Pacific Journal of Mathematics

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Vol. 15, No. 1 September 1965

PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 1, 1965

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If $\{a_n\}$ is a moment sequence and (Δa) is the difference matrix having base sequence $\{a_n\}$, then (Δa) is symmetric about the main diagonal if and only if the function $\alpha(x)$ such that $a_n = \int_0^1 x^n d\alpha(x)$, $n = 0, 1, 2, \cdots$, is symmetric in the sense that $\alpha(x) + \alpha(1 + x) = \alpha(1) + \alpha(0)$ except for at most countably many x in $[0, 1]$. This property is related to the "fixed points" of the matrix H, where HaH is the Hausdorff matrix determined by the moment sequence $\{a_n\}$.

In each of the papers [2], [3] and [5], there is reference to difference matrices of the form

$$(\Delta d) = \begin{bmatrix} \Delta^0 d_0 & \Delta^0 d_1 & \Delta^0 d_2 & \\ \Delta^1 d_0 & \Delta^1 d_1 & \Delta^1 d_2 & \vdots \\ \Delta^2 d_0 & \Delta^2 d_1 & \Delta^2 d_2 & \\ & \cdots & & \end{bmatrix}$$

where $\{d_n\}$ is a moment sequence, $\Delta^0 d_n = d_n$, $n = 0, 1, 2, \cdots$ and $\Delta^m d_n = \Delta^{m-1} d_n - \Delta^{m-1} d_{n+1}$, for $n = 0, 1, 2, \cdots$ and $m = 1, 2, 3, \cdots$. In [2], Garabedian and Wall discussed the importance of (Δd) having the property of being symmetric about the main diagonal, i.e. $\Delta^m d_n = \Delta^n d_m$. They also showed that if $\{d_n\}$ is a totally monotone sequence, then (Δd) is symmetric about the main diagonal if and only if the function $f(x)$ which generates $\{d_n\}$ has a certain type continued fraction expansion.

In this paper, the symmetry of (Δd) is investigated with the restriction of total monotonicity removed and a collection of necessary and sufficient conditions are given, Theorem 3, for moment sequences in general. A relation is established between the symmetry of (Δd) and the "fixed points" of the difference matrix

$$H = \begin{bmatrix} \binom{0}{0} & & \\ \binom{1}{0} & -\binom{1}{1} & \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} \\ & \cdots & \end{bmatrix}. \tag{1}$$

Received March 15, 1964. This work was performed under the auspices of the United States Atomic Energy Commission.

2. Notation, definitions, and examples. Except for some notation and definitions introduced for convenience, the notation and definitions of this paper will follow [6].

NOTATION. If $\{d_n\}$ is an infinite sequence, d^* and d' denote respectively the diagonal and column matrices determined by $\{d_n\}$.

DEFINITION 1. If $\{d_n\}$ is a number sequence such that for some function $f(x)$ on $[0, 1]$,

$$d_p = \int_0^1 x^p df(x) = \int_0^1 (1-x)^p df(x) \; ; \qquad p = 0, 1, 2, \cdots ,$$

then $\{d_n\}$ is called a symmetric moment sequence.

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots$ provides an example of a moment sequence satisfying Definition 1 since for $p = 0, 1, 2, \cdots$

$$(2) \qquad c_p = \int_0^1 x^p dx = x^{p+1}/p + 1 \Big]_0^1$$
$$= \int_0^1 (1-x)^p dx = -(1-p)^{p+1}/p + 1 \Big]_0^1 = \frac{1}{p+1} \, .$$

DEFINITION 2. If A is a semi-infinite, lower triangular, matrix having inverse and $\{a_n\}$ and $\{d_n\}$ are sequences such that $A^{-1}d^*Aa' = A^{-1}a^*Ad'$, then $\{a_n\}$ and $\{d_n\}$ are symmetric relative to A.

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, c_p$ of (2), and the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ are symmetric relative to the matrix H of (1).

