*(\lambda y + xy) \in R_0 \). Thus \( y^*R_0 y \subset R_0 \) for each \( y \in A \). From Lemma 2.2 it can be seen that, for \( h \in H \), the mapping \( x \to hxh \) is continuous on \( H \). It is easily shown that \( x \to hxh \) is also continuous on \( H \). Then it follows that \( hP_ih \subset P \). This shows that \( h[P_1 \cap (-P) + iP_1 \cap (-P)]h \subset P \cap (-P) + iP \cap (-P) \). By (c) this gives

\[
(4.1) 
\mathcal{R}^*h \subset \mathcal{Z}(\mathcal{D}), h \in H.
\]

From (4.1) and (4.2) we have \( h\mathcal{R}^*h \subset \mathcal{Z}(\mathcal{D}) \). It follows readily that \( uzw + wzu \in \mathcal{Z}(\mathcal{D}) \) for all \( u, w \in H \) and \( z \in \mathcal{R}^* \). Let \( x = u + iv \in \mathcal{R}^* \), \( u, v \in H \) and note that \( u, v \in \mathcal{R}^* \). Writing \( x^3 = u^3 - iv^3 + i(u^2v + uv^2) + ivuv - vu - (v^2u + uv^2) \) we see that the individual terms of the expansion lie in \( \mathcal{Z}(\mathcal{D}) \).

We turn to (e). Let \( \gamma \) be the natural homomorphism of \( A \) onto \( A/\mathcal{Z}(\mathcal{D}) \). For \( x \in \mathcal{R}^* \), \( |\gamma(x)|^2 = 0 \) by (d) so that \( \gamma(\mathcal{R}^*) \subset \mathcal{W} \), the radical of \( A/\mathcal{Z}(\mathcal{D}) \). Inasmuch as \( A/\mathcal{R}^* \) is semi-simple by Lemma 3.1, so is \( [A/\mathcal{Z}(\mathcal{D})]/[\mathcal{R}^*/\mathcal{Z}(\mathcal{D})] \). Therefore \( \mathcal{R}^*/\mathcal{Z}(\mathcal{D}) \subset \mathcal{W} \).

4.2. Theorem. The following statements are equivalent.

(a) There exists a faithful \(*\)-representation of \( A \) continuous on \( H \).

(b) \( A \) is semi-simple and \( P \cap (-P) = 0 \).

(c) \( A \) is semi-simple and \( RI(\mathcal{D}) = 0 \).

Suppose (a). \( A \) is semi-simple by \([8, \text{Theorem 4.1.19}] \). Lemma 3.2 gives \( \mathcal{R}^* = 0 \) so that \( P \cap (-P) = 0 \) from Theorem 4.1 (b).
Suppose (b). Then \( Z(\mathfrak{D}) = 0 \) so that, by Theorem 4.1 (d), \( x^* = 0 \) for each \( x \in \mathfrak{R}^* \). Since \( A \) is semi-simple, \( \mathfrak{R}^* = (0) \). Then Theorem 4.1 (b) shows that \( RI(\mathfrak{D}) = (0) \).

Suppose (c). Again \( Z(\mathfrak{D}) = 0 \) by Theorem 4.1 (b). As just seen this implies that \( \mathfrak{R}^* = (0) \) so that (a) follows from Lemma 3.2.

4.3. COROLLARY. Let \( A \) be a normed *-Q-algebra. Then \( A \) has faithful *-representation if and only if \( A \) is semi-simple and \( P \cap (-P) = (0) \).

This follows immediately from Theorem 4.2 and Lemma 2.4.

We now exhibit a normed *-algebra with a faithful *-representation but for which \( P \cap (-P) \neq (0) \). Let \( A \) be the algebra of all polynomials in the complex variable \( z \). If \( p(z) = \sum \alpha_n z^n \) set \( p^*(z) = \sum \bar{\alpha}_n z^n \).

First consider \( A \) in the norm

\[
||p|| = \sup_{t \in [0,1]} |p(t)|.
\]

Here, for each \( t \), \( 0 \leq t \leq 1 \) the functional \( f_t(p) = p(t) \) is a positive linear functional continuous on \( A \) and real-valued on \( H \). Thus \( Z(\mathfrak{D}) = (0) \). By Theorem 4.2 we see that \( A \) has a faithful *-representation. This also justifies a remark following Lemma 2.5.

