

# Pacific Journal of Mathematics

**TRANSITIVE GROUPS OF COLLINEATIONS ON CERTAIN  
DESIGNS**

RICHARD EARL BLOCK

## TRANSITIVE GROUPS OF COLLINEATIONS ON CERTAIN DESIGNS

RICHARD E. BLOCK

Let  $M = (a_{ij})$  be an  $m \times n$  matrix with entries in  $\{1, -1\}$ . Suppose that there is a positive integer  $d$  such that the inner product of every pair of distinct rows of  $M$  is  $n - 2d$ ; this is equivalent to assuming that any two distinct rows have Hamming distance  $d$ , i.e. differ in exactly  $d$  places. The rows of  $M$  form the code words of a binary code; such a code is called a (binary) *constant-distance code*, of length  $n$  and distance  $d$ . Special cases of matrices which may be taken to be  $M$  are the Hadamard matrices, which are defined by the condition that  $m = n = 2d$ , and the incidence matrices (written with  $\pm 1$ ) of balanced incomplete block designs, which are characterized by the property that all column sums are equal and all row sums are equal.

Suppose that  $\pi$  is a permutation of  $\{1, \dots, n\}$  such that replacement, for  $i = 1, \dots, n$ , of the  $\pi(i)$ th column of  $M$  by the  $i$ th column of  $M$  sends each row of  $M$  into a row of  $M$ . Then  $\pi$  induces a permutation of the rows of  $M$ . Call such a pair of permutations of the columns and of the rows a *collineation* of  $M$ , or of the code. We shall examine constant-distance codes with a group  $G$  of collineations which is transitive on the columns. We shall show that  $G$  has at most two orbits on the rows (just one orbit if and only if  $M$  comes from a balanced incomplete block design), and that if  $G$  is nilpotent then at most one of these orbits contains more than a constant row.

Moreover, it will be shown that this last conclusion need not hold if  $G$  is not assumed nilpotent; this will be done by giving an infinite class of Hadamard matrices with doubly transitive collineation groups.

One way of obtaining a constant-distance code with a transitive group on the columns is the following. Given a (cyclic)  $(v, k, \lambda)$  difference set, write a  $v$ -tuple of 1's and -1's with 1 in the  $k$  places which corresponds to elements of the difference set, and repeat this  $v$ -tuple  $s$  times to obtain a  $vs$ -tuple. The set of all cyclic permutations of this  $vs$ -tuple forms constant-distance code with  $v$  code words and distance  $d = 2(k - \lambda)s$ . Call such a code an *iterated difference set code*. The code is closed under the cyclic shift (the permutation  $\pi = (1, 2, \dots, vs)$  on the columns).

Our results imply that, conversely, any constant-distance code which is closed under the cyclic shift consists of repeated cyclic shifts of

some single word, plus possibly a single constant word. The main part of the code is thus an iterated difference set code; the extra word can occur if and only if the parameters  $(v, k, \lambda)$  are of Hadamard type.

## 2. The number of orbits on the rows.

**THEOREM 1.** *Suppose that  $G$  is a group of collineations of a constant-distance code. If  $G$  is transitive on the columns then  $G$  has at most two orbits on the rows.*

*Proof.* Suppose that  $G$  has  $t$  orbits  $T_1, \dots, T_t$  on the rows. Then there are integers  $r_i$  such that each row in  $T_i$  has exactly  $r_i$  1's,  $i = 1, \dots, t$ . It follows that if  $\alpha_i$  and  $\alpha_j$  are rows and  $\alpha_i \in T_i, \alpha_j \in T_j$ , and if  $c(\alpha_i, \alpha_j)$  is the number of places in which both  $\alpha_i$  and  $\alpha_j$  have 1, then  $r_i + r_j = d + 2c(\alpha_i, \alpha_j)$ , or  $c(\alpha_i, \alpha_j) = (r_i + r_j - d)/2$ . Let  $v_i$  denote the number of words in  $T_i$ . Since  $G$  is transitive on the columns, for each column there are the same number  $k_i$  of words in  $T_i$  with 1 in that place; we have  $k_i = v_i r_i / n$ , where  $n$  is the length of the words. Thus the words in  $T_i$  form the incidence matrix of a balanced incomplete block design with  $\lambda = r_i - (d/2)$ . Now suppose that  $t \geq 2$ , that  $T_i$  and  $T_j$  are distinct orbits and that  $\alpha \in T_j$ . Counting in two ways the total number of times in which words in  $T_i$  have a 1 in the same place as a 1 in  $\alpha$ , we have  $v_i(r_i + r_j - d)/2 = r_j k_i$ . Thus, since  $k_i = v_i r_i / n$ ,