3. THEOREMS. LEMMA. *Suppose $\{s_n\}$ is a sequence such that $s_p \neq 0$ for $p = 0, 1, 2, \cdots$ and suppose that A is a semi-infinite matrix having inverse such that $As' = s'$; then,*

(i) $A^{-1}s' = s'$,

(ii) *$\{x_n\}$ and $\{s_n\}$ are symmetric with respect to A if and only if $Ax' = x'$, and*

(iii) *if $A^{-1}a^*As' = A^{-1}s^*Aa'$ and $A^{-1}b^*As' = A^{-1}s^*Ab'$, then $A^{-1}b^*Aa' = A^{-1}a^*Ab'$.*

Proof. (i) is obvious. For the proof of (ii), we first suppose $\{x_n\}$ is symmetric with $\{s_n\}$ relative to A so that $A^{-1}x^*As' = A^{-1}s^*Ax'$. Multiplying both sides on the left by A and using $As' = s'$ gives $x^*s' = s^*Ax'$. Under the hypothesis, s^* has inverse s^{*-1} so that

$$(3) \qquad s^{*-1}x^*s' = s^{*-1}s^*Ax' = Ax' \, .$$

Since $x^*s' = s^*x'$, it follows from (3) that $x' = Ax'$.

On the other hand, if $Ax' = x'$,

$$A^{-1}x^*As' = A^{-1}x^*s' \tag{4}$$

and

$$A^{-1}s^*Ax' = A^{-1}s^*x' .$$

Since $s^*x' = x^*s'$, it follows from (4) that x and s are symmetric relative to A.

For the proof of (iii), we suppose that $a' = s^{*-1}a^*s'$ and $b' = s^{*-1}b^*s'$, from which it follows that

$$A^{-1}a^*Ab' = A^{-1}a^*s^{*-1}b^*s' \tag{5}$$

and

$$A^{-1}b^*Aa' = A^{-1}b^*s^{*-1}a^*s' . \tag{6}$$

Since diagonal matrices permute, it follows that (5) and (6) are equal establishing (iii).

THEOREM 1. *If $\{b_n\}$ is a moment sequence, i.e.,*

$$b_p = \int_0^1 x^p dg(x) , \tag{7}$$

$\{b_n\}$ and the Cesàro sequence (2) *are symmetric relative to H if and only if $\{b_n\}$ is a symmetric moment sequence.*

Proof. Let

$$f_n(x) = \begin{cases} \sum_{p=0}^{n-1} \binom{n}{p}(-1)^p x^p & \text{for } n = 2, 4, 6, \cdots \\ \sum_{p=0}^{n-1} \binom{n}{p}(-1)^p x^p - 2x^n & \text{for } n = 1, 3, 5, \cdots . \end{cases}$$

Clearly, if $\{t_n\}$ is any number sequence, $Ht' = t'$ if and only if

$$\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p t_p = 0 \qquad \text{for } n = 2, 4, 6, \cdots$$

and

$$\sum_{p=0}^{n-1} \binom{n}{p}(-1)^p t_p - 2t_n = 0 \qquad \text{for } n = 1, 3, 5, \cdots .$$

Thus if $\{b_n\}$ is defined as in (7), $Hb' = b'$ if and only if

$$\int_0^1 f_n(x) dg(x) = 0 \qquad \text{for } n = 1, 2, 3, \cdots . \tag{8}$$

But, $f_n(x) = (1 - x)^n - x^n$ for $n = 1, 2, 3, \cdots$ so that

(9) $$\int_0^1 f_n(x)dg(x) = \int_0^1 (1-x)^n dg(x) - \int_0^1 x^n dg(x) ,$$

and consequently (8) holds if and only if $\{b_n\}$ is a symmetric moment sequence. It follows from (9) and (2) that $Hc' = c'$ and from the preceding Lemma that $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H.

Conversely, if $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H, it follows that $Hb' = b'$, and if $\{b_n\}$ is defined as in (7), then $\{b_n\}$ is a symmetric moment sequence.