Next consider \( A \) in the norm \( ||p|| = \sum |\alpha_k|/k! \) (see §2). For \( p(z) = \alpha_0 + \cdots + \alpha_n z^n \) let \( f(p) = \alpha_0 \). This gives us a continuous *-representation of \( A \) as operators on one-dimensional Hilbert space with kernel \( M = \{p: p(0) = 0\} \) so that \( M \supseteq \mathfrak{R}^* \). The arguments of §2 following Lemma 2.5 show that any *-representation of \( A \) continuous on \( H \) must vanish on \( M \). Therefore \( M = \mathfrak{R}^* \). Via Theorem 4.1 we see that \( P \cap (-P) \) is the set of all polynomials with real coefficients vanishing at the origin. We investigate the commutative case more closely in §5.

4.4. LEMMA. \( \mathfrak{R}^* = A \cap \mathfrak{R}_i^* \).

We already have \( \mathfrak{R}^* \subset A \cap \mathfrak{R}_i^* \) by (4.1). Let \( \mathfrak{F} = A \cap \mathfrak{R}_i^* \). By (4.2), \( h\mathfrak{F}h \subset Z(\mathfrak{D}) \) for each \( h \in H \). Reasoning exactly as in the proof of Theorem 4.1 (d) we obtain \( x^* \in Z(\mathfrak{D}) \) for each \( x \in \mathfrak{F} \). Let \( \beta \) be the natural homomorphism of \( A \) onto \( A/\mathfrak{R}^* \). Since \( Z(\mathfrak{D}) \subset \mathfrak{R}^* \) by Theorem 4.1 (b), we see that \( [\beta(x)]^* = 0 \) for each \( x \in \mathfrak{F} \). From Lemma 3.1 we obtain \( \beta(\mathfrak{F}) = (0) \). We now derive another formula for \( \mathfrak{R}^* \).

4.5. THEOREM. \( \mathfrak{R}^* = Z(\mathfrak{E}) \).

As noted in the proof of Theorem 3.4, \( \mathfrak{E} \) is the set of positive
linear functionals on $A$ which are extendable to positive linear functionals $f$ on $A_1$, lying in $\mathcal{D}_1$, with $f(e) \leq 1$. Now any $g \in \mathcal{D}_i$ is a multiple of such a functional. Therefore $Z(\mathcal{G}) = A \cap Z(\mathcal{D}_i) = A \cap \mathbb{R}_i^* = \mathbb{R}^*$ by way of Theorem 4.1 and Lemma 4.4.

We have $\mathbb{R}^* = RI(\mathcal{G}) = Z(\mathcal{G})$, a situation which differs from what can happen for $\mathcal{D}$. In particular $Z(\mathcal{G}) \neq Z(\mathcal{D})$ can occur.

5. The commutative case. Let $A$ be a commutative algebra with an involution. By commutativity, $H$ is a real subalgebra of $A$. We suppose in § 5 that $H$ has a norm in terms of which it is a real normed algebra. Let $\mathcal{M}$ be the set of modular maximal ideals of $A$. We call $M \in \mathcal{M}$ symmetric if $M = M^*$ and single out for special attention the set of symmetric $M$ for which $M \cap H$ is closed in $H$.

5.1. Lemma. Let $\mu$ be a homomorphism of $H$ into the reals. Define, for each $x = h + ik$, $h, k \in H$ the functional $\mu_x$ by the rule $\mu_x(x) = \mu(h) + i\mu(k)$. Then $\mu_x$ is a multiplicative (complex) linear functional on $A$.

This can be verified in a straightforward way.

5.2. Lemma. Let $M$ be a symmetric modular maximal ideal of $A$ where $M \cap H$ is closed in $H$. Then there exists a continuous homomorphism $\mu$ of $H$ onto the reals such that $\mu^{-1}(0) = M$.

Let $j$ be an identity for $A$ modulo $M$. Then so is $(j + j^*)/2$ so without loss of generality we can take $j$ s.a. Then $ju - u \in M \cap H$ for all $u \in H$ and therefore $M \cap H$ is a modular ideal of $H$. Since $M = M \cap H \oplus i(M \cap H)$ it is clear the $M \cap H \neq H$. We claim that $M \cap H$ is a modular maximal ideal of $H$. For otherwise there exists a modular maximal ideal $K$ of $H$ containing $M \cap H$, $K = M \cap H$. An easy computation shows that $K \oplus iK$ is an ideal of $A$ containing $M$. Then $K \oplus iK = A$ which is impossible as $j \in K$ (otherwise $K = H$). Inasmuch as $M \cap H$ is closed in $H$, $H/M \cap H$ is a normed field in the quotient algebra norm. By Mazur's theorem, $H/M \cap H$ is a copy of the real or complex field. We rule out the latter possibility. If $H/M \cap H$ were a copy of the complexes then it would be two-dimensional over the real field and there would be a two-dimensional real subspace $L$ of $H$ such that $H = M \cap H \oplus L$. Then $A = H \oplus iH = M \oplus L \oplus iL$ which compels $A/M$ to be four-dimensional over the reals. But surely $A/M$ is a division algebra over the reals. Thus a well-known theorem of Frobenius makes $A/M$ a copy of the quaternions. This is impossible in view of commutativity. Consequently there is a continuous homomorphism $\mu$ of $H$ onto the reals with kernel $M \cap H$. 
5.3. **Theorem.** \( \mathbb{R}^* \) is the intersection of those symmetric modular maximal ideals \( M \) of \( A \) such that \( M \cap H \) is closed in \( H \).

