A NOTE ON HAUSDORFF’S SUMMATION METHODS

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If \( \{a_n\} \) is a moment sequence and \( (\Delta a) \) is the difference matrix having base sequence \( \{a_n\} \), then \( (\Delta a) \) is symmetric about the main diagonal if and only if the function \( \alpha(x) \) such that 
\[
a_n = \int_0^1 x^n d\alpha(x), \quad n = 0, 1, 2, \ldots,
\]
is symmetric in the sense that 
\[
\alpha(x) + \alpha(1 + x) = \alpha(1) + \alpha(0)
\]
extcept for at most countably many \( x \) in \([0, 1]\). This property is related to the “fixed points” of the matrix \( H \), where \( HaH \) is the Hausdorff matrix determined by the moment sequence \( \{a_n\} \).

In each of the papers [2], [3] and [5], there is reference to difference matrices of the form
\[
(\Delta d) = \begin{bmatrix}
\Delta^0 d_0 & \Delta^1 d_1 & \Delta^2 d_2 & \\
\Delta^0 d_0 & \Delta^1 d_1 & \Delta^2 d_2 & \\
\Delta^0 d_0 & \Delta^1 d_1 & \Delta^2 d_2 & \\
\vdots & & & \\
\end{bmatrix}
\]
where \( \{d_n\} \) is a moment sequence, \( \Delta^m d_n = d_n, \quad n = 0, 1, 2, \ldots \) and \( \Delta^m d_n = \Delta^{m-1} d_n - \Delta^{m-1} d_{n+1} \), for \( n = 0, 1, 2, \ldots \) and \( m = 1, 2, 3, \ldots \). In [2], Garabedian and Wall discussed the importance of \( (\Delta d) \) having the property of being symmetric about the main diagonal, i.e. \( \Delta^m d_n = \Delta^n d_m \). They also showed that if \( \{d_n\} \) is a totally monotone sequence, then \( (\Delta d) \) is symmetric about the main diagonal if and only if the function \( f(x) \) which generates \( \{d_n\} \) has a certain type continued fraction expansion.

In this paper, the symmetry of \( (\Delta d) \) is investigated with the restriction of total monotonicity removed and a collection of necessary and sufficient conditions are given, Theorem 3, for moment sequences in general. A relation is established between the symmetry of \( (\Delta d) \) and the “fixed points” of the difference matrix
\[
H = \begin{bmatrix}
(0) & (0) & (0) & \\
(1) & (1) & (1) & \\
(2) & (2) & (2) & \\
\vdots & & & \\
\end{bmatrix}.
\]

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2. **Notation, definitions, and examples.** Except for some notation and definitions introduced for convenience, the notation and definitions of this paper will follow [6].

**Notation.** If \( \{d_n\} \) is an infinite sequence, \( d^* \) and \( d' \) denote respectively the diagonal and column matrices determined by \( \{d_n\} \).

**Definition 1.** If \( \{d_n\} \) is a number sequence such that for some function \( f(x) \) on \([0, 1]\),

\[
d_p = \int_0^1 x^p f(x) \, dx = \int_0^1 (1 - x)^p f(x) \, dx ; \quad p = 0, 1, 2, \cdots ,
\]

then \( \{d_n\} \) is called a symmetric moment sequence.

The Cesàro moment sequence 1, \( \frac{1}{2} \), \( \frac{1}{3} \), \( \cdots \) provides an example of a moment sequence satisfying Definition 1 since for \( p = 0, 1, 2, \cdots \)

\[
c_p = \int_0^1 x^p dx = \frac{x^{p+1}}{p+1} \bigg|_0^1 = \frac{1}{p+1}.
\]

**Definition 2.** If \( A \) is a semi-infinite, lower triangular, matrix having inverse and \( \{a_j\} \) and \( \{d_n\} \) are sequences such that \( A^{-1}d^*Aa' = A^{-1}a^*Ad' \), then \( \{a_j\} \) and \( \{d_n\} \) are symmetric relative to \( A \).

The Cesàro moment sequence 1, \( \frac{1}{2} \), \( \frac{1}{3} \), \( \cdots \), \( c_p \) of (2), and the sequence 1, \( \frac{1}{2} \), \( \frac{1}{3} \), \( \cdots \) are symmetric relative to the matrix \( H \) of (1).

3. **Theorems. Lemma.** Suppose \( \{s_n\} \) is a sequence such that \( s_p \neq 0 \) for \( p = 0, 1, 2, \cdots \) and suppose that \( A \) is a semi-infinite matrix having inverse such that \( As' = s' \); then,

(i) \( A^{-1}s' = s' \),

(ii) \( \{x_n\} \) and \( \{s_n\} \) are symmetric with respect to \( A \) if and only if \( Ax' = x' \), and

(iii) if \( A^{-1}a^*Aa' = A^{-1}s^*Aa' \) and \( A^{-1}b^*As' = A^{-1}s^*Ab' \), then \( A^{-1}b^*Aa' = A^{-1}a^*Ab' \).

**Proof.** (i) is obvious. For the proof of (ii), we first suppose \( \{x_n\} \) is symmetric with \( \{s_n\} \) relative to \( A \) so that \( A^{-1}x^*As' = A^{-1}s^*Ax' \). Multiplying both sides on the left by \( A \) and using \( As' = s' \) gives \( x^*s' = s^*Ax' \). Under the hypothesis, \( s^* \) has inverse \( s_-^* \) so that

\[
s_-^{-1}x^*s' = s_-^{-1}s^*Ax' = Ax'.
\]

Since \( x^*s' = s^*x' \), it follows from (3) that \( x' = Ax' \).
On the other hand, if \( Ax' = x' \),

\[
(4) \quad A^{-1}x^*A's' = A^{-1}x^*s'
\]

and

\[
A^{-1}s^*Ax' = A^{-1}s^*x'.
\]

Since \( s^*x' = x^*s' \), it follows from (4) that \( x \) and \( s \) are symmetric relative to \( A \).

