

# Pacific Journal of Mathematics

## **DOUBLY STOCHASTIC OPERATORS OBTAINED FROM POSITIVE OPERATORS**

CHARLES RAY HOBBY AND RONALD PYKE

## DOUBLY STOCHASTIC OPERATORS OBTAINED FROM POSITIVE OPERATORS

CHARLES HOBBY AND RONALD PYKE

A recent result of Sinkhorn [3] states that for any square matrix  $A$  of positive elements, there exist diagonal matrices  $D_1$  and  $D_2$  with positive diagonal elements for which  $D_1 A D_2$  is doubly stochastic. In the present paper, this result is generalized to a wide class of positive operators as follows.

Let  $(\Omega, \mathfrak{A}, \lambda)$  be the product space of two probability measure spaces  $(\Omega_i, \mathfrak{A}_i, \lambda_i)$ . Let  $f$  denote a measurable function on  $(\Omega, \mathfrak{A})$  for which there exist constants  $c, C$  such that  $0 < c \leq f \leq C < \infty$ . Let  $K$  be any nonnegative, two-dimensional real valued continuous function defined on the open unit square,  $(0,1) \times (0,1)$ , for which the functions  $K(u, \cdot)$  and  $K(\cdot, v)$  are strictly increasing functions with strict ranges  $(0, \infty)$  for each  $u$  or  $v$  in  $(0,1)$ . Then there exist functions  $h: \Omega_1 \rightarrow E_1$  and  $g: \Omega_2 \rightarrow E_1$  such that

$$\int_{\Omega_2} f(x, v) K(h(x), g(v)) d\lambda_2(v) = 1 = \int_{\Omega_1} f(u, y) K(h(u), g(y)) d\lambda_1(u),$$

almost everywhere  $-\lambda$ .

Let  $(\Omega, \mathfrak{A}, \lambda)$  be the product space of two probability measure spaces  $(\Omega_i, \mathfrak{A}_i, \lambda_i)$ . Let  $f$  denote a measurable function on  $(\Omega, \mathfrak{A})$  for which there exist constants  $c, C$  such that

$$(1) \quad 0 < c \leq f \leq C < \infty.$$

Let  $K$  be any nonnegative, real valued continuous function defined on the open unit square,  $(0,1) \times (0,1)$ , for which the functions  $K(u, \cdot)$  and  $K(\cdot, v)$  are strictly increasing functions with strict ranges  $(0, \infty)$  for each  $u$  or  $v$  in  $(0,1)$ .

In what follows,  $h$  and  $g$  will denote measurable, real valued, functions defined on  $\Omega_1$ , and  $\Omega_2$ , respectively. Whenever well defined, set

$$(2) \quad \begin{aligned} R(x; h, g) &= \int_{\Omega_2} f(x, v) K(h(x), g(v)) d\lambda_2(v) \\ C(y; h, g) &= \int_{\Omega_1} f(u, y) K(h(u), g(y)) d\lambda_1(u) \end{aligned}$$

for  $(x, y) \in \Omega$ .

---

Received February 13, 1964. The authors research was supported in part by the National Science Foundation, Grant G-25211 and Grant G-19915.

For a fixed choice of  $h, g$  we can think of  $R$  and  $C$  as defining positive operators. The main result of this paper is that  $R$  and  $C$  can be made doubly stochastic by choosing  $h$  and  $g$  appropriately. One immediate consequence of this result is a recent theorem of Sinkhorn [3] on doubly stochastic matrices.

**THEOREM.** *There exist functions  $h: \Omega_1 \rightarrow (0,1)$  and  $g: \Omega_2 \rightarrow (0,1)$  for which*

$$(3) \quad R(x; h, g) = 1 = C(y; h, g),$$

almost everywhere  $-\lambda$ .

*Proof.* We shall obtain  $h$  and  $g$  as the limits of two sequences of functions,  $\{h_n\}$  and  $\{g_n\}$ . The  $h_n$  and  $g_n$  are defined recursively as follows.

