

Pacific Journal of Mathematics

A NOTE ON HAUSDORFF'S SUMMATION METHODS

JOSEPH PATRICK BRANNEN

A NOTE ON HAUSDORFF'S SUMMATION METHODS

J. P. BRANNEN

If $\{a_n\}$ is a moment sequence and (Δa) is the difference matrix having base sequence $\{a_n\}$, then (Δa) is symmetric about the main diagonal if and only if the function $\alpha(x)$ such that $a_n = \int_0^1 x^n d\alpha(x)$, $n = 0, 1, 2, \dots$, is symmetric in the sense that $\alpha(x) + \alpha(1+x) = \alpha(1) + \alpha(0)$ except for at most countably many x in $[0, 1]$. This property is related to the "fixed points" of the matrix H , where HaH is the Hausdorff matrix determined by the moment sequence $\{a_n\}$.

In each of the papers [2], [3] and [5], there is reference to difference matrices of the form

$$(\Delta d) = \begin{bmatrix} \Delta^0 d_0 & \Delta^0 d_1 & \Delta^0 d_2 & \vdots \\ \Delta^1 d_0 & \Delta^1 d_1 & \Delta^1 d_2 & \vdots \\ \Delta^2 d_0 & \Delta^2 d_1 & \Delta^2 d_2 & \vdots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

where $\{d_n\}$ is a moment sequence, $\Delta^0 d_n = d_n$, $n = 0, 1, 2, \dots$ and $\Delta^m d_n = \Delta^{m-1} d_n - \Delta^{m-1} d_{n+1}$, for $n = 0, 1, 2, \dots$ and $m = 1, 2, 3, \dots$. In [2], Garabedian and Wall discussed the importance of (Δd) having the property of being symmetric about the main diagonal, i.e. $\Delta^m d_n = \Delta^n d_m$. They also showed that if $\{d_n\}$ is a totally monotone sequence, then (Δd) is symmetric about the main diagonal if and only if the function $f(x)$ which generates $\{d_n\}$ has a certain type continued fraction expansion.

In this paper, the symmetry of (Δd) is investigated with the restriction of total monotonicity removed and a collection of necessary and sufficient conditions are given, Theorem 3, for moment sequences in general. A relation is established between the symmetry of (Δd) and the "fixed points" of the difference matrix

$$(1) \quad H = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & -\begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ \dots & \dots & \dots \end{bmatrix}.$$

Received March 15, 1964. This work was performed under the auspices of the United States Atomic Energy Commission.

2. Notation, definitions, and examples. Except for some notation and definitions introduced for convenience, the notation and definitions of this paper will follow [6].

NOTATION. If $\{d_n\}$ is an infinite sequence, d^* and d' denote respectively the diagonal and column matrices determined by $\{d_n\}$.

DEFINITION 1. If $\{d_n\}$ is a number sequence such that for some function $f(x)$ on $[0, 1]$,

$$d_p = \int_0^1 x^p df(x) = \int_0^1 (1-x)^p df(x); \quad p = 0, 1, 2, \dots,$$

then $\{d_n\}$ is called a symmetric moment sequence.

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \dots$ provides an example of a moment sequence satisfying Definition 1 since for $p = 0, 1, 2, \dots$

$$(2) \quad \begin{aligned} c_p &= \int_0^1 x^p dx = x^{p+1}/p + 1 \Big|_0^1 \\ &= \int_0^1 (1-x)^p dx = -(1-x)^{p+1}/p + 1 \Big|_0^1 = \frac{1}{p+1}. \end{aligned}$$

DEFINITION 2. If A is a semi-infinite, lower triangular, matrix having inverse and $\{a_n\}$ and $\{d_n\}$ are sequences such that $A^{-1}d^*Aa' = A^{-1}a^*Ad'$, then $\{a_n\}$ and $\{d_n\}$ are symmetric relative to A .

The Cesàro moment sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, c_p$ of (2), and the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ are symmetric relative to the matrix H of (1).

3. THEOREMS. LEMMA. Suppose $\{s_n\}$ is a sequence such that $s_p \neq 0$ for $p = 0, 1, 2, \dots$ and suppose that A is a semi-infinite matrix having inverse such that $As' = s'$; then,

- (i) $A^{-1}s' = s'$,
- (ii) $\{x_n\}$ and $\{s_n\}$ are symmetric with respect to A if and only if $Ax' = x'$, and
- (iii) if $A^{-1}a^*As' = A^{-1}s^*Aa'$ and $A^{-1}b^*As' = A^{-1}s^*Ab'$, then $A^{-1}b^*Aa' = A^{-1}a^*Ab'$.

Proof. (i) is obvious. For the proof of (ii), we first suppose $\{x_n\}$ is symmetric with $\{s_n\}$ relative to A so that $A^{-1}x^*As' = A^{-1}s^*Ax'$. Multiplying both sides on the left by A and using $As' = s'$ gives $x^*s' = s^*Ax'$. Under the hypothesis, s^* has inverse s^{*-1} so that

$$(3) \quad s^{*-1}x^*s' = s^{*-1}s^*Ax' = Ax'.$$

Since $x^*s' = s^*x'$, it follows from (3) that $x' = Ax'$.