Take \( x = u + iv, u, v \in H \). By commutativity, \( x^*x = (u^2 + v^2)/2 \). Thus \( P \) is the closure in \( H \) of finite sums of squares of elements of \( H \). Suppose first that \( A \) has an identity \( e \). The proof of Theorem 3.4 shows that \( e \in \text{Int}(P) \). Let \( \Sigma \) represent the set of all continuous real linear functionals \( g \) on \( H \) where \( g(P) \geq 0 \) and \( g(e) \leq 1 \). The arguments of [5, Theorem 2.1] show that the set \( \Sigma_* \) of extreme points of \( \Sigma \) is the set the continuous homomorphisms of \( H \) into the reals. As in [5, Remark 2.3] at follows that \( P \cap (-P) = \bigcap f^{-1}(0) \) where \( f \) ranges over \( \Sigma_* \). Let \( S \) be the intersection of the symmetric modular maximal ideals \( M \) of \( A \) with \( M \cap H \) closed in \( H \). Lemmas 5.1 and 5.2 show that \( H \cap S = \bigcap f^{-1}(0) = P \cap (-P) \) and Lemma 4.1 (b) shows that \( S = \mathbb{R}^* \).

Now suppose that \( A \) has no identity. Each multiplicative linear functional on \( A \) which is real and continuous on \( H \) extends, as is easily verified, to a multiplicative linear functional on \( A \), which is real and continuous on \( H \). Applying the result for the case with the identity we get \( S = A \cap \mathbb{R}^*_\ast = \mathbb{R}^* \) with the aid of Lemma 4.4.

6. **An example.** We give an example of a normed *-algebra \( A \) which has a continuous faithful *-representation and a continuous involution but for which the completion\(^2\) \( A_\ast \) has no faithful *-representation. This demonstrates conclusively that our results in the case of a normed *-algebra (e.g. Theorem 4.2 and Corollary 4.3) cannot possibly deduced from the theory of Banach algebras.

The algebra \( A \) which we use is a subalgebra of an algebra devised for other purposes by C. Feldman [2]. His algebra is the commutative algebra \( B \) which is the completion of the algebra of all finite sums

\[
\sum_{i=1}^{n} \alpha_i e_i + \beta r
\]

where \( \alpha_i \) and \( \beta \) are complex, the \( e_i \) are mutually orthogonal idempotents, \( r^2 = 0 = e_i r = re_i \) for all \( i \) and

\[
\| \Sigma \alpha_i e_i + \beta r \| = \max \{ (\Sigma | \alpha_i |) \|^{1/2}, | \beta - \Sigma \alpha_i | \}.
\]

Consider the subalgebra \( A \) consisting of all finite sums \( \Sigma \alpha_i e_i \). The involution

\[
(\Sigma \alpha_i e_i)^* = \Sigma \bar{\alpha}_i e_i
\]
on \( A \) is an isometry. For each integer \( n > 1 \) let \( s(n) \) be the smallest

---

The involution on \( A \) extends to an involution on \( A_\ast \).
integer of the form $n+k$ such that $\sum_{j=0}^{k} (n+j)^{-1} > 1$. Let $z_n = n^{-1}e_n + (n+1)^{-1}e_{n+1} + \cdots + [s(n)]^{-1}e_{s(n)}$. It is readily verified that $\| r - z_n \| \to 0$. Therefore $B$ is the completion of $A$. For each $n = 1, 2, \cdots$ the functional $f_n$ defined on $A$ by the rule $f_n(\sum \alpha_i e_i) = \alpha_n$ is a continuous multiplicative linear functional on $A$. Moreover $\bigcap f_n^{-1}(0) = (0)$ so that $A$ is semi-simple and, by Theorem 5.3 and Lemma 3.2, $A$ has a faithful *-representation continuous on $H$. The continuity of the involution allows us to assert that this *-representation is continuous on $A$. However, the completion $B$ of $A$ is not semi-simple [2] and so has no faithful *-representation [8, Theorem 4.1.19].

REFERENCES


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