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It is well known that if a series of real numbers $\sum_{n=1}^{\infty} a_n$ converges, but not absolutely, then for any b, there exists a sequence $\{x_i\}, x_i = \pm 1$, such that $\sum_{n=1}^{\infty} a_n x_n = b$. In § 1, a criterion is given on a system of denumerably many equations of this type, with real coefficients, so that solutions $x_i = \pm 1$ exist for arbitrary right hand sides. A sequence $\{x_i\}$ such that $x_i = \pm 1$ will be called unimodular. In § 2, there results are extended to finite systems, and it is shown that an infinite system has unimodular solutions for arbitrary right hand sides if and only if every finite subsystem has this property. § 3 shows that if a system satisfies the criterion of § 1, then, in a certain sense, "almost any" sequence $\{x_i\}$, $x_i = \pm 1$, "satisfies" the system for any choice of right hand sides. In § 4, conditions are given whereby infinite systems can be constructed which satisfy the criterion of § 2. It follows, for example, that the system

$$\sum_{j=1}^{\infty} (-1)^{[j/2^i]} j^{-\alpha} x_j = b_i\,, \qquad i = 1, 2, \cdots; 0 < \alpha \leqq 1$$

has solutions ($x_i = \pm 1$) for any b_i ($i = 1, 2, \cdots$). The b_i are allowed to be real numbers or $\pm\infty$.

1. The main theorem.

THEOREM 1. *Let a_{ij} $(i, j = 1, 2, 3, \cdots)$ be real numbers such that there exist x_{jkl} $(j = 1, 2, \cdots; k = 0, 1, 2, \cdots; l = 1, 2, \cdots)$ which satisfy the following conditions:*

1. *Each x_{jkl} is equal to $+1$ or -1.*
2. *$\sum_{j=1}^{\infty} a_{ij} x_{jkl}$ converges for all i such that $i \neq k$ and $i \leqq l$.*
3. *$\sum_{j=1}^{\infty} a_{ij} x_{jil}$ diverges to $+\infty$.*

Then, for any sequence $\{b_i\}$, the infinite system of equations

$$\sum_{j=1}^{\infty} a_{ij} x_j = b_i \tag{1}$$

can be solved such that for each i, $x_i = \pm 1$. Here, b_i is allowed to be either a real number or $\pm\infty$.

Proof. If $k \neq i \leqq l$, for any $\varepsilon > 0$ there exists $N(\varepsilon; i, k, l)$ such

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that

$$\left|\sum_{j=m+1}^{n} a_{ij} x_{jkl}\right| < \varepsilon \text{ and } |a_{in}| < \varepsilon/2 , \tag{2}$$

provided $m, n > N(\varepsilon; i, k, l)$.

We define the solution $\{x_i\}$ inductively along with positive integers M_{nm} which will be defined whenever n is a positive integer and m is a nonnegative integer such that $m \leqq n$. The ordered pairs (n, m) are ordered lexicographically, i. e., $(n, m) < (n_1, m_1)$ if and only if either $n < n_1$, or $n = n_1$ and $m < m_1$. The induction will be with respect to this order.

The following definitions will be used with $m \leqq n$ and $i \leqq n$:

$$B_{inm} = \begin{cases} b_i \text{ if } b_i \text{ is finite} \\ \pm n \text{ if } b_i = \pm\infty \text{ and } i \leqq m , \\ \pm(n-1) \text{ if } b_i = \pm\infty \text{ and } i > m \end{cases} \tag{3}$$

$$A_{inm} = \begin{cases} b_i \text{ if } b_i \text{ is finite} \\ B_{inm} \text{ if } i \neq m , \\ \pm(n-1) \text{ if } i = m \text{ and } b_i = \pm\infty \end{cases} \tag{4}$$

$$\delta_{inm} = \begin{cases} (2(m-i)+1)/n^2 \text{ if } i < m \\ 2(n-i)/(n-1)^2 + 2m/n^2 \text{ if } i \geqq m . \end{cases} \tag{5}$$

Let us suppose that positive integers M_{nm} have been defined for $(n, m) \leqq (s, t)$, and x_i for $i \leqq M_{st}$ such that the following conditions are satisfied:

(A) $M_{nm} < M_{pq}$ if and only if $(n, m) < (p, q)$.