Set  $h_0(x) = \alpha$  for all  $x \in \Omega_1$ , where  $\alpha$  is any number in  $(0,1)$ . If  $h_n$  has been defined, let  $g_n$  be the function defined by the equation  $C(y; h_n, g_n) = 1$ . That is,  $g_n(y)$  is the solution of the equation

$$(4) \quad 1 = \int_{\Omega_1} f(x, y) K(h_n(x), g_n(y)) d\lambda_1(x).$$

This solution exists and is unique since  $C(y; h_n, t)$  is a strictly increasing continuous function of  $t$  with range  $(0, \infty)$ . Furthermore,  $g_n$  is easily seen to be measurable if  $h_n$  is measurable (certainly the case for  $h_0$ ), since  $\{y \in \Omega_2: g_n(y) \leq t\} = \{y \in \Omega_2: C(y; h_n, t) \geq 1\}$  and since  $C(y; h_n, t)$  is a measurable function of  $y$  for each fixed  $t$ . By Fubini's theorem

$$(5) \quad \int_{\Omega_1} R(x; h_n, g_n) d\lambda_1(x) = \int_{\Omega_2} C(y; h_n, g_n) d\lambda_2(y) = 1.$$

Thus if  $R(x; h_n, g_n) \geq 1$  for all  $x$  in  $\Omega_1$ , then  $R(x; h_n, g_n) = 1$  almost everywhere  $-\lambda_1$ , and the proof is complete. If for some  $x \in \Omega_1$ ,  $R(x; h_n, g_n) < 1$ , we define  $h_{n+1}(x)$  to be the number  $t$  for which  $R(x; t, g_n) = 1$ . The existence and uniqueness of  $h_{n+1}(x)$  follow from our assumptions on  $K$ . We set  $h_{n+1}(x) = h_n(x)$  at every  $x$  where  $R(x; h_n, g_n) \geq 1$ . Just as for  $g_n$ , we see that  $h_{n+1}$  is measurable (since  $g_n$  is measurable).

Let  $A_n = \{x \in \Omega_1 \mid R(x; h_n, g_n) \leq 1\}$ . If for some  $n \geq 0$ ,  $\lambda_1(A_n) = 1$  we stop our iteration since this implies that  $R(x; h_n, g_n) = 1$  a.e.  $-\lambda_1$ , so we can take  $h_n$  and  $g_n$  to be  $h$  and  $g$  of the theorem. We shall assume henceforth that  $\lambda_1(A_n) < 1$  for every  $n$ .

Observe that  $h_{n+1}(x) \geq h_n(x)$  for every  $x$ , thus

$$(6) \quad 1 = C(y; h_n, g_n) \leq C(y; h_{n+1}, g_n).$$

Consequently  $g_{n+1}(y) \leq g_n(y)$  for every  $y$ . It follows from this mono-

tonicity that the limits  $h = \lim_{n \rightarrow \infty} h_n$  and  $g = \lim_{n \rightarrow \infty} g_n$  exist. We shall now show that this choice of  $h$  and  $g$  satisfies the theorem.

By our construction,  $\{A_n\}$  is a nondecreasing sequence of sets. Set  $A = \lim_{n \rightarrow \infty} A_n$ . Since  $\lambda_1(A_n) < 1$ , the complementary set  $A_n^c$  is a set of positive measure for each  $n$ . For  $x \in A_n^c$ ,  $h_n(x) = \alpha$  whence

$$\begin{aligned} 1 &\leq R(x: h_n, g_n) = \int_{\Omega_2} f(x, y) K(\alpha, g_n(y)) d\lambda_2(y) \\ &\leq C \int_{\Omega_2} K(\alpha, g_n(y)) d\lambda_2(y) . \end{aligned}$$

This inequality holds for each  $n$ , so one may take limits to obtain

$$1 \leq C \int_{\Omega_2} K(\alpha, g(y)) d\lambda_2(y) .$$

Thus there are positive numbers  $r$  and  $\sigma$  such that  $\lambda_2\{y \in \Omega_2: g(y) \geq r\} > \sigma$ . Then for arbitrary  $n$  and  $x \in A_n$ ,

$$1 \geq c \int_{\Omega_2} K(h_n, g_n) d\lambda_2(y) \geq c\sigma K(h_n(x), r) .$$

Hence, by taking limits on  $n$ , one obtains  $1 \geq c\sigma K(h(x), r)$  for each  $x \in A$ . Let  $t$  be a number for which  $1 = c\sigma K(t, r)$ . Then  $h(x) \leq t$  for  $x \in A$ , and  $h(x) = \alpha$  for  $x \in A^c$ , whence  $h(x) \leq \beta = \max(\alpha, t) < 1$  for all  $x \in \Omega_1$ . But for all  $y \in \Omega_2$  and all  $n$ ,

$$\begin{aligned} 1 &= \int_{\Omega_1} f(x, y) K(h_n(x), g_n(y)) d\lambda_1(x) \\ &\leq CK(\beta, g_n(y)) , \end{aligned}$$

thus  $g(y) \geq \gamma > 0$  where  $\gamma$  satisfies  $C^{-1} = K(\beta, \gamma)$ .