On the other hand, if $Ax' = x'$,

$$(4) \quad A^{-1}x^*As' = A^{-1}x^*s'$$

and

$$A^{-1}s^*Ax' = A^{-1}s^*x'.$$

Since $s^*x' = x^*s'$, it follows from (4) that x and s are symmetric relative to A .

For the proof of (iii), we suppose that $a' = s^{*-1}a^*s'$ and $b' = s^{*-1}b^*s'$, from which it follows that

$$(5) \quad A^{-1}a^*Ab' = A^{-1}a^*s^{*-1}b^*s'$$

and

$$(6) \quad A^{-1}b^*Aa' = A^{-1}b^*s^{*-1}a^*s'.$$

Since diagonal matrices permute, it follows that (5) and (6) are equal establishing (iii).

THEOREM 1. *If $\{b_n\}$ is a moment sequence, i.e.,*

$$(7) \quad b_p = \int_0^1 x^p dg(x),$$

$\{b_n\}$ and the Cesàro sequence (2) are symmetric relative to H if and only if $\{b_n\}$ is a symmetric moment sequence.

Proof. Let

$$f_n(x) = \begin{cases} \sum_{p=0}^{n-1} \binom{n}{p} (-1)^p x^p & \text{for } n = 2, 4, 6, \dots \\ \sum_{p=0}^{n-1} \binom{n}{p} (-1)^p x^p - 2x^n & \text{for } n = 1, 3, 5, \dots \end{cases}$$

Clearly, if $\{t_n\}$ is any number sequence, $Ht' = t'$ if and only if

$$\sum_{p=0}^{n-1} \binom{n}{p} (-1)^p t_p = 0 \quad \text{for } n = 2, 4, 6, \dots$$

and

$$\sum_{p=0}^{n-1} \binom{n}{p} (-1)^p t_p - 2t_n = 0 \quad \text{for } n = 1, 3, 5, \dots$$

Thus if $\{b_n\}$ is defined as in (7), $Hb' = b'$ if and only if

$$(8) \quad \int_0^1 f_n(x) dg(x) = 0 \quad \text{for } n = 1, 2, 3, \dots$$

But, $f_n(x) = (1-x)^n - x^n$ for $n = 1, 2, 3, \dots$ so that

$$(9) \quad \int_0^1 f_n(x) dg(x) = \int_0^1 (1-x)^n dg(x) - \int_0^1 x^n dg(x),$$

and consequently (8) holds if and only if $\{b_n\}$ is a symmetric moment sequence. It follows from (9) and (2) that $Hc' = c'$ and from the preceding Lemma that $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H .

Conversely, if $\{b_n\}$ and $\{c_n\}$ are symmetric relative to H , it follows that $Hb' = b'$, and if $\{b_n\}$ is defined as in (7), then $\{b_n\}$ is a symmetric moment sequence.

THEOREM 2. *If $g(x)$ is of bounded variation on $[0, 1]$ and $\{z_n\}$ is the moment sequence determined by $g(x)$, the following two statements are equivalent:*

- (i) $\{z_n\}$ is a symmetric moment sequence, and
- (ii) there do not exist uncountably many x in $[0, 1]$ for which $g(x) + g(1-x) \neq g(1) + g(0)$.

Proof. Suppose (i). Then let $u = 1 - x$ so that,

$$z_p = \int_0^1 (1-x)^p dg(x) = \int_0^1 u^p dg(1-x) = -\int_0^1 u^p dg(1-u).$$

Thus, $\int_0^1 (1-x)^p dg(x) = -\int_0^1 x^p dg(1-x)$ so that for $p = 0, 1, 2, \dots$,

$$(10) \quad \int_0^1 x^p d[g(x) + g(1-x)] = 0.$$

Since $g(x) - g(1-x)$ is of bounded variation on $[0, 1]$, (10) implies that for every $k(x)$ continuous on $[0, 1]$, $\int_0^1 k(x) d[g(x) + g(1-x)] = 0$. This, [4, p. 69], implies (ii). Reversing the steps leading to (10) shows that (ii) implies (i).

An interesting example of a function satisfying (ii) is provided by Evans in [1].

THEOREM 3. *Suppose $g(x)$ is of bounded variation on $[0, 1]$ and suppose $\{a_n\}$ is the moment sequence generated by $g(x)$. The following statements are equivalent:*

- (i) $\{a_n\}$ is a symmetric moment sequence,
- (ii) $Ha' = a'$,
- (iii) $\{a_n\}$ and the Cesàro moment sequence $\{c_n\}$ are symmetric relative to H , and
- (iv) the difference matrix (Δa) having base sequence $\{a_n\}$ is symmetric about the main diagonal.

Proof. Theorem 1 implies the equivalence of (i), (ii), and (iii).