(B) $M_{n0} \geqq N(1/n^2; i, k, n)$ for all $i, k \leqq n$ $(i \neq k)$.

(C) $\left|\sum_{j=1}^{M_{nm}} a_{mj} x_m - B_{mnm}\right| < 1/n^2$ where $m \neq 0$.

(D) If $i \leqq n-1$ and $M_{nm} \geqq p > \begin{cases} M_{n,m-1} \text{ if } m \neq 0 \\ M_{n-1,n-1} \text{ if } m = 0, \end{cases}$

then

$$\text{(D1)} \quad A_{inm} - \delta_{inm} < \sum_{j=1}^{p} a_{ij} x_j < B_{inm} + \delta_{inm} \text{ if } A_{inm} \leqq B_{inm}$$

and

$$\text{(D2)} \quad B_{inm} - \delta_{inm} < \sum_{j=1}^{p} a_{ij} x_j < A_{inm} + \delta_{inm} \text{ if } B_{inm} \leqq A_{inm}.$$

We wish to determine M_{uv} where (u, v) is the immediate successor of (s, t), and x_i, $i = M_{st}+1, \cdots, M_{uv}$, such that the conditions (A)–(D) are valid for all $(n, m) \leqq (u, v)$. There are two cases to consider. Either we have $s = t$, or $s > t$.

Case I. $s = t$. In this case the immediate successor of (s, t) is $(s+1, 0)$. Putting $M_{s+1,0}$ equal to the largest of the numbers $M_{ss}+1$ and $N(1/(s+1)^2; i, k, s+1)$ for all $i, k \leqq s+1$ $(i \neq k)$, we see that (A) and (B) will be satisfied.

We now put $x_j = x_{jos}$, $(j = M_{ss}+1, \cdots, M_{s+1.0})$. Condition (C) remains satisfied because the newly defined quantities do not occur in (C). Condition (B) holds with $n = s$ by the inductive assumption. Therefore, $|\sum_{j=k+1}^{l} a_{ij}x_{jos}| < 1/s^2$, provided $k, l > M_{ss}$ and $i \leqq s$. We have $\delta_{iss} = (2(s-i)+1)/s^2$, $\delta_{i,s+1,0} = 2(s+1-i)/s^2$, and hence $\delta_{i,s+1,0} - \delta_{iss} = 1/s^2$. From the equality $A_{iss} = B_{iss} = A_{i,s+1,0} = B_{i,s+1,0}$ for $i < s$, we see that (D) holds with $i < s$. It must be shown that (D) holds with $i = s$. We have also $B_{sss} = B_{s,s+1,0} = A_{s,s+1,0}$. Recall that

$$\left|\sum_{j=M_{ss}}^{l} a_{ij}x_{jos}\right| < 1/s^2 .$$

Since $\delta_{s,s+1,0} = 2/s^2$ and (C) holds with $i = s$, the result follows, namely that (D) holds with $i = s$. This disposes of Case I.

Case II. $s > t$. The immediate successor of (s, t) is $(s, t+1)$. We use the fact that $\sum_{j=1}^{\infty} a_{t+1,j}x_{j,t+1,s} = +\infty$.

Subcase IIA. For some $l > M_{st}$ we have

$$\left|\sum_{j=1}^{l} a_{t+1,j}x_{jos} - B_{t+1,s,t+1}\right| < 1/s^2 .$$

In this case, we put $l = M_{s,t+1}$ and $x_j = x_{jos}$, $j = M_{st}+1, \cdots, M_{s,t+1}$.

Subcase IIB. If the above never happens, then

$$\sum_{j=1}^{l} a_{t+1,j}x_{jos} - B_{t+1,s,t+1}$$

must keep the same sign for all $l > M_{st}$, because, for $j > M_{st}$, we have $|a_{t+1,j}| < 1/2s^2$, from inductive assumption (B).