The import of the above is that the set  $\{(h_n(x), g_n(y)): (x, y) \in \Omega, n \geq 0\}$  is contained in a compact subset of the interior of  $[0, 1] \times [0, 1]$ , on which  $K$  is continuous, and hence bounded. Therefore, by the Lebesgue dominated convergence theorem

$$1 = \lim_{n \rightarrow \infty} C(y: h_n, g_n) = \int_{\Omega_1} f(x, y) K(h(x), g(y)) d\lambda_1(x)$$

and

$$1 = \lim_{n \rightarrow \infty} R(x: h_{n+1}, g_n) = \int_{\Omega_2} f(x, y) K(h(x), g(y)) d\lambda_2(y) ,$$

for  $x \in A$ . Moreover

$$1 \leq \lim_{n \rightarrow \infty} R(x: h_n, g_n) = \int_{\Omega_2} f(x, y) K(h(x), g(y)) d\lambda_2(y) ,$$

for  $x \notin A$ . But an inequality here on a set of positive  $\lambda_1$ -measure is

impossible by (5), thereby completing the proof.

**COROLLARY** (Sinkhorn [3]). *Let  $A = (a_{ij})$  be an  $m$  by  $m$  matrix of positive elements. There exist diagonal matrices  $D_1$  and  $D_2$  of positive diagonal elements for which the matrix  $D_1AD_2$  is doubly stochastic.*

*Proof.* In the above theorem let  $\Omega_1 = \Omega_2 = \{1, 2, \dots, m\}$  and let  $\lambda_1 = \lambda_2$  be the uniform measure,  $\lambda_1(\{j\}) = 1/m$ . Set  $K(u, v) = uv(1-u)^{-1}(1-v)^{-1}$  and  $f(i, j) = a_{ij}$ . By the theorem there exist functions  $h$  and  $g$  such that

$$\begin{aligned} m^{-1} \sum_{i=1}^m a_{ij} h(i) g(j) [1 - h(i)]^{-1} [1 - g(j)]^{-1} &= 1 \\ &= m^{-1} \sum_{j=1}^m a_{ij} h(i) g(j) [1 - h(i)]^{-1} [1 - g(j)]^{-1}. \end{aligned}$$

The corollary is then proved if one lets  $d_{1i} = m^{-1/2}[1 - h(i)]^{-1}h(i)$  and  $d_{2i} = m^{-1/2}[1 - g(i)]^{-1}g(i)$  be the diagonal elements of  $D_1$  and  $D_2$  respectively.

The above result for symmetric matrices has also been obtained by Marcus and Newman [1] and Maxfield and Minc [2].

The application which motivated Sinkhorn's theorem was the case in which  $A$  is the matrix of maximum likelihood estimates of a stochastic transition matrix  $P$  of a Markov Chain. When it is further known that  $P$  is actually doubly stochastic, then Sinkhorn's result shows that numbers  $\{x_1, \dots, x_n; y_1, \dots, y_n\}$  exist such that  $A$  can be renormalized by dividing the  $i$ th row by  $x_i$  and the  $j$ th column by  $y_j$  to obtain a doubly stochastic matrix. However, if one considers the maximum likelihood equations for the restricted case in which  $P$  is known to be doubly stochastic one observes that the proper normalized form of  $A$  (relative to the maximum likelihood approach) is a doubly stochastic matrix  $B = (b_{ij})$  with  $b_{ij} = a_{ij}(x_i + y_j)^{-1}$ . The existence of such a normalization follows straightforwardly from the proof of the above theorem. To see this, consider the function  $K(u, v) = [v^{-1} - (1-u)^{-1}]^{-1}$  defined on the triangular region  $u > 0$ ,  $v > 0$ ,  $u + v < 1$ . This function is nonnegative and continuous on this triangle. Moreover, both  $K(u, \cdot)$  and  $K(\cdot, v)$  are strictly increasing functions wherever defined and the ranges of  $K(u, \cdot)$  and  $K(\cdot, v)$  are respectively  $(0, \infty)$  and  $(v[1-v]^{-1}, \infty)$  for each fixed  $u$  and  $v$ . Let  $\lambda_1$  and  $\lambda_2$  be the same discrete measures as used in the proof of the above corollary. The functions  $R(x: h_n, g_n)$  and  $C(y: h_n, g_n)$  then become finite sums. The only change required in the proof is that one must show that the points  $(h_n(x), g_n(y))$ , for all  $n \geq 1$  and all  $x$  and  $y$ , are well defined and contained in a compact subset of the domain of  $K$ . That this is

true follows from the assumptions on the monotonicity, continuity and range of  $K$ , combined with the fact that the integrals are finite sums. Actually, because of these properties, it is clear that  $K(h_n(x), g_n(y))$  is bounded by  $mc^{-1}$  for all  $n$  and  $y$ .