$$(1) \quad n \frac{(r_i + r_j - d)}{2} = r_i r_j .$$

Suppose that,  $r_i \neq n$ . Then for some prime  $p$ , with  $p^e$  and  $p^f$  the highest powers of  $p$  dividing  $n$  and  $r_i$ , respectively, one has  $e > f$ . Since  $v_i r_i = n k_i$  and

$$(2) \quad r_i(k_i - 1) = \left(r_i - \frac{d}{2}\right)(v_i - 1),$$

$p \nmid (v_i - 1)$  and  $p^f \mid r_i - (d/2)$ . If  $r_i = r_j$  then the left side of (1) is divisible by  $p^{e+f}$ , the right side only by  $p^{2f}$ , a contradiction. Hence  $r_i \neq r_j$  if  $i \neq j$ . Also  $r_i \neq n/2$ , since otherwise, by (1),  $r_i = n/2 = d$  and  $k_i = v_i/2$ , contradicting (2). Thus  $r_j$  is uniquely determined in terms of  $r_i$  by (1). It follows that  $t \leq 2$ , and the theorem is proved.

If there is only one orbit, then, as shown in the above proof,  $M$  is the incidence matrix of a balanced incomplete block design. The next result is the converse.

**THEOREM 2.** *Suppose that  $G$  is a group of collineations of a balanced incomplete block design. If  $G$  is transitive on the blocks then  $G$  is also transitive on the points.*

*Proof.* The incidence matrix of the design is a constant-distance code with  $d = 2(r - \lambda)$ . If  $G$  had two orbits on the points, then  $r_1 = r_2 = r$ . But by the proof of Theorem 1,  $r_1 \neq r_2$ , a contradiction. This proves Theorem 2.

**COROLLARY 1.** *Let  $G$  be a group of collineations of a constant-distance code. Suppose that  $G$  fixes  $c$  columns and is transitive on the remaining columns. Let  $q$  be the number of different  $c$ -tuples in the rows of the submatrix formed by the  $c$  fixed columns. Then  $G$  has at most  $2q$  orbits on the rows; if moreover the code corresponds to a balanced incomplete block design, then  $G$  has exactly  $q$  orbits on the rows (points).*

*Proof.* The set of rows with a given  $c$ -tuple in the fixed columns must be closed under  $G$ ; deleting the fixed columns from these rows, one obtains a constant distance code with a transitive group of collineations. The result now follows immediately from Theorems 1 and 2.

These results are a partial generalization to nonsymmetric designs of a theorem proved by Dembowski [2], Hughes [3], and Parker [4], which says that for a symmetric design, the number of orbits on the points is the same as the number of orbits on the lines. However there are balanced incomplete block designs with a group of collineations which is transitive, even cyclic, on the points, but not transitive on the lines.

**3. Codes with a nilpotent transitive group.** In this section we assume that  $M$  is an  $m \times n$  matrix whose rows form a constant-distance code with distance  $d$ , and that  $G$  is a group of collineations which is transitive on the columns. Let  $H$  denote the subgroup of  $G$  fixing the first column. We shall continue using the notation  $T_i$ ,  $v_i$ ,  $r_i$  and  $k_i$  introduced in the above proofs.

**THEOREM 3.** *Suppose that  $T_1$  and  $T_2$  are distinct orbits of  $G$  (on the rows). For  $i = 1, 2$ , take  $\alpha_i$  in  $T_i$  and let  $S_i$  be the subgroup of  $G$  fixing  $\alpha_i$ . Suppose that  $p$  is any prime such that the highest power  $p^j$  of  $p$  dividing  $n$  does not divide  $d$ . Then, either for  $i = 1$  or  $2$ ,  $S_i$  contains the normalizer of a Sylow  $p$ -subgroup of  $G$ ,  $p \mid v_i - 1$ , and  $p^j \mid r_i$ .*

*Proof.* If the orbit  $T_i$  is trivial (consists of a constant word) then  $S_i = G$  and the conclusion is obvious. Thus suppose that both orbits

are nontrivial. Take a prime  $p$  such that  $p^j$ , the highest power of  $p$  dividing  $n$ , does not divide  $d$ . Let  $p^e$  and  $p^f$  be the highest powers of  $p$  dividing  $r_1$  and  $r_2$ , respectively; by choice of notation we may suppose that  $e \leq f$ . By (1),  $p^i | r_1 r_2$ .