Let $\sigma = \pm 1$, depending upon whether the sign stays $+$ or $-$. Because the series $\sum_{j=1}^{\infty} a_{t+1,j}x_{j,t+1,s}$ diverges to $+\infty$, there exists $K > M_{st}$ such that

$$(6) \qquad \sum_{j=M_{st}+1}^{l} a_{t+1,j}x_{jos} + \sum_{j=M_{st}+1}^{l} a_{t+1,j}x_{j,t+1,s} > 0$$

for all $l > K$. Let K_0 be the smallest number K with this property. We put $x_j = x_{jos}$, $M_{st} < j < K_0$, and $x_j = -x_{j,t+1,s}$, $j = K_0, \cdots, M_{s,t+1}$, where the integer $M_{s,t+1}$ will now be defined. Because $\sum_{j=1}^{\infty} a_{t+1,j}x_{j,t+1,s} = +\infty$, and $|a_{t+1,j}| < 1/2s^2$ for $j > M_{st}$, there exist integers $M > M_{st}$

such that (C) is satisfied for $n = s$, $m = t+1$ and $M_{s,t+1} = M$. Let $M_{s,t+1}$ be the smallest integer with the above property.

It will be shown that conditions (A)–(D) hold for both subcases. Conditions (A) and (B) are evident. Condition (C) follows immediately from the construction above. Condition (D) is somewhat more difficult.

It will be shown first that (D) holds with $i = t+1$, $n = s$, $m = t+1$. We may suppose $i \leqq n-1$, i.e., $t+1 \leqq s-1$. We have

$$\delta_{t+1,s,t+1} - \delta_{t+1,s,t} = 2/s^2 , \tag{7}$$

and

$$B_{t+1,s,t} = A_{t+1,s,t} = A_{t+1,s,t+1} .$$

From inductive assumption (D), we have

$$\left|\sum_{j=1}^{M_{st}} a_{t+1,j}x_j - A_{t+1,s,t+1}\right| < \delta_{t+1,s,t+1} - 2/s^2 . \tag{8}$$

Thus, in Subcase IIA with $M_{st} < p \leqq M_{s,t+1}$, and in Subcase IIB, with $M_{st} < p < K_0 < M_{s,t+1}$, we have

$$\left|\sum_{j=1}^{p} a_{t+1,j}x_j - A_{t+1,s,t+1}\right| < \delta_{t+1,s,t+1} - 1/s^2 .$$

This disposes of Subcase IIA, because the above inequality is stronger than (D). It must be shown now that (D) holds for $K_0 \leqq p \leqq M_{s,t+1}$, with $i = t+1$, $n = s$, $m = t+1$.

From the inequalities,

$$\sigma \sum_{j=M_{st}+1}^{K_0} a_{t+1,j}x_{jos} + \sum_{j=M_{st}+1}^{K_0} a_{t+1,j}x_{j,t+1,s} > 0 ,$$

$$\sigma \sum_{j=M_{st}+1}^{K_0-1} a_{t+1,j}x_{jos} + \sum_{j=M_{st}+1}^{K_0-1} a_{t+1,j}x_{j,t+1,s} \leqq 0 ,$$

(where an empty sum is taken to equal zero), it follows that we have $(p \leqq M_{s,t+1})$

$$\begin{aligned} \sum_{j=M_{st}+1}^{K_0-1} a_{t+1,j}x_{jos} - \sum_{j=K_0}^{p} a_{t+1,j}x_{j,t+1,s} \\ &= \sigma \sum_{j=M_{st}+1}^{p} a_{t+1,j}x_j \\ &< - \sum_{j=M_{st}+1}^{p} a_{t+1,j}x_{j,t+1,s} + 2\,|a_{t+1,K_0}| \\ &< - \sum_{j=M_{st}+1}^{p} a_{t+1,j}x_{j,t+1,s} + 1/s^2 \end{aligned} \tag{10}$$

using (B) and the definition of x_j, $M_{st} < j \leqq M_{s,t+1}$. Now, from (6), we have

$$(11)\qquad -\sum_{j=M_{st}+1}^{p} a_{t+1,j}x_{j,t+1,s} < -\sigma \sum_{j=M_{st}+1}^{p} a_{t+1,j}x_{jos} .$$

Combining (1), (11) and (B), we have

$$(12)\qquad \sigma \sum_{j=M_{st}+1}^{p} a_{t+1,j}x_j < -\sigma \sum_{j=M_{st}+1}^{p} a_{t+1,j}x_{jos} + 1/s^2 < 2/s^2 .$$

From (8), we have

$$(13)\qquad \sigma \sum_{j=1}^{M_{st}} a_{t+1,j}x_j < \sigma A_{t+1,s,t+1} + \delta_{t+1,s,t+1} - 1/s^2 .$$

Putting (12) and (13) together, we have

$$(14)\qquad \sigma \sum_{j=1}^{p} a_{t+1,j}x_j < \sigma A_{t+1,s,t+1} + \delta_{t+1,s,t+1} .$$

From the definition of σ and the fact that if $A_{t+1,s,t+1} \neq B_{t+1,s,t+1}$, then $|A_{t+1,s,t+1} - B_{t+1,s,t+1}| = 1$, it follows that $\sigma(A_{t+1,s,t+1} - B_{t+1,s,t+1}) \geqq 0$. Thus (D1) or (D2) must be demonstrated, depending on whether $\sigma = -1$ or $+1$.

