DOUBLY STOCHASTIC OPERATORS OBTAINED FROM POSITIVE OPERATORS

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A recent result of Sinkhorn [3] states that for any square matrix $A$ of positive elements, there exist diagonal matrices $D_1$ and $D_2$ with positive diagonal elements for which $D_1 A D_2$ is doubly stochastic. In the present paper, this result is generalized to a wide class of positive operators as follows.

Let $(\Omega, \mathcal{A}, \lambda)$ be the product space of two probability measure spaces $(\Omega_i, \mathcal{A}_i, \lambda_i)$. Let $f$ denote a measurable function on $(\Omega, \mathcal{A})$ for which there exist constants $c, C$ such that $0 < c \leq f \leq C < \infty$. Let $K$ be any nonnegative, two-dimensional real valued continuous function defined on the open unit square, $(0,1) \times (0,1)$, for which the functions $K(u, \cdot)$ and $K(\cdot, v)$ are strictly increasing functions with strict ranges $(0, \infty)$ for each $u$ or $v$ in $(0,1)$. Then there exist functions $h: \Omega_i \to E_i$ and $g: \Omega_2 \to E_1$ such that

$$\int_{\Omega_2} f(x, v) K(h(x), g(v)) \, d\lambda_2(v) = 1 = \int_{\Omega_1} f(u, y) K(h(u), g(y)) \, d\lambda_1(u),$$

almost everywhere $- (\lambda)$.

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$$0 < c \leq f \leq C < \infty. \quad (1)$$

Let $K$ be any nonnegative, real valued continuous function defined on the open unit square, $(0,1) \times (0,1)$, for which the functions $K(u, \cdot)$ and $K(\cdot, v)$ are strictly increasing functions with strict ranges $(0, \infty)$ for each $u$ or $v$ in $(0,1)$.

In what follows, $h$ and $g$ will denote measurable, real valued, functions defined on $\Omega_i$, and $\Omega_2$, respectively. Whenever well defined, set

$$R(x; h, g) = \int_{\Omega_2} f(x, v) K(h(x), g(v)) \, d\lambda_2(v) \quad (2)$$

$$C(y; h, g) = \int_{\Omega_1} f(u, y) K(h(u), g(y)) \, d\lambda_1(u)$$

for $(x,y) \in \Omega$.

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For a fixed choice of \( h, g \) we can think of \( R \) and \( C \) as defining positive operators. The main result of this paper is that \( R \) and \( C \) can be made doubly stochastic by choosing \( h \) and \( g \) appropriately. One immediate consequence of this result is a recent theorem of Sinkhorn [3] on doubly stochastic matrices.

**Theorem.** There exist functions \( h: \Omega_1 \rightarrow (0,1) \) and \( g: \Omega_2 \rightarrow (0,1) \) for which

\[
R(x; h, g) = 1 = C(y; h, g),
\]

almost everywhere \( - (\lambda) \).

**Proof.** We shall obtain \( h \) and \( g \) as the limits of two sequences of functions, \( \{h_n\} \) and \( \{g_n\} \). The \( h_n \) and \( g_n \) are defined recursively as follows.

Set \( h_0(x) = \alpha \) for all \( x \in \Omega_1 \), where \( \alpha \) is any number in \((0,1)\). If \( h_n \) has been defined, let \( g_n \) be the function defined by the equation

\[
C(y; h_n, g_n) = 1.
\]

That is, \( g_n(y) \) is the solution of the equation

\[
1 = \int_{\Omega_1} f(x, y) K(h_n(x), g_n(y))d\lambda_1(x).
\]

This solution exists and is unique since \( C(y; h_n, t) \) is a strictly increasing continuous function of \( t \) with range \((0, \infty)\). Furthermore, \( g_n \) is easily seen to be measurable if \( h_n \) is measurable (certainly the case for \( h_0 \)), since \( \{y \in \Omega_2 : g_n(y) \leq t\} = \{y \in \Omega_2 : C(y; h_n, t) \geq 1\} \) and since \( C(y; h_n, t) \) is a measurable function of \( y \) for each fixed \( t \). By Fubini's theorem

\[
\int_{\Omega_1} R(x; h_n, g_n)d\lambda_1(x) = \int_{\Omega_2} C(y; h_n, g_n)d\lambda_2(y) = 1.
\]