## REFERENCES

1. M. Marcus and M. Newman. *The permanent of a symmetric matrix*, to appear. [cf. Notices of Amer. Math. Soc. **8** (1961), Abstract 587-85, p. 595.]
2. H. Minc and J. E. Maxfield, *A doubly stochastic matrix equivalent to a given matrix*, Notices of Amer. Math. Soc. **9** (1962), Abstract 592-81, p. 309.
3. R. Sinkhorn, *A relationship between arbitrary positive matrices and doubly stochastic matrices*, Ann. Math. Stat. **35** (1964), 876-879.

UNIVERSITY OF WASHINGTON



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON

Stanford University  
Stanford, California

R. M. BLUMENTHAL

University of Washington  
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California  
Los Angeles, California 90007

\*RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION



Donald Charles Benson, <i>Unimodular solutions of infinite systems of linear equations</i> .....	1
Richard Earl Block, <i>Transitive groups of collineations on certain designs</i> .....	13
Barry William Boehm, <i>Existence of best rational Tchebycheff approximations</i> .....	19
Joseph Patrick Brannen, <i>A note on Hausdorff's summation methods</i> .....	29
Dennison Robert Brown, <i>Topological semilattices on the two-cell</i> .....	35
Peter Southcott Bullen, <i>Some inequalities for symmetric means</i> .....	47
David Geoffrey Cantor, <i>On arithmetic properties of coefficients of rational functions</i> .....	55
Luther Elic Claborn, <i>Dedekind domains and rings of quotients</i> .....	59
Allan Clark, <i>Homotopy commutativity and the Moore spectral sequence</i> .....	65
Allen Devinatz, <i>The asymptotic nature of the solutions of certain linear systems of differential equations</i> .....	75
Robert E. Edwards, <i>Approximation by convolutions</i> .....	85
Theodore William Gamelin, <i>Decomposition theorems for Fredholm operators</i> .....	97
Edmond E. Granirer, <i>On the invariant mean on topological semigroups and on topological groups</i> .....	107
Noel Justin Hicks, <i>Closed vector fields</i> .....	141
Charles Ray Hobby and Ronald Pyke, <i>Doubly stochastic operators obtained from positive operators</i> .....	153
Robert Franklin Jolly, <i>Concerning periodic subadditive functions</i> .....	159
Tosio Kato, <i>Wave operators and unitary equivalence</i> .....	171
Paul Katz and Ernst Gabor Straus, <i>Infinite sums in algebraic structures</i> .....	181
Herbert Frederick Kreimer, Jr., <i>On an extension of the Picard-Vessiot theory</i> .....	191
Radha Govinda Laha and Eugene Lukacs, <i>On a linear form whose distribution is identical with that of a monomial</i> .....	207
Donald A. Ludwig, <i>Singularities of superpositions of distributions</i> .....	215
Albert W. Marshall and Ingram Olkin, <i>Norms and inequalities for condition numbers</i> .....	241
Horace Yomishi Mochizuki, <i>Finitistic global dimension for rings</i> .....	249
Robert Harvey Oehmke and Reuben Sandler, <i>The collineation groups of division ring planes. II. Jordan division rings</i> .....	259
George H. Orland, <i>On non-convex polyhedral surfaces in <math>E^3</math></i> .....	267
Theodore G. Ostrom, <i>Collineation groups of semi-translation planes</i> .....	273
Arthur Argyle Sagle, <i>On anti-commutative algebras and general Lie triple systems</i> .....	281
Laurent Siebenmann, <i>A characterization of free projective planes</i> .....	293
Edward Silverman, <i>Simple areas</i> .....	299
James McLean Sloss, <i>Chebyshev approximation to zero</i> .....	305
Robert S. Strichartz, <i>Isometric isomorphisms of measure algebras</i> .....	315
Richard Joseph Turyn, <i>Character sums and difference sets</i> .....	319
L. E. Ward, <i>Concerning Koch's theorem on the existence of arcs</i> .....	347
Israel Zuckerman, <i>A new measure of a partial differential field extension</i> .....	357