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A NOTE ON HAUSDORFF'S SUMMATION METHODS

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If $\{a_n\}$ is a moment sequence and (Δa) is the difference matrix having base sequence $\{a_n\}$, then (Δa) is symmetric about the main diagonal if and only if the function $\alpha(x)$ such that $a_n = \int_0^1 x^n d\alpha(x)$, $n = 0, 1, 2, \dots$, is symmetric in the sense that $\alpha(x) + \alpha(1+x) = \alpha(1) + \alpha(0)$ except for at most countably many x in $[0, 1]$. This property is related to the "fixed points" of the matrix H , where HaH is the Hausdorff matrix determined by the moment sequence $\{a_n\}$.

In each of the papers [2], [3] and [5], there is reference to difference matrices of the form

$$(\Delta d) = \begin{bmatrix} \Delta^0 d_0 & \Delta^0 d_1 & \Delta^0 d_2 & & \\ \Delta^1 d_0 & \Delta^1 d_1 & \Delta^1 d_2 & \vdots & \\ \Delta^2 d_0 & \Delta^2 d_1 & \Delta^2 d_2 & & \\ & \dots & & & \end{bmatrix}$$

where $\{d_n\}$ is a moment sequence, $\Delta^0 d_n = d_n$, $n = 0, 1, 2, \dots$ and $\Delta^m d_n = \Delta^{m-1} d_n - \Delta^{m-1} d_{n+1}$, for $n = 0, 1, 2, \dots$ and $m = 1, 2, 3, \dots$. In [2], Garabedian and Wall discussed the importance of (Δd) having the property of being symmetric about the main diagonal, i.e. $\Delta^m d_n = \Delta^n d_m$. They also showed that if $\{d_n\}$ is a totally monotone sequence, then (Δd) is symmetric about the main diagonal if and only if the function $f(x)$ which generates $\{d_n\}$ has a certain type continued fraction expansion.

In this paper, the symmetry of (Δd) is investigated with the restriction of total monotonicity removed and a collection of necessary and sufficient conditions are given, Theorem 3, for moment sequences in general. A relation is established between the symmetry of (Δd) and the "fixed points" of the difference matrix

$$(1) \quad H = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & & & \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & -\begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & -\begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 2 \end{pmatrix} & \\ & \dots & & \end{bmatrix}.$$

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2. Notation, definitions, and examples. Except for some notation and definitions introduced for convenience, the notation and definitions of this paper will follow [6].

NOTATION. If $\{d_n\}$ is an infinite sequence, d^* and d' denote respectively the diagonal and column matrices determined by $\{d_n\}$.

DEFINITION 1. If $\{d_n\}$ is a number sequence such that for some function $f(x)$ on $[0, 1]$,

$$d_p = \int_0^1 x^p df(x) = \int_0^1 (1-x)^p df(x); \quad p = 0, 1, 2, \dots,$$

then $\{d_n\}$ is called a symmetric moment sequence.

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \dots$ provides an example of a moment sequence satisfying Definition 1 since for $p = 0, 1, 2, \dots$

$$(2) \quad \begin{aligned} c_p &= \int_0^1 x^p dx = \left[x^{p+1}/p + 1 \right]_0^1 \\ &= \int_0^1 (1-x)^p dx = \left[-(1-p)^{p+1}/p + 1 \right]_0^1 = \frac{1}{p+1}. \end{aligned}$$

DEFINITION 2. If A is a semi-infinite, lower triangular, matrix having inverse and $\{a_n\}$ and $\{d_n\}$ are sequences such that $A^{-1}d^*Aa' = A^{-1}a^*Ad'$, then $\{a_n\}$ and $\{d_n\}$ are symmetric relative to A .

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, c_p$ of (2), and the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ are symmetric relative to the matrix H of (1).

3. THEOREMS. LEMMA. Suppose $\{s_n\}$ is a sequence such that $s_p \neq 0$ for $p = 0, 1, 2, \dots$ and suppose that A is a semi-infinite matrix having inverse such that $As' = s'$; then,

- (i) $A^{-1}s' = s'$,
- (ii) $\{x_n\}$ and $\{s_n\}$ are symmetric with respect to A if and only if $Ax' = x'$, and
- (iii) if $A^{-1}a^*As' = A^{-1}s^*Aa'$ and $A^{-1}b^*As' = A^{-1}s^*Ab'$, then $A^{-1}b^*Aa' = A^{-1}a^*Ab'$.