Thus if \( R(x; h_n, g_n) \geq 1 \) for all \( x \) in \( \Omega_1 \), then \( R(x; h_n, g_n) = 1 \) almost everywhere \( - \lambda_1 \), and the proof is complete. If for some \( x \in \Omega_1 \), \( R(x; h_n, g_n) < 1 \), we define \( h_{n+1}(x) \) to be the number for \( t \) which \( R(x; t, g_n) = 1 \). The existence and uniqueness of \( h_{n+1}(x) \) follow from our assumptions on \( K \). We set \( h_{n+1}(x) = h_n(x) \) at every \( x \) where \( R(x; h_n, g_n) \geq 1 \). Just as for \( g_n \), we see that \( h_{n+1} \) is measurable (since \( g_n \) is measurable).

Let \( A_n = \{x \in \Omega_1 | R(x; h_n, g_n) \leq 1\} \). If for some \( n \geq 0, \lambda(A_n) = 1 \) we stop our iteration since this implies that \( R(x; h_n, g_n) = 1 \) a.e. \( - \lambda_1 \), so we can take \( h_n \) and \( g_n \) to be \( h \) and \( g \) of the theorem. We shall assume henceforth that \( \lambda(A_n) < 1 \) for every \( n \).

Observe that \( h_{n+1}(x) \geq h_n(x) \) for every \( x \), thus

\[
1 = C(y; h_n, g_n) \leq C(y; h_{n+1}, g_n).
\]

Consequently \( g_{n+1}(y) \leq g_n(y) \) for every \( y \). It follows from this mono-
tonicity that the limits \( h = \lim_{n \to \infty} h_n \) and \( g = \lim_{n \to \infty} g_n \) exist. We shall now show that this choice of \( h \) and \( g \) satisfies the theorem.

By our construction, \( \{A_n\} \) is a nondecreasing sequence of sets. Set \( A = \lim_{n \to \infty} A_n \). Since \( \lambda_1(A_n) < 1 \), the complementary set \( A_n^c \) is a set of positive measure for each \( n \). For \( x \in A_n^c \), \( h_n(x) = \alpha \) whence

\[
1 \leq R(x; h_n, g_n) = \int_{\Omega_2} f(x, y) K(\alpha, g_n(y)) d\lambda_2(y)
\]

\[
\leq C \int_{\Omega_2} K(\alpha, g_n(y)) d\lambda_2(y) .
\]

This inequality holds for each \( n \), so one may take limits to obtain

\[
1 \leq C \int_{\Omega_2} K(\alpha, g(y)) d\lambda_2(y) .
\]

Thus there are positive numbers \( r \) and \( \sigma \) such that \( \lambda_2 \{ y \in \Omega_2 : g(y) \geq r \} > \sigma \). Then for arbitrary \( n \) and \( x \in A_n \),

\[
1 \geq c \int_{\Omega_2} K(h_n, g_n) d\lambda_2(y) \geq c\sigma K(h_n(x), r) .
\]

Hence, by taking limits on \( n \), one obtains \( 1 \geq c\sigma K(h(x), r) \) for each \( x \in A \). Let \( t \) be a number for which \( 1 = c\sigma K(t, r) \). Then \( h(x) \leq t \) for \( x \in A \), and \( h(x) = \alpha \) for \( x \in A_n^c \), whence \( h(x) \leq \beta = \max(\alpha, t) < 1 \) for all \( x \in \Omega_1 \). But for all \( y \in \Omega_2 \) and all \( n \),

\[
1 = \int_{\Omega_1} f(x, y) K(h_n(x), g_n(y)) d\lambda_1(x)
\]

\[
\leq CK(\beta, g_n(y)) ,
\]

thus \( g(y) \geq \gamma > 0 \) where \( \gamma \) satisfies \( C^{-1} = K(\beta, \gamma) \).