If $\sigma = -1$, we have from (14)

$$A_{t+1,s,t+1} - \delta_{t+1,s,t+1} < \sum_{j=1}^{p} a_{t+1,j}x_j$$

which is half of (D1). From the definition of $M_{s,t+1}$, we have

$$\sum_{j=1} a_{t+1,j}x_j < B_{t+1,s,t+1} + 1/s^2 < B_{t+1,s,t+1} + \delta_{t+1,s,t+1},$$

provided $M_{st} < p \leqq M_{s,t+1}$, which gives the remaining half of (D1). Similar considerations show that if $\sigma = 1$, then (D2) holds. This concludes the demonstration that (D) holds with $i = t+1, n = s, m = t+1$.

It remains to show that (D) holds for $i \neq t+1, n = s, m = t+1$, $i \leqq s-1$.

We have $\delta_{i,s,t+1} - \delta_{i,s,t} = 2/s^2$, so that it is sufficient to show

$$(15)\qquad \left| \sum_{j=M_{st}+1}^{p} a_{ij}x_j \right| < 2/s^2$$

for $M_{st} < p \leqq M_{s,t+1}$.

In Subcase IIA, we have, using (B),

$$(16)\qquad \left| \sum_{j=M_{st}+1}^{p} a_{ij}x_j \right| = \left| \sum_{j=M_{st}+1}^{p} a_{ij}x_{jos} \right| < 1/s^2 .$$

In Subcase IIB with $p < K_0$, (15) is valid once again because (16) holds.

In Subcase IIB with $p \geqq K_0$, we have

$$
\left|\sum_{j=M_{st}+1}^{p} a_{ij}x_j\right| \leqq \left|\sum_{j=M_{st}+1}^{K_0-1} a_{ij}x_{jos}\right| + \left|\sum_{j=K_0}^{p} a_{ij}x_{j,t+1,s}\right| < (1/s^2) + (1/s^2) \tag{17}
$$

because of (B). This concludes the demonstration of Case II.

The definition by induction is completed by setting $M_{01} = N(1; 1, 0, 1)$ and observing that thereby (A)–(D) are satisfied with $n = 1, m = 0, i = 1$.

It remains to show the sequence $\{x_i\}$ constructed in this way satisfies the infinite system of equations (1). However, this follows from the fact that $\{x_i\}$ satisfies condition (D).

2. Systems with finitely many equations. The extension of Theorem 1 to systems with finitely many equations is accomplished by producing an infinite system which can be treated by Theorem 1 and which is equivalent to the given finite system.

THEOREM 2. *Let a_{ij} $(i = 1, \cdots, R; j = 1, 2, \cdots)$ be real numbers such that there exist x_{jk} $(j = 1, 2, \cdots; k = 0, 1, \cdots, R)$ which satisfy the following conditions:*

1. *Each x_{jk} is equal to $+1$ or -1.*
2. *$\sum_{j=1}^{\infty} a_{ij}x_{jk}$ converges for all i such that $i \neq k$.*
3. *$\sum_{j=1}^{\infty} a_{ij}x_{ji}$ diverges to $+\infty$.*

Then, for any numbers $b_1, \cdots, b_R$, each of which is a real number or $\pm\infty$, the equations

$$
\sum_{j=1}^{\infty} a_{ij}x_j = b_i\,, \qquad (i = 1, \cdots, R) \tag{18}
$$

can be solved such that for each i, $x_i = \pm 1$.

