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**DOUBLY STOCHASTIC OPERATORS OBTAINED FROM
POSITIVE OPERATORS**

CHARLES RAY HOBBY AND RONALD PYKE

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A recent result of Sinkhorn [3] states that for any square matrix A of positive elements, there exist diagonal matrices D_1 and D_2 with positive diagonal elements for which $D_1 A D_2$ is doubly stochastic. In the present paper, this result is generalized to a wide class of positive operators as follows.

Let $(\Omega, \mathfrak{A}, \lambda)$ be the product space of two probability measure spaces $(\Omega_i, \mathfrak{A}_i, \lambda_i)$. Let f denote a measurable function on (Ω, \mathfrak{A}) for which there exist constants c, C such that $0 < c \leq f \leq C < \infty$. Let K be any nonnegative, two-dimensional real valued continuous function defined on the open unit square, $(0,1) \times (0,1)$, for which the functions $K(u, \cdot)$ and $K(\cdot, v)$ are strictly increasing functions with strict ranges $(0, \infty)$ for each u or v in $(0,1)$. Then there exist functions $h: \Omega_1 \rightarrow E_1$ and $g: \Omega_2 \rightarrow E_1$ such that

$$\int_{\Omega_2} f(x, v) K(h(x), g(v)) d\lambda_2(v) = 1 = \int_{\Omega_1} f(u, y) K(h(u), g(y)) d\lambda_1(u),$$

almost everywhere $-(\lambda)$.

Let $(\Omega, \mathfrak{A}, \lambda)$ be the product space of two probability measure spaces $(\Omega_i, \mathfrak{A}_i, \lambda_i)$. Let f denote a measurable function on (Ω, \mathfrak{A}) for which there exist constants c, C such that

$$(1) \quad 0 < c \leq f \leq C < \infty .$$

Let K be any nonnegative, real valued continuous function defined on the open unit square, $(0,1) \times (0,1)$, for which the functions $K(u, \cdot)$ and $K(\cdot, v)$ are strictly increasing functions with strict ranges $(0, \infty)$ for each u or v in $(0,1)$.

In what follows, h and g will denote measurable, real valued, functions defined on Ω_1 , and Ω_2 , respectively. Whenever well defined, set

$$(2) \quad \begin{aligned} R(x; h, g) &= \int_{\Omega_2} f(x, v) K(h(x), g(v)) d\lambda_2(v) \\ C(y; h, g) &= \int_{\Omega_1} f(u, y) K(h(u), g(y)) d\lambda_1(u) \end{aligned}$$

for $(x, y) \in \Omega$.

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For a fixed choice of h, g we can think of R and C as defining positive operators. The main result of this paper is that R and C can be made doubly stochastic by choosing h and g appropriately. One immediate consequence of this result is a recent theorem of Sinkhorn [3] on doubly stochastic matrices.

THEOREM. *There exist functions $h: \Omega_1 \rightarrow (0,1)$ and $g: \Omega_2 \rightarrow (0,1)$ for which*

$$(3) \quad R(x; h, g) = 1 = C(y; h, g),$$

almost everywhere $-\lambda$.

Proof. We shall obtain h and g as the limits of two sequences of functions, $\{h_n\}$ and $\{g_n\}$. The h_n and g_n are defined recursively as follows.

Set $h_0(x) = \alpha$ for all $x \in \Omega_1$, where α is any number in $(0,1)$. If h_n has been defined, let g_n be the function defined by the equation $C(y; h_n, g_n) = 1$. That is, $g_n(y)$ is the solution of the equation

$$(4) \quad 1 = \int_{\Omega_1} f(x, y) K(h_n(x), g_n(y)) d\lambda_1(x).$$

This solution exists and is unique since $C(y; h_n, t)$ is a strictly increasing continuous function of t with range $(0, \infty)$. Furthermore, g_n is easily seen to be measurable if h_n is measurable (certainly the case for h_0), since $\{y \in \Omega_2: g_n(y) \leq t\} = \{y \in \Omega_2: C(y; h_n, t) \geq 1\}$ and since $C(y; h_n, t)$ is a measurable function of y for each fixed t . By Fubini's theorem

$$(5) \quad \int_{\Omega_1} R(x; h_n, g_n) d\lambda_1(x) = \int_{\Omega_2} C(y; h_n, g_n) d\lambda_2(y) = 1.$$