For the proof of (iii), we suppose that \( a' = s^{-1}a^*s' \) and \( b' = s^{-1}b^*s' \), from which it follows that

\[
(5) \quad A^{-1}a^*Ab' = A^{-1}a^*s^{-1}b^*s' \]

and

\[
(6) \quad A^{-1}b^*Aa' = A^{-1}b^*s^{-1}a^*s'.
\]

Since diagonal matrices permute, it follows that (5) and (6) are equal establishing (iii).

**Theorem 1.** If \( \{b_n\} \) is a moment sequence, i.e.,

\[
(7) \quad b_p = \int_0^1 x^p dg(x),
\]

\( \{b_n\} \) and the Cesàro sequence (2) are symmetric relative to \( H \) if and only if \( \{b_n\} \) is a symmetric moment sequence.

**Proof.** Let

\[
f_n(x) = \begin{cases} 
\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p x^p & \text{for } n = 2, 4, 6, \cdots \\
\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p x^p - 2x^n & \text{for } n = 1, 3, 5, \cdots.
\end{cases}
\]

Clearly, if \( \{t_n\} \) is any number sequence, \( Ht' = t' \) if and only if

\[
\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p t_p = 0 \quad \text{for } n = 2, 4, 6, \cdots
\]

and

\[
\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p t_p - 2t_n = 0 \quad \text{for } n = 1, 3, 5, \cdots.
\]

Thus if \( \{b_n\} \) is defined as in (7), \( Hb' = b' \) if and only if

\[
(8) \quad \int_0^1 f_n(x) dg(x) = 0 \quad \text{for } n = 1, 2, 3, \cdots.
\]

But, \( f_n(x) = (1 - x)^n - x^n \) for \( n = 1, 2, 3, \cdots \) so that
\[ \int_0^1 f_n(x) dg(x) = \int_0^1 (1 - x)^n dg(x) - \int_0^1 x^n dg(x) , \]

and consequently (8) holds if and only if \( \{b_n\} \) is a symmetric moment sequence. It follows from (9) and (2) that \( Hc' = c' \) and from the preceding Lemma that \( \{b_n\} \) and \( \{c_n\} \) are symmetric relative to \( H \).

Conversely, if \( \{b_n\} \) and \( \{c_n\} \) are symmetric relative to \( H \), it follows that \( Hb' = b' \), and if \( \{b_n\} \) is defined as in (7), then \( \{b_n\} \) is a symmetric moment sequence.

**Theorem 2.** If \( g(x) \) is of bounded variation on \([0, 1]\) and \( \{z_n\} \) is the moment sequence determined by \( g(x) \), the following two statements are equivalent:

(i) \( \{z_n\} \) is a symmetric moment sequence, and

(ii) there do not exist uncountably many \( x \) in \([0, 1]\) for which \( g(x) + g(1 - x) \neq g(1) + g(0) \).

**Proof.** Suppose (i). Then let \( u = 1 - x \) so that,

\[ z_p = \int_0^1 (1 - x)^p dg(x) = \int_0^1 u^p dg(1 - x) = -\int_0^1 u^p dg(1 - u) . \]

Thus, \( \int_0^1 (1 - x)^p dg(x) = -\int_0^1 x^p dg(1 - x) \) so that for \( p = 0, 1, 2, \ldots \),

\[ \int_0^1 x^p [g(x) + g(1 - x)] = 0 . \]

Since \( g(x) - g(1 - x) \) is of bounded variation on \([0, 1]\), (10) implies that for every \( k(x) \) continuous on \([0, 1]\),

\[ \int_0^1 k(x) d[g(x) + g(1 - x)] = 0 . \]

This, [4, p. 69], implies (ii). Reversing the steps leading to (10) shows that (ii) implies (i).

An interesting example of a function satisfying (ii) is provided by Evans in [1].

**Theorem 3.** Suppose \( g(x) \) is of bounded variation on \([0, 1]\) and suppose \( \{a_n\} \) is the moment sequence generated by \( g(x) \). The following statements are equivalent:

(i) \( \{a_n\} \) is a symmetric moment sequence,

(ii) \( H\alpha' = \alpha' \),

(iii) \( \{a_n\} \) and the Cesàro moment sequence \( \{c_n\} \) are symmetric relative to \( H \), and

(iv) the difference matrix \( (\Delta a) \) having base sequence \( \{a_n\} \) is symmetric about the main diagonal.

**Proof.** Theorem 1 implies the equivalence of (i), (ii), and (iii).
(i) implies (iv) provided

\[(11) \int_0^1 x^n(1-x)^m dg(x) = \int_0^1 x^n(1-x)^m dg(x) \quad \text{for } m, n = 0, 1, 2, \cdots .\]

Let \( u = 1 - x \) so that \( \int_0^1 x^n(1-x)^m dg(x) = \int_1^0 (1-u)^m u^n dg(1-u) \). Thus (11) may be rewritten as

\[(12) - \int_0^1 (1-x)^m x^n dg(1-x) = \int_0^1 x^n(1-x)^m dg(x) = \int_0^1 x^n(1-x)^m d[g(x) + g(1-x)] = 0 .\]

That (12) is the case for \( \{a_n\} \) a symmetric moment sequence follows from (ii) of Theorem 2. (iv) implies (ii) since (iv) implies that \( a_n = A^n A_n \), which is the same as saying that \( Ha' = a' \). Thus the equivalence of the four statements is established.

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