(i) implies (iv) provided

$$(11) \quad \int_0^1 x^m(1-x)^n dg(x) = \int_0^1 x^n(1-x)^m dg(x) \quad \text{for } m, n = 0, 1, 2, \dots$$

Let $u = 1 - x$ so that $\int_0^1 x^m(1-x)^n dg(x) = \int_1^0 (1-u)^m u^n dg(1-u)$. Thus (11) may be rewritten as

$$(12) \quad -\int_0^1 (1-x)^m x^n dg(1-x) = \int_0^1 x^n(1-x)^m dg(x) \\ = \int_0^1 x^n(1-x)^m d[g(x) + g(1-x)] = 0.$$

That (12) is the case for $\{a_n\}$ a symmetric moment sequence follows from (ii) of Theorem 2. (iv) implies (ii) since (iv) implies that $a_n = A^n A_0$, which is the same as saying that $Ha' = a'$. Thus the equivalence of the four statements is established.

I am grateful to Professor H. S. Wall for some comments which have been of considerable value in the preparation of this paper.

REFERENCES

1. G. C. Evans, *Calculation of moments for a Cantor-Vitali function*, Sixth Herbert Ellsworth Slaughter Memorial Paper, Mathematical Association of America, (1957), 22-27.
2. H. L. Garabedian and H. S. Wall, *Hausdorff methods of summation and continued fractions*, Trans. Amer. Math. Soc. **48** (1940), 185-207.
3. H. L. Garabedian, Einar Hille, and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Math. J. **8** (1941), 193-213.
4. T. H. Hildebrandt, *Introduction to the theory of integration*, Academic-Press, New York, 1963.
5. W. T. Scott and H. S. Wall, *The transformation of series and sequences*, Trans. Amer. Math. Soc. **51** (1942), 255-279.
6. H. S. Wall, *Analytic theory of continued fractions*, van Nostrand, New York, 1948.

SANDHIA LABORATORY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California

R. M. BLUMENTHAL

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California
Los Angeles, California 90007

*RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Donald Charles Benson, <i>Unimodular solutions of infinite systems of linear equations</i>	1
Richard Earl Block, <i>Transitive groups of collineations on certain designs</i>	13
Barry William Boehm, <i>Existence of best rational Tchebycheff approximations</i>	19
Joseph Patrick Brannen, <i>A note on Hausdorff's summation methods</i>	29
Dennison Robert Brown, <i>Topological semilattices on the two-cell</i>	35
Peter Southcott Bullen, <i>Some inequalities for symmetric means</i>	47
David Geoffrey Cantor, <i>On arithmetic properties of coefficients of rational functions</i>	55
Luther Elic Claborn, <i>Dedekind domains and rings of quotients</i>	59
Allan Clark, <i>Homotopy commutativity and the Moore spectral sequence</i>	65
Allen Devinatz, <i>The asymptotic nature of the solutions of certain linear systems of differential equations</i>	75
Robert E. Edwards, <i>Approximation by convolutions</i>	85
Theodore William Gamelin, <i>Decomposition theorems for Fredholm operators</i>	97
Edmond E. Granirer, <i>On the invariant mean on topological semigroups and on topological groups</i>	107
Noel Justin Hicks, <i>Closed vector fields</i>	141
Charles Ray Hobby and Ronald Pyke, <i>Doubly stochastic operators obtained from positive operators</i>	153
Robert Franklin Jolly, <i>Concerning periodic subadditive functions</i>	159
Tosio Kato, <i>Wave operators and unitary equivalence</i>	171
Paul Katz and Ernst Gabor Straus, <i>Infinite sums in algebraic structures</i>	181
Herbert Frederick Kreimer, Jr., <i>On an extension of the Picard-Vessiot theory</i>	191
Radha Govinda Laha and Eugene Lukacs, <i>On a linear form whose distribution is identical with that of a monomial</i>	207
Donald A. Ludwig, <i>Singularities of superpositions of distributions</i>	215
Albert W. Marshall and Ingram Olkin, <i>Norms and inequalities for condition numbers</i>	241
Horace Yomishi Mochizuki, <i>Finitistic global dimension for rings</i>	249
Robert Harvey Oehmke and Reuben Sandler, <i>The collineation groups of division ring planes. II. Jordan division rings</i>	259
George H. Orland, <i>On non-convex polyhedral surfaces in E^3</i>	267
Theodore G. Ostrom, <i>Collineation groups of semi-translation planes</i>	273
Arthur Argyle Sagle, <i>On anti-commutative algebras and general Lie triple systems</i>	281
Laurent Siebenmann, <i>A characterization of free projective planes</i>	293
Edward Silverman, <i>Simple areas</i>	299
James McLean Sloss, <i>Chebyshev approximation to zero</i>	305
Robert S. Strichartz, <i>Isometric isomorphisms of measure algebras</i>	315
Richard Joseph Turyn, <i>Character sums and difference sets</i>	319
L. E. Ward, <i>Concerning Koch's theorem on the existence of arcs</i>	347
Israel Zuckerman, <i>A new measure of a partial differential field extension</i>	357