Proof. (i) is obvious. For the proof of (ii), we first suppose $\{x_n\}$ is symmetric with $\{s_n\}$ relative to A so that $A^{-1}x^*As' = A^{-1}s^*Ax'$. Multiplying both sides on the left by A and using $As' = s'$ gives $x^*s' = s^*Ax'$. Under the hypothesis, s^* has inverse s^{*-1} so that

$$(3) \quad s^{*-1}x^*s' = s^{*-1}s^*Ax' = Ax'.$$

Since $x^*s' = s^*x'$, it follows from (3) that $x' = Ax'$.

On the other hand, if $Ax' = x'$,

$$(4) \quad A^{-1}x^*As' = A^{-1}x^*s'$$

and

$$A^{-1}s^*Ax' = A^{-1}s^*x' .$$

Since $s^*x' = x^*s'$, it follows from (4) that x and s are symmetric relative to A .

For the proof of (iii), we suppose that $a' = s^{*-1}a^*s'$ and $b' = s^{*-1}b^*s'$, from which it follows that

$$(5) \quad A^{-1}a^*Ab' = A^{-1}a^*s^{*-1}b^*s'$$

and

$$(6) \quad A^{-1}b^*Aa' = A^{-1}b^*s^{*-1}a^*s' .$$

Since diagonal matrices permute, it follows that (5) and (6) are equal establishing (iii).

THEOREM 1. *If $\{b_n\}$ is a moment sequence, i.e.,*

$$(7) \quad b_p = \int_0^1 x^p dg(x) ,$$

$\{b_n\}$ and the Cesàro sequence (2) are symmetric relative to H if and only if $\{b_n\}$ is a symmetric moment sequence.

Proof. Let

$$f_n(x) = \begin{cases} \sum_{p=0}^{n-1} \binom{n}{p} (-1)^p x^p & \text{for } n = 2, 4, 6, \dots \\ \sum_{p=0}^{n-1} \binom{n}{p} (-1)^p x^p - 2x^n & \text{for } n = 1, 3, 5, \dots . \end{cases}$$

Clearly, if $\{t_n\}$ is any number sequence, $Ht' = t'$ if and only if

$$\sum_{p=0}^{n-1} \binom{n}{p} (-1)^p t_p = 0 \quad \text{for } n = 2, 4, 6, \dots$$

and

$$\sum_{p=0}^{n-1} \binom{n}{p} (-1)^p t_p - 2t_n = 0 \quad \text{for } n = 1, 3, 5, \dots .$$

Thus if $\{b_n\}$ is defined as in (7), $Hb' = b'$ if and only if

$$(8) \quad \int_0^1 f_n(x) dg(x) = 0 \quad \text{for } n = 1, 2, 3, \dots .$$

But, $f_n(x) = (1 - x)^n - x^n$ for $n = 1, 2, 3, \dots$ so that

$$(9) \quad \int_0^1 f_n(x)dg(x) = \int_0^1 (1-x)^n dg(x) - \int_0^1 x^n dg(x),$$

and consequently (8) holds if and only if $\{b_n\}$ is a symmetric moment sequence. It follows from (9) and (2) that $Hc' = c'$ and from the preceding Lemma that $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H .

Conversely, if $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H , it follows that $Hb' = b'$, and if $\{b_n\}$ is defined as in (7), then $\{b_n\}$ is a symmetric moment sequence.

THEOREM 2. *If $g(x)$ is of bounded variation on $[0, 1]$ and $\{z_n\}$ is the moment sequence determined by $g(x)$, the following two statements are equivalent:*

- (i) $\{z_n\}$ is a symmetric moment sequence, and
- (ii) there do not exist uncountably many x in $[0, 1]$ for which $g(x) + g(1-x) \neq g(1) + g(0)$.

Proof. Suppose (i). Then let $u = 1 - x$ so that,

$$z_p = \int_0^1 (1-x)^p dg(x) = \int_0^1 u^p dg(1-x) = -\int_0^1 u^p dg(1-u).$$

Thus, $\int_0^1 (1-x)^p dg(x) = -\int_0^1 x^p dg(1-x)$ so that for $p = 0, 1, 2, \dots$,

$$(10) \quad \int_0^1 x^p d[g(x) + g(1-x)] = 0.$$

Since $g(x) - g(1-x)$ is of bounded variation on $[0, 1]$, (10) implies that for every $k(x)$ continuous on $[0, 1]$, $\int_0^1 k(x)d[g(x) + g(1-x)] = 0$. This, [4, p. 69], implies (ii). Reversing the steps leading to (10) shows that (ii) implies (i).

An interesting example of a function satisfying (ii) is provided by Evans in [1].