Thus if $R(x; h_n, g_n) \geq 1$ for all x in Ω_1 , then $R(x; h_n, g_n) = 1$ almost everywhere $-\lambda_1$, and the proof is complete. If for some $x \in \Omega_1$, $R(x; h_n, g_n) < 1$, we define $h_{n+1}(x)$ to be the number t for which $R(x; t, g_n) = 1$. The existence and uniqueness of $h_{n+1}(x)$ follow from our assumptions on K . We set $h_{n+1}(x) = h_n(x)$ at every x where $R(x; h_n, g_n) \geq 1$. Just as for g_n , we see that h_{n+1} is measurable (since g_n is measurable).

Let $A_n = \{x \in \Omega_1 \mid R(x; h_n, g_n) \leq 1\}$. If for some $n \geq 0$, $\lambda_1(A_n) = 1$ we stop our iteration since this implies that $R(x; h_n, g_n) = 1$ a.e. $-\lambda_1$, so we can take h_n and g_n to be h and g of the theorem. We shall assume henceforth that $\lambda_1(A_n) < 1$ for every n .

Observe that $h_{n+1}(x) \geq h_n(x)$ for every x , thus

$$(6) \quad 1 = C(y; h_n, g_n) \leq C(y; h_{n+1}, g_n).$$

Consequently $g_{n+1}(y) \leq g_n(y)$ for every y . It follows from this mono-

tonicity that the limits $h = \lim_{n \rightarrow \infty} h_n$ and $g = \lim_{n \rightarrow \infty} g_n$ exist. We shall now show that this choice of h and g satisfies the theorem.

By our construction, $\{A_n\}$ is a nondecreasing sequence of sets. Set $A = \lim_{n \rightarrow \infty} A_n$. Since $\lambda_1(A_n) < 1$, the complementary set A_n^c is a set of positive measure for each n . For $x \in A_n^c$, $h_n(x) = \alpha$ whence

$$\begin{aligned} 1 &\leq R(x: h_n, g_n) = \int_{\Omega_2} f(x, y) K(\alpha, g_n(y)) d\lambda_2(y) \\ &\leq C \int_{\Omega_2} K(\alpha, g_n(y)) d\lambda_2(y). \end{aligned}$$

This inequality holds for each n , so one may take limits to obtain

$$1 \leq C \int_{\Omega_2} K(\alpha, g(y)) d\lambda_2(y).$$

Thus there are positive numbers r and σ such that $\lambda_2\{y \in \Omega_2: g(y) \geq r\} > \sigma$. Then for arbitrary n and $x \in A_n$,

$$1 \geq c \int_{\Omega_2} K(h_n, g_n) d\lambda_2(y) \geq c\sigma K(h_n(x), r).$$

Hence, by taking limits on n , one obtains $1 \geq c\sigma K(h(x), r)$ for each $x \in A$. Let t be a number for which $1 = c\sigma K(t, r)$. Then $h(x) \leq t$ for $x \in A$, and $h(x) = \alpha$ for $x \in A^c$, whence $h(x) \leq \beta = \max(\alpha, t) < 1$ for all $x \in \Omega_1$. But for all $y \in \Omega_2$ and all n ,

$$\begin{aligned} 1 &= \int_{\Omega_1} f(x, y) K(h_n(x), g_n(y)) d\lambda_1(x) \\ &\leq CK(\beta, g_n(y)), \end{aligned}$$

thus $g(y) \geq \gamma > 0$ where γ satisfies $C^{-1} = K(\beta, \gamma)$.