Proof. We construct an auxiliary infinite system of equations

$$
\sum_{j=1}^{\infty} \alpha_{ij}\xi_j = \beta_i \qquad (i = 1, 2, \cdots)\,. \tag{19}
$$

We define $\beta_{i+nR} = b_i$ for any nonnegative integer n, and

$$
\alpha_{i+nR,l} = \begin{cases} a_{ik} \text{ if there exists } k > 0 \text{ such that} \\ \quad l = T(k+n-1) + n - 1 \text{ and} \\ 0 \text{ otherwise,} \end{cases} \tag{20}
$$

where $T(n) = n(n+1)/2$ is the nth triangular number. The fact to be used about $T(n)$ is that each positive integer has one and only one representation in the form $T(k+n-1)+n-1=S(k,n)$, where k and n are positive integers.

Let us define $\xi_{jkl}=\xi_{jk}$ as follows:

$$\xi_{l,i+nR}=\begin{cases} x_{ki} \text{ if there exists } k>0 \text{ such that } l=S(k,n), \text{ and} \\ x_{k0} \text{ if } l=S(k,m),\ n\neq m>0\ . \end{cases} \tag{21}$$

Then we have

1. Each ξ_{jk} is equal to $+1$ or -1.
2. $\sum_{j=1}^{\infty}\alpha_{ij}\xi_{jk}$ converges for all i such that $i\neq k$.
3. $\sum_{j=1}^{\infty}\alpha_{ij}\xi_{ji}$ diverges to $+\infty$.

The hypotheses of Theorem 1 are satisfied, and therefore the system (19) has a solution $\{\xi_j\}$. Then $x_j=\xi_{T(j)}$, $j=1,2,\cdots$, is a solution of (18).

COROLLARY. *The system* (1), *with arbitrary right hand sides, has a unimodular solution if and only if every finite subsystem of* (1), *with arbitrary right hand sides has a unimodular solution.*

A system of nondenumerably many equations of the type described in Theorem 1 will never have unimodular solutions for all possible right hand sides, because the number of ways in which the right hand sides could be prescribed would have cardinality greater than C, whereas the cardinality of all unimodular sequences, $x_i=\pm 1$, $i=1,2,\cdots$, is equal to C. (Here C denotes the cardinality of the continuum.)

3. The metric space $\mathscr{M}$. The set of sequences $\{x_i\}$, $x_i=\pm 1$ form a complete metric space under the metric

$$d(\{x_i\},\{x_i'\})=1/l\ ,$$

where $l=\min\{i: x_i\neq x_i'\}$.

Let a_{ij} satisfy the hypotheses of Theorem 1. Let U_i, $i=1,2,\cdots$, be nonempty open sets of extended real numbers. (U_i may contain $+\infty$ or $-\infty$.) Let $\mathscr{N}_M$ be the set of sequences $\{x_i\}$ such that for all $N\geqq M$, $\sum_{j=1}^{N}a_{ij}x_j\notin U_i$ for some i $(0<i\leqq M)$.

$\mathscr{N}_M$ is closed. For suppose $\{x_i^n\}\in\mathscr{N}_M$ and $\lim_{n\to\infty}d(\{x_i^n\},\{x_i\})=0$. Also, suppose there exists $N\geqq M$ such that $\sum_{j=1}^{N}a_{ij}x_j\in U_i$ for each i $(0<i\leqq M)$. For sufficiently large n, we have $x_j=x_j^n$, $j=1,\cdots,N$, and hence we get $\sum_{j=1}^{N}a_{ij}x_j^n\in U_i$ $(0<i\leqq M)$, contrary to the assumption $\{x_i^n\}\in\mathscr{N}_M$.

$\mathscr{N}_M$ is nowhere dense. For suppose $\{x_i\}\in\mathscr{N}_M$. Let b_i be an arbitrary element of U_i. For any $P > 0$, there exists, because of Theorem 1, a sequence $\{x_i\}$ such that

1. $x'_i = x_i$, $i = 1, 2, \cdots, P$, and
2. $\sum_{j=1}^{\infty} a_{ij}x'_j = b_i$.

Clearly, $\{x'_i\} \notin \mathscr{N}_M$ and $d(\{x_i\}, \{x'_i\}) < 1/P$.