Suppose first that  $p \nmid v_1 - 1$  and  $p \nmid v_2 - 1$ . Then by (2),  $p^e | [r_1 - (d/2)]$  and  $p^f | [r_2 - (d/2)]$ , so that  $p^j | (d/2)$  and  $p^e | r_1 + r_2 - d$ . If  $p > 2$  then  $p^{j+e}$  divides the left side of (1) while  $p^{e+f}$  is the highest power of  $p$  dividing the right side; hence  $f \geq j$ , so that  $p^j | d$ , a contradiction. If  $p = 2$  then  $p^{e-1} | [(r_1 + r_2 - d)/2]$  and  $p^{j+e-1}$  divides the left side of (1), so that  $f \geq j - 1$ ,  $p^{j-1} | (d/2)$  and  $p^j | d$ , again a contradiction.

Hence  $p | v_i - 1$  for some  $i$ , with  $i = 1$  or  $2$ . Then since  $p | ([G : S_i] - 1)$ ,  $p \nmid [G : S_i]$  and  $S_i$  contains a Sylow  $p$ -subgroup of  $G$ . Suppose that  $K$  is any subgroup of  $G$ , and consider the orbits of  $K$  when  $K$  is regarded as a permutation group on the columns. For each of these orbits there is an  $x$  in  $G$  such that the number of elements in the orbit is  $[K : K \cap xHx^{-1}]$ . If  $p^l$  is the highest power of  $p$  dividing  $|H|$  then  $p^{l+1}$  is the highest power of  $p$  dividing  $|G|$ . Hence if  $K$  contains a Sylow  $p$ -subgroup of  $G$  then  $p^j | [K : K \cap xHx^{-1}]$  for any  $x$ . Taking  $K = S_i$  we see that  $p^j | r_i$ , since the set of places where  $\alpha_i$  has 1 is a union of orbits of  $S_i$  (on the columns). If  $g \in G$  and  $g \notin S_i$  then  $g\alpha_i \neq \alpha_i$ , and  $gS_i g^{-1}$  is the subgroup of  $G$  fixing  $g\alpha_i$ . If moreover  $gS_i g^{-1}$  contains a Sylow  $p$ -subgroup of  $S_i$ , then  $p^j$  divides the number of elements in each orbit (on the columns) of  $S_i \cap gS_i g^{-1}$ . But the set of places where  $\alpha_i$  and  $g\alpha_i$  disagree is a union of orbits of  $S_i \cap gS_i g^{-1}$ , so that  $p^j | d$ , a contradiction. Therefore no Sylow  $p$ -subgroup of  $S_i$  is contained in a conjugate of  $S_i$ . Suppose that  $P$  is a Sylow  $p$ -subgroup of  $S_i$  (and so also of  $G$ ), and that  $x \in N_G(P)$ , the normalizer of  $P$ . If  $x \notin S_i$  then  $xS_i x^{-1} \neq S_i$  but  $P = xPx^{-1} \subseteq xS_i x^{-1}$ , a contradiction. Hence  $N_G(P) \subseteq S_i$ , and the theorem is proved.

**COROLLARY 2.** *If  $G$  is a nilpotent group of collineations of  $M$  which is transitive on the columns, then either  $G$  is transitive on the rows or one of the two orbits of  $G$  on the rows consists of one trivial row.*

*Proof.* Unless  $M$  has only the two trivial rows, there is a prime  $p$  such that the highest power of  $p$  dividing  $n$  does not divide  $d$ . Since a Sylow  $p$ -subgroup of a nilpotent group is normal, if  $G$  is not transitive on the rows then by Theorem 3,  $G$  fixes a row. This proves the result.

Now suppose the constant distance code is closed under the cyclic shift  $\pi = (1, 2, \dots, n)$ . If  $\alpha$  is a code word with  $r$  ones, then  $\alpha$  must be periodic of (minimal) period  $v$ , a divisor of  $n$ ; write  $v = n/s$ .

A single period of  $\alpha$  gives a  $(v, k, \lambda)$  difference set with  $k = r/s$  and  $\lambda = [r - (d/2)]/s$ . Thus the set of cyclic shifts  $\pi^t\alpha$  or  $\alpha$  forms an  $s$ -times iterated  $(v, k, \lambda)$ -difference set code; solving  $k(k - 1) = \lambda(v - 1)$  for  $s$ , one has  $s = n + [2r(r - n)/d]$ . By Corollary a, either this set is the entire code or there is one more word, with all 1's or all -1's. If the extra word has all -1's then  $r = d$ ,  $\lambda = d/2s$ , and from  $k(k - 1) = \lambda(v - 1)$  one obtains  $n/s = 2d/s$ . Hence, with  $d/2s = u$ , one would have  $v = 4u - 1$ ,  $k = 2u$  and  $\lambda = u$ . If on the other hand the extra word has all 1's, then we have the complement of a code of the above type, and  $v = 4u - 1$ ,  $k = 2u - 1$  and  $\lambda = u - 1$ .