The import of the above is that the set $\{(h_n(x), g_n(y)): (x, y) \in \Omega, n \geq 0\}$ is contained in a compact subset of the interior of $[0, 1] \times [0, 1]$, on which K is continuous, and hence bounded. Therefore, by the Lebesgue dominated convergence theorem

$$1 = \lim_{n \rightarrow \infty} C(y: h_n, g_n) = \int_{\Omega_1} f(x, y) K(h(x), g(y)) d\lambda_1(x)$$

and

$$1 = \lim_{n \rightarrow \infty} R(x: h_{n+1}, g_n) = \int_{\Omega_2} f(x, y) K(h(x), g(y)) d\lambda_2(y),$$

for $x \in A$. Moreover

$$1 \leq \lim_{n \rightarrow \infty} R(x: h_n, g_n) = \int_{\Omega_2} f(x, y) K(h(x), g(y)) d\lambda_2(y),$$

for $x \notin A$. But an inequality here on a set of positive λ_1 -measure is

impossible by (5), thereby completing the proof.

COROLLARY (Sinkhorn [3]). *Let $A = (a_{ij})$ be an m by m matrix of positive elements. There exist diagonal matrices D_1 and D_2 of positive diagonal elements for which the matrix D_1AD_2 is doubly stochastic.*

Proof. In the above theorem let $\Omega_1 = \Omega_2 = \{1, 2, \dots, m\}$ and let $\lambda_1 = \lambda_2$ be the uniform measure, $\lambda_i(\{j\}) = 1/m$. Set $K(u, v) = uv(1-u)^{-1}(1-v)^{-1}$ and $f(i, j) = a_{ij}$. By the theorem there exist functions h and g such that

$$\begin{aligned} m^{-1} \sum_{i=1}^m a_{ij} h(i) g(j) [1 - h(i)]^{-1} [1 - g(j)]^{-1} &= 1 \\ &= m^{-1} \sum_{j=1}^m a_{ij} h(i) g(j) [1 - h(i)]^{-1} [1 - g(j)]^{-1}. \end{aligned}$$

The corollary is then proved if one lets $d_{1i} = m^{-1/2} [1 - h(i)]^{-1} h(i)$ and $d_{2i} = m^{-1/2} [1 - g(i)]^{-1} g(i)$ be the diagonal elements of D_1 and D_2 respectively.

The above result for symmetric matrices has also been obtained by Marcus and Newman [1] and Maxfield and Minc [2].

The application which motivated Sinkhorn's theorem was the case in which A is the matrix of maximum likelihood estimates of a stochastic transition matrix P of a Markov Chain. When it is further known that P is actually doubly stochastic, then Sinkhorn's result shows that numbers $\{x_1, \dots, x_n; y_1, \dots, y_n\}$ exist such that A can be renormalized by dividing the i th row by x_i and the j th column by y_j to obtain a doubly stochastic matrix. However, if one considers the maximum likelihood equations for the restricted case in which P is known to be doubly stochastic one observes that the proper normalized form of A (relative to the maximum likelihood approach) is a doubly stochastic matrix $B = (b_{ij})$ with $b_{ij} = a_{ij}(x_i + y_j)^{-1}$. The existence of such a normalization follows straightforwardly from the proof of the above theorem. To see this, consider the function $K(u, v) = [v^{-1} - (1-u)^{-1}]^{-1}$ defined on the triangular region $u > 0$, $v > 0$, $u + v < 1$. This function is nonnegative and continuous on this triangle. Moreover, both $K(u, \cdot)$ and $K(\cdot, v)$ are strictly increasing functions wherever defined and the ranges of $K(u, \cdot)$ and $K(\cdot, v)$ are respectively $(0, \infty)$ and $(v[1-v]^{-1}, \infty)$ for each fixed u and v . Let λ_1 and λ_2 be the same discrete measures as used in the proof of the above corollary. The functions $R(x; h_n, g_n)$ and $C(y; h_n, g_n)$ then become finite sums. The only change required in the proof is that one must show that the points $(h_n(x), g_n(y))$, for all $n \geq 1$ and all x and y , are well defined and contained in a compact subset of the domain of K . That this is

true follows from the assumptions on the monotonicity, continuity and range of K , combined with the fact that the integrals are finite sums. Actually, because of these properties, it is clear that $K(h_n(x), g_n(y))$ is bounded by mc^{-1} for all n and y .

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