THEOREM 2. *If $g(x)$ is of bounded variation on* $[0, 1]$ *and* $\{z_n\}$ *is the moment sequence determined by $g(x)$, the following two statements are equivalent*:

(i) *$\{z_n\}$ is a symmetric moment sequence, and*

(ii) *there do not exist uncountably many x in* $[0, 1]$ *for which* $g(x) + g(1-x) \neq g(1) + g(0)$.

Proof. Suppose (i). Then let $u = 1 - x$ so that,

$$z_p = \int_0^1 (1-x)^p dg(x) = \int_0^1 u^p dg(1-x) = -\int_0^1 u^p dg(1-u) .$$

Thus, $\int_0^1 (1-x)^p dg(x) = -\int_0^1 x^p dg(1-x)$ so that for $p = 0, 1, 2, \cdots$,

(10) $$\int_0^1 x^p d[g(x) + g(1-x)] = 0 .$$

Since $g(x) - g(1-x)$ is of bounded variation on $[0, 1]$, (10) implies that for every $k(x)$ continuous on $[0, 1]$, $\int_0^1 k(x)d[g(x) + g(1-x)] = 0$. This, [4, p. 69], implies (ii). Reversing the steps leading to (10) shows that (ii) implies (i).

An interesting example of a function satisfying (ii) is provided by Evans in [1].

THEOREM 3. *Suppose $g(x)$ is of bounded variation on* $[0, 1]$ *and suppose $\{a_n\}$ is the moment sequence generated by $g(x)$. The following statements are equivalent*:

(i) *$\{a_n\}$ is a symmetric moment sequence,*

(ii) *$Ha' = a'$,*

(iii) *$\{a_n\}$ and the Cesàro moment sequence $\{c_n\}$ are symmetric relative to H, and*

(iv) *the difference matrix (Δa) having base sequence $\{a_n\}$ is symmetric about the main diagonal.*

Proof. Theorem 1 implies the equivalence of (i), (ii), and (iii).

(i) implies (iv) provided

$$(11)\quad \int_0^1 x^m(1-x)^n dg(x) = \int_0^1 x^n(1-x)^m dg(x) \qquad \text{for } m, n = 0, 1, 2, \cdots .$$

Let $u = 1 - x$ so that $\int_0^1 x^m(1-x)^n dg(x) = \int_1^0 (1-u)^m u^n dg(1-u)$. Thus (11) may be rewritten as

$$(12)\qquad -\int_0^1 (1-x)^m x^n dg(1-x) = \int_0^1 x^n(1-x)^m dg(x)$$
$$= \int_0^1 x^n(1-x)^m d[g(x) + g(1-x)] = 0 .$$

That (12) is the case for $\{a_n\}$ a symmetric moment sequence follows from (ii) of Theorem 2. (iv) implies (ii) since (iv) implies that $a_n = \Delta^n A_0$, which is the same as saying that $Ha' = a'$. Thus the equivalence of the four statements is established.

I am grateful to Professor H. S. Wall for some comments which have been of considerable value in the preparation of this paper.

References

1. G. C. Evans, *Calculation of moments for a Cantor-Vitali function*, Sixth Herbert Ellsworth Slaught Memorial Paper, Mathematical Association of America, (1957), 22-27.

2. H. L. Garabedian and H. S. Wall, *Hausdorff methods of summation and continued fractions*, Trans. Amer. Math. Soc. **48** (1940), 185-207.

3. H. L. Garabedian, Einar Hille, and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Math. J. **8** (1941), 193-213.

4. T. H. Hildebrandt, *Introduction to the theory of integration*, Academic-Press, New York, 1963.

5. W. T. Scott and H. S. Wall, *The transformation of series and sequences*, Trans. Amer. Math. Soc. **51** (1942), 255-279.

6. H. S. Wall, *Analytic theory of continued fractions*, van Nostrand, New York, 1948.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

Pacific Journal of Mathematics

Vol. 15, No. 1 September, 1965