The above characterization of constant-distance code closed under the cyclic shift was conjectured by the writer and proved independently at the same time by the writer [1] and R.C. Titsworth [5]. Titsworth's proof uses arguments on polynomials dividing  $x^n - 1$ .

**3. Hadamard matrices and codes with two orbits.** In this section we give a class of Hadamard matrices with doubly transitive collineation groups, and use these matrices to obtain a class of constant-distance codes with a transitive group on the columns for which the conclusion of Corollary 2 does not hold.

Let  $A$  be the Hadamard matrix of order 4 with 1 on the diagonal, -1 elsewhere, and let  $B = B(s)$  be the tensor product of  $s$  copies of  $A$ .

**THEOREM 4.** *For any  $s$ , the group  $G$  of collineations of  $B(s)$  is doubly transitive on the columns (and also on the rows).*

*Proof.* Denote the rows and columns of  $B$  by  $s$ -tuples, so that

$$b_{i_1} \cdots, i_s; j_1, \cdots, j_s = a_{i_1, j_1} a_{i_2, j_2} \cdots a_{i_s, j_s}.$$

The result is obvious when  $s = 1$ . Suppose  $s = 2$ . We shall show that the subgroup  $H$  of  $G$  fixing the column  $(1, 1)$  is transitive on the remaining columns. If  $\tau_1$  and  $\tau_2$  are any permutations on four letters then the permutation of columns sending  $(i_1, i_2)$  to  $(\tau_1(i_1), \tau_2(i_2))$  is a collineation of  $B$ , sending row  $(i_1, i_2)$  to row  $(\tau_1(i_1), \tau_2(i_2))$ ; denote this collineation by  $(\tau_1, \tau_2)$ . It can be verified that the product of four transpositions of columns  $\sigma = ((1, 4) (2, 3))((4, 1) (3, 2))((1, 3) (2, 4))((3, 1) (4, 2))$  is a collineation of  $B$ ; also,  $\sigma \in H$ . Taking  $\sigma$  and its products with various  $(\tau_1, \tau_2)$ , we see that all columns other than  $(1, 1)$  form a single orbit of  $H$ . Moreover some  $(\tau_1, \tau_2)$  moves column  $(1, 1)$ , so that  $G$  is transitive, and hence doubly transitive. Now suppose that  $s > 2$ . If  $\tau$  is a collineation of  $B(2)$  and if a set of two column coordinates of  $B(s)$  is given, then a collineation of  $B(s)$  is obtained by applying  $\tau$  to the given

column coordinates while keeping the remaining ones fixed. Using this type of collineation, we see that the subgroup of  $G$  fixing column  $(1, \dots, 1)$  is transitive on the remaining columns. Hence  $G$  is always doubly transitive on the columns, and, by symmetry, also on the rows. This completes the proof.

**COROLLARY 3.** *For every power  $4^s$  of  $4$  ( $s > 1$ ), there is a constant-distance code with  $4^s$  words of length  $4^s - 1$ , such that the group of collineations is transitive on the columns but has two nontrivial orbits on the rows.*

*Proof.* The matrix  $B(s)$  is Hadamard, and hence its rows form a constant-distance code. Complement the rows with  $a + 1$  in column  $(1, \dots, 1)$  and then delete this column. What remains is still a constant-distance code; call it  $C$ . The subgroup of  $G$  fixing  $(1, \dots, 1)$  clearly gives a group of collineations of  $C$  which is transitive on the columns. Moreover the set of uncomplemented rows is closed under the group, so the group has two nontrivial orbits. This completes the proof.

Let  $G$  and  $H$  continue to have the same meanings as in Theorem 4. It follows from Corollary 2 and the proof of Corollary 3 that  $H$  is not nilpotent. However it can actually be shown that the subgroup  $K$  of  $H$  fixing column  $(1, 2)$  is isomorphic to  $S_6$ , being generated by  $\sigma$  and certain  $(\tau_1, \tau_2)$ 's. Hence when  $s = 2$ ,  $G$  has order  $16 \cdot 15 \cdot 720$ . Also it follows that if  $s > 1$  then  $G$  contains a subgroup isomorphic to  $S_6$  which fixes  $2 \cdot 4^{s-2}$  columns.

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# Pacific Journal of Mathematics

Vol. 15, No. 1

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