THEOREM 3. *Suppose $g(x)$ is of bounded variation on $[0, 1]$ and suppose $\{a_n\}$ is the moment sequence generated by $g(x)$. The following statements are equivalent:*

- (i) $\{a_n\}$ is a symmetric moment sequence,
- (ii) $Ha' = a'$,
- (iii) $\{a_n\}$ and the Cesàro moment sequence $\{c_n\}$ are symmetric relative to H , and
- (iv) the difference matrix (Δa) having base sequence $\{a_n\}$ is symmetric about the main diagonal.

Proof. Theorem 1 implies the equivalence of (i), (ii), and (iii).

(i) implies (iv) provided

$$(11) \quad \int_0^1 x^m(1-x)^n dg(x) = \int_0^1 x^n(1-x)^m dg(x) \quad \text{for } m, n = 0, 1, 2, \dots$$

Let $u = 1 - x$ so that $\int_0^1 x^m(1-x)^n dg(x) = \int_1^0 (1-u)^m u^n dg(1-u)$. Thus (11) may be rewritten as

$$(12) \quad -\int_0^1 (1-x)^m x^n dg(1-x) = \int_0^1 x^n(1-x)^m dg(x) \\ = \int_0^1 x^n(1-x)^m d[g(x) + g(1-x)] = 0.$$

That (12) is the case for $\{a_n\}$ a symmetric moment sequence follows from (ii) of Theorem 2. (iv) implies (ii) since (iv) implies that $a_n = A^n A_0$, which is the same as saying that $Ha' = a'$. Thus the equivalence of the four statements is established.

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Vol. 15, No. 1

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Donald Charles Benson, <i>Unimodular solutions of infinite systems of linear equations</i>	1
Richard Earl Block, <i>Transitive groups of collineations on certain designs</i>	13
Barry William Boehm, <i>Existence of best rational Tchebycheff approximations</i>	19
Joseph Patrick Brannen, <i>A note on Hausdorff's summation methods</i>	29
Dennison Robert Brown, <i>Topological semilattices on the two-cell</i>	35
Peter Southcott Bullen, <i>Some inequalities for symmetric means</i>	47
David Geoffrey Cantor, <i>On arithmetic properties of coefficients of rational functions</i>	55
Luther Elic Claborn, <i>Dedekind domains and rings of quotients</i>	59
Allan Clark, <i>Homotopy commutativity and the Moore spectral sequence</i>	65
Allen Devinatz, <i>The asymptotic nature of the solutions of certain linear systems of differential equations</i>	75
Robert E. Edwards, <i>Approximation by convolutions</i>	85
Theodore William Gamelin, <i>Decomposition theorems for Fredholm operators</i>	97
Edmond E. Granirer, <i>On the invariant mean on topological semigroups and on topological groups</i>	107
Noel Justin Hicks, <i>Closed vector fields</i>	141
Charles Ray Hobby and Ronald Pyke, <i>Doubly stochastic operators obtained from positive operators</i>	153
Robert Franklin Jolly, <i>Concerning periodic subadditive functions</i>	159
Tosio Kato, <i>Wave operators and unitary equivalence</i>	171
Paul Katz and Ernst Gabor Straus, <i>Infinite sums in algebraic structures</i>	181
Herbert Frederick Kreimer, Jr., <i>On an extension of the Picard-Vessiot theory</i>	191
Radha Govinda Laha and Eugene Lukacs, <i>On a linear form whose distribution is identical with that of a monomial</i>	207
Donald A. Ludwig, <i>Singularities of superpositions of distributions</i>	215
Albert W. Marshall and Ingram Olkin, <i>Norms and inequalities for condition numbers</i>	241
Horace Yomishi Mochizuki, <i>Finitistic global dimension for rings</i>	249
Robert Harvey Oehmke and Reuben Sandler, <i>The collineation groups of division ring planes. II. Jordan division rings</i>	259
George H. Orland, <i>On non-convex polyhedral surfaces in E^3</i>	267
Theodore G. Ostrom, <i>Collineation groups of semi-translation planes</i>	273
Arthur Argyle Sagle, <i>On anti-commutative algebras and general Lie triple systems</i>	281
Laurent Siebenmann, <i>A characterization of free projective planes</i>	293
Edward Silverman, <i>Simple areas</i>	299
James McLean Sloss, <i>Chebyshev approximation to zero</i>	305
Robert S. Strichartz, <i>Isometric isomorphisms of measure algebras</i>	315
Richard Joseph Turyn, <i>Character sums and difference sets</i>	319
L. E. Ward, <i>Concerning Koch's theorem on the existence of arcs</i>	347
Israel Zuckerman, <i>A new measure of a partial differential field extension</i>	357