Thus the set $\bigcup_{M=1}^{\infty} \mathscr{N}_M = \mathscr{N}$ is of the first category, and since $\mathscr{M}$ is a complete metric space, $\mathscr{M} - \mathscr{N}$ is of the second category. We have proved the following:

LEMMA 1. *For any sequence $\{x_n\}$ in $\mathscr{M} - \mathscr{N}$ there exists an infinite monotone increasing sequence $\{N_k\}$ of positive integers such that for each k, $\sum_{j=1}^{N_k} a_{ij}x_j \in U_i$ for $i \leqq k$.*

For any sequence $\{b_i\}$ of extended real numbers we may take U_i^n as follows:

$$U_i^n = \begin{cases} \{x: |x - b_i| < 1/n\} \text{ if } b_i \text{ is finite} \\ \{x: \pm(x - b_i) > n\} \text{ if } b_i = \pm\infty \,. \end{cases}$$

By applying the lemma for each n to $\{U_i^n\}$ $0 < i < \infty$, we find that there exists a monotone increasing sequence of positive integers $\{S_k\}$ such that

$$\sum_{j=1}^{S_k} a_{ij}x_j \in U_i^k \qquad \text{for } i \leqq k \,. \tag{23}$$

From (22), it now follows that we have

$$\lim_{k\to\infty} \sum_{j=1}^{S_k} a_{ij}x_j = b_i \qquad \text{for every } i > 0 \,.$$

In summary, this proves the following:

THEOREM 3. *Let a_{ij} satisfy the hypotheses of Theorem* 1. *Then there exists a sequence $\{x_i\}$, $x_i = \pm 1$, with the following property. (Indeed, any sequence $\{x_i\}$ in the complete metric space $\mathscr{M}$, apart from a certain set of first category, has this property.) For any sequence $\{b_i\}$ of extended real numbers, there exists a sequence of positive integers $\{S_k\}$ such that for each i,*

$$\lim_{k\to\infty} \sum_{j=1}^{S_k} a_{ij}x_j = b_i \,.$$

4. Sufficient conditions. In this section we shall find sufficient conditions on the coefficients a_{ij} so that the hypotheses of Theorem 1 are satisfied.

THEOREM 4. *Let $\{a_i\}$ be a sequence of real numbers such that*

(i) $a_i > 0$

(ii) $\sum_{i=1}^{\infty} a_i = \infty$.

(iii) *For every* $k \geqq 1$, $a_i - a_{j+k}$ *is monotone decreasing in* i.

(iv) a_i *tends monotonically to zero as* $i \to \infty$.

Let $a_{ij} = (-1)^{[j/2^i]} a_j$. *Then* a_{ij} *satisfies the hypotheses of Theorem* 1.

Proof. We must find sequences $\{x_{jkl}\}$, $1 \leqq j < \infty$, $0 \leqq k < \infty$, $1 \leqq l < \infty$, such that conditions (1), (2) and (3) of Theorem 1 are fulfilled.

$k = 0$. First we show that by putting $x_{j0l} = 1$, the conditions are satisfied for $k = 0$. Condition (3) is fulfilled vacuously and condition (1) is trivial.

It will be shown that condition (2) holds, i.e., that $\sum_{j=1}^{\infty} (-1)^{[j/2^i]} a_j$ converges for each i. Let

(24) $$(-1)^k b_k = \sum_{j=k\cdot 2^i}^{(k+1)2^i - 1} (-1)^{[j/2^i]} a_j \ .$$

Then we have

(25) $$b_k = \sum_{j=k\cdot 2^i}^{(k+1)2^i - 1} a_j > 0 \ .$$

From (iv), b_k is monotone decreasing, and hence $\sum (-1)^k b_k$ converges. The condition (2) follows because

(26) $$\sum_{j=1}^{k} (-1)^i b_j = \sum_{j=0}^{(k+1)2^i - 1} (-1)^{[j/2^i]} a_j \ .$$

$k \neq 0$. Let $x_{jkl} = (-1)^{[j/2^k]}$. Since it is assumed that $\sum_{i=1}^{\infty} a_i = \infty$, we have $\sum_{j=1}^{\infty} a_{ij} x_{jil} = \infty$, and thus condition (3) holds.