The import of the above is that the set \( \{(h_n(x), g_n(y)) : (x, y) \in \Omega, \quad n \geq 0 \} \) is contained in a compact subset of the interior of \([0,1] \times [0,1]\), on which \( K \) is continuous, and hence bounded. Therefore, by the Lebesgue dominated convergence theorem

\[
1 = \lim_{n \to \infty} C(y; h_n, g_n) = \int_{\Omega_1} f(x, y) K(h(x), g(y)) d\lambda_1(x)
\]

and

\[
1 = \lim_{n \to \infty} R(x; h_{n+1}, g_n) = \int_{\Omega_2} f(x, y) K(h(x), g(y)) d\lambda_2(y) ,
\]

for \( x \in A \). Moreover

\[
1 \leq \lim_{n \to \infty} R(x; h_n, g_n) = \int_{\Omega_2} f(x, y) K(h(x), g(y)) d\lambda_2(y) ,
\]

for \( x \in A \). But an inequality here on a set of positive \( \lambda_1 \)-measure is
impossible by (5), thereby completing the proof.

**Corollary (Sinkhorn [3]).** Let \( A = (a_{ij}) \) be an \( m \) by \( m \) matrix of positive elements. There exist diagonal matrices \( D_1 \) and \( D_2 \) of positive diagonal elements for which the matrix \( D_1 AD_2 \) is doubly stochastic.

**Proof.** In the above theorem let \( \Omega_1 = \Omega_2 = \{1,2, \cdots, m\} \) and let \( \lambda_1 = \lambda_2 \) be the uniform measure, \( \lambda_1(\{j\}) = 1/m. \) Set \( K(u,v) = uv(1-u)^{-1}(1-v)^{-1} \) and \( f(i,j) = a_{ij}. \) By the theorem there exist functions \( h \) and \( g \) such that

\[
m^{-1} \sum_{i=1}^{m} a_{ij} h(i) g(j) [1 - h(i)]^{-1} [1 - g(j)]^{-1} = 1
\]

\[
= m^{-1} \sum_{j=1}^{m} a_{ij} h(i) g(j) [1 - h(i)]^{-1} [1 - g(j)]^{-1}.
\]

The corollary is then proved if one lets \( d_{1i} = m^{-1/2}[1 - h(i)]^{-1} h(i) \) and \( d_{2i} = m^{-1/2}[1 - g(i)]^{-1} g(i) \) be the diagonal elements of \( D_1 \) and \( D_2 \) respectively.

The above result for symmetric matrices has also been obtained by Marcus and Newman [1] and Maxfield and Minc [2].

The application which motivated Sinkhorn's theorem was the case in which \( A \) is the matrix of maximum likelihood estimates of a stochastic transition matrix \( P \) of a Markov Chain. When it is further known that \( P \) is actually doubly stochastic, then Sinkhorn's result shows that numbers \( \{x_1, \cdots, x_n; y_1, \cdots, y_n\} \) exist such that \( A \) can be renormalized by dividing the \( i \)th row by \( x_i \) and the \( j \)th column by \( y_j \) to obtain a doubly stochastic matrix. However, if one considers the maximum likelihood equations for the restricted case in which \( P \) is known to be doubly stochastic one observes that the proper normalized form of \( A \) (relative to the maximum likelihood approach) is a doubly stochastic matrix \( B = (b_{ij}) \) with \( b_{ij} = a_{ij}(x_i + y_j)^{-1}. \) The existence of such a normalization follows straightforwardly from the proof of the above theorem. To see this, consider the function \( K(u,v) = [v^{-1} - (1 - u)^{-1}]^{-1} \) defined on the triangular region \( u > 0, v > 0, u + v < 1. \) This function is nonnegative and continuous on this triangle. Moreover, both \( K(u, \cdot) \) and \( K(\cdot, v) \) are strictly increasing functions wherever defined and the ranges of \( K(u, \cdot) \) and \( K(\cdot, v) \) are respectively \((0, \infty)\) and \((v^{-1}[1 - v], \infty)\) for each fixed \( u \) and \( v. \) Let \( \lambda_1 \) and \( \lambda_2 \) be the same discrete measures as used in the proof of the above corollary. The functions \( R(x; h_n, g_n) \) and \( C(y; h_n, g_n) \) then become finite sums. The only change required in the proof is that one must show that the points \( (h_n(x), g_n(y)) \), for all \( n \geq 1 \) and all \( x \) and \( y, \) are well defined and contained in a compact subset of the domain of \( K. \) That this is
true follows from the assumptions on the monotonicity, continuity and range of $K$, combined with the fact that the integrals are finite sums. Actually, because of these properties, it is clear that $K(h_n(x), g_n(y))$ is bounded by $mc^{-1}$ for all $n$ and $y$.

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