We will show (2) holds and thereby complete the proof by showing that

(27) $$\sum_{j=1}^{\infty} (-1)^{[j/2^i] + [j/2^k]} a_j$$

converges if $i > k$. We have

(28) $$[j/2^i] + [j/2^k] = [(j + 2^i)/2^i] + [(j + 2^i)/2^i] - 1 - 2^{i-k}$$

and

(29) $$(-1)^{[j/2^i] + [j/2^k]} = -(-1)^{[(j+2^i)/2^i] + [(j+2^i)/2^k]} \ .$$

Putting

(30) $$(-1)^n c_n = \sum_{j=n\cdot 2^i}^{(n+1)2^i - 1} (-1)^{[j/2^i] + [j/2^k]} a_j \ ,$$

we have

$$c_n = \sum_{j=n\cdot 2^i}^{(n+1)2^i-1} (-1)^{[j/2^k]} a_j \tag{31}$$
$$= \sum_{j=n\cdot 2^i}^{n\cdot 2^i+2^k-1} \sum_{m=0}^{2^{i-k-1}-1} (a_{j+2m\cdot 2k} - a_{j+(2m+1)\cdot 2k}) .$$

Evidently c_n is positive. Also, c_n is monotone because, from (iii), $a_i - a_{i+2^k}$ is monotone decreasing in i. Thus $\sum_{n=1}^{\infty}(-1)^n c_n$ converges. Since we have

$$\sum_{j=1}^{n} (-1)^j c_j = \sum_{j=0}^{n\cdot 2^i-1} (-1)^{[j/2^i]+[j/2^k]} a_j \tag{32}$$

it follows that (27) converges if $i > k$. This concludes the proof of Theorem 4.

The sequence $a_i = 1/i^\alpha$, for positive $\alpha \leqq 1$, is an example satisfying (i)–(iv) of Theorem 4.

This result can be extended with the help of Abel's test for convergence.

THEOREM 5. *Let $\{a_i\}$ satisfy the hypotheses of Theorem 4. Let $\{v_{ij}\}$, $i, j = 1, 2, \cdots$ satisfy the following*:

1. $v_{ij} > 0$.
2. *For each i, $\{v_{ij}\}$ is monotone (increasing or decreasing) with respect to j.*
3. $\sum_{j=1}^{\infty} a_j v_{ij} = \infty$ *for each i.*

Then $(-1)^{[j/2^i]} a_j v_{ij}$ satisfies the hypotheses of Theorem 1.

Proof. We take the same definition for x_{jk} as in Theorem 4. Then $\sum_{j=1}^{\infty} a_{ij} x_{jk}$ converges for $i \neq k$ by Abel's test. Further, we have

$$\sum_{j=1}^{\infty} a_{ij} x_{jil} = \sum_{i=1}^{\infty} a_j v_{ji} = +\infty .$$

We obtain a result which allows us to transform any array of coefficients a_{ij} which satisfies the hypotheses of Theorem 1 into a different array satisfying the same conditions. First we need a lemma which is related to Abel's test for convergence.

LEMMA 2. *Let $\{v_i\}$ be a monotone decreasing sequence of real numbers which is bounded away from zero; i.e., there exists b such that $0 < b \leqq v_i$ for all i. Suppose $\sum_{i=1}^{\infty} a_i = +\infty$. Then $\sum_{i=1}^{\infty} v_i a_i = +\infty$.*

Proof. Let $s_n = \sum_{i=1}^{n} a_i$, and let $h_n = \inf_{k \geqq n} s_k$. We have, if $m \leqq p$,

$$
\begin{aligned}
(33) \qquad \sum_{i=1}^{p} a_i v_i &= s_1(v_1 - v_2) + \cdots + s_{p-1}(v_{p-1} - v_p) + s_p v_p \\
&\geqq h_1(v_1 - v_m) + h_m v_m \\
&\geqq -|h_1| v_1 + h_m b\,.
\end{aligned}
$$

The result follows because $h_m \to \infty$ as $m \to \infty$.

THEOREM 6. *Let a_{ij} satisfy the hypotheses of Theorem* 1. *Let v_{ij} satisfy the following*:

1. *There exist c_i such that $0 < c_i \leqq v_{ij}$ for all positive integers i and j.*
2. *For each i, $\{v_{ij}\}$ is monotone decreasing with respect to j.*

Then $a_{ij}v_{ij}$ satisfy the hypotheses of Theorem 1.

Proof. The conditions are satisfied by using the x_{jkl} which are assumed to exist in Theorem 1. We have that $\sum_{j=1}^{\infty} a_{ij}v_{ij}x_{jkl}$ converges if $i \neq k$ by Abel's test and diverges to $+\infty$ for $i = k$ by Lemma 2.

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