INFINITE SUMS IN ALGEBRAIC STRUCTURES

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The purpose of this note is an outline of an algebraic theory of summability in algebraic structures like abelian groups, ordered abelian groups, modules, and rings. "Infinite sums" of elements of these structures will be defined by means of homomorphisms satisfying some weak requirements of permanency which hold in all usual linear summability methods. It will turn out that several elementary well known theorems from the theory of infinite series, proved ordinarily by methods of analysis, (i.e. by use of some concept of a limit) are consequences of algebraic properties.

1. Definitions and existence theorems. Let $G$ be an abelian group with a ring $T$ operating from the left; we assume, without loss of generality, that $T$ contains the integers. Denote by $G^\omega$ the strong direct sum of countably many copies of $G$, i.e., the set of all infinite sequences $s = (g_i)_{i=1}^\infty = (g_1, g_2, \ldots, g_i, \ldots)$ of elements of $G$, with the natural definitions of addition and of left multiplication by elements of $T$. Let $\Gamma$ be the weak direct sum of countably many copies of $G$, i.e., the subgroup of $G^\omega$ consisting of all infinite sequences with at most a finite number of coordinates different from 0 (the neutral element of $G$). For $s = (g_1, g_2, \ldots, g_i, \ldots) \in G^\omega$, denote by $s'$ the element $(0, g_1, g_2, \ldots, g_i-1, g_i, \ldots)$; $s'$ will be called the translate of $s$.

**Definition 1.** The $T$-subgroup $S$ of $G^\omega$ will be called admissible if

1. $\Gamma \subseteq S$
2. $s \in S$ if and only if $s' \in S$, where $s'$ is the translate of $s$.

Obviously, both $\Gamma$ and $G^\omega$ are admissible, and any subset $K$ of $G^\omega$ can be completed in a unique way to a minimal admissible subgroup containing $K$.

**Definition 2.** Let $S$ be admissible, and $\varphi$ a $T$-homomorphism $S \to G$ with the following properties:

1. $\varphi(g, 0, 0, \ldots) = g$, $(g \in G)$
2. $\varphi(s) = \varphi(s')$, $(s \in S)$.
Then $\varphi$ will be called a summation method on $G$ with domain $S$, and we shall refer to it briefly as the summation method $[S, \varphi]$.

Using the fact that $S$ is admissible, and by properties (3) and (4) of the homomorphism $\varphi$, it follows immediately that

$$\varphi(g_1, g_2, \ldots, g_n, 0, 0, \ldots) = \sum_{i=1}^{n} g_i$$

for any summation method $[S, \varphi]$. Therefore, the (unique) summation method $[\Gamma, \varphi]$ shall be called trivial. Furthermore, by (3), $\varphi$ is always a $T$-homomorphism onto $G$.

We ask first the following question:

When does there exist a summation method containing in its domain a given $s \in G^\omega$?

Denote by $s^{(n)}$ for any integer $n$ the $n$th translate of $s$, i.e.

$$s^{(0)} = s$$

$$s^{(n)} = (s^{(n-1)})' = (0, \ldots, 0, g_1, g_2, \ldots)$$

for $n > 0$

$$s^{(n)} = (g_{-n+1}, g_{-n+2}, \ldots)$$

for $n < 0$.

Let $S$ be the minimal admissible subgroup of $G^\omega$ containing $s$. The elements of $S$ have the form

$$\sum_{k=-m}^{n} t_k s^{(k)} + \gamma,$$

where $m, n$ are nonnegative integers, the $t_k$ are arbitrary elements of $T$, and $\gamma$ is an element of $\Gamma$. This representation of an element of $S$ is not necessarily unique, and a $T$-homomorphism $\varphi: S \to G$ has to be independent of it. But, since all summation methods agree on $\Gamma$, one has to answer first the question when an expression of type (5) will be in $\Gamma$. We may evidently assume $m = 0$ (changing $s$ if necessary); hence we shall study linearly independent expressions of type

$$l_i = \sum_{k=0}^{i} t_{i,k} s^{(k)} = \gamma_i \in \Gamma$$

where $t_{i,k} \in T$. For each $i$, the coefficients $t_{i,k}$ appearing in (6) form a left ideal $T_i$ of $T$, and $T_1 \subset T_2 \subset T_3 \subset \cdots$

We now assume that $T$ satisfies an ascending chain condition, so that each $T_i$ is finitely generated, and that there exists an index $m$ such that $T_m = T_{m+1} = \cdots$. Let $t_{j,i}^{(j=1, \ldots, n_i)}$ be a system of generators of $T_i$ ($i = 1, \ldots, m$). Then a finite system of equations
implies all equations of (6) in the sense that each $l_i$ in (6) is a linear combination over $T$ of the $l_i$ and their translates.

A summation method $\varphi$ on $G$ with domain $S$ exists if and only if $\varphi(s)$ satisfies all the equations

\[
(6'') \quad \left( \sum_{k=0}^i t_{ik} \right) \varphi(s) = \varphi(\gamma_i),
\]

where the right side is independent of $\varphi$, since on $I$ the homomorphism $\varphi$ is the ordinary sum of finitely many elements of $G$. Once $\varphi(s)$ is determined it extends by linearity (over $T$) to all of $S$.

This may be generalized easily for any finite number of elements $s_1, s_2, \ldots, s_r$. Assume a summation method $\varphi$ defined for the minimal admissible subgroup $I'$, containing $s_i$. We can now obtain a finite system of relations of the type $(6')$, with $s$ replaced by $s_i$, and $I'$ by $I'$. This leads to a system of necessary and sufficient conditions for $\varphi(s)$ compatible with $\varphi(s_i)$, which is analogous to $(6'')$ (the right side there being already defined by the previous step). Proceed by induction.

As a consequence we can prove the following existence theorem:

**Theorem 1.** For any abelian group $G \neq \{0\}$ with ring of operators $T$ satisfying an ascending chain condition, there exists a nontrivial summation method.

**Proof.** Let $g \in G$ be $\neq 0$. Define $s = (g_n)_{n=1}^\infty$ by

\[
g_n = \begin{cases} g & \text{if } n = 2^k \\ 0 & \text{otherwise} \end{cases}
\]

Let $S$ be the minimal admissible subgroup containing $s$, and $\bar{g}$ any element of $G$ such that $tg = 0$ implies $t\bar{g} = 0$ for all $t \in T$ (for example $\bar{g} = g$). Then obviously the only relations of type $(6)$ are of the form $ts = 0$ (because $tg = 0$), so that $(6'')$ reduces to $t\varphi(s) = 0$ whenever $tg = 0$. These conditions are satisfied by setting $\varphi(s) = \bar{g}$.

**Remark 1.** From the $2^{\aleph_0}$ sequences in $G^\omega$ whose elements are $g$ or 0 one can pick a subset $R$, of power $2^{\aleph_0}$ so that any relation $\sum_{j=0}^{\infty} \sum_{i=1}^{\infty} t_{ij} r_{ij}^{(j)} \in I'$ for elements $t_{ij}$ of $T$ and $r_{ij} \in R$ implies $t_{ij} g = 0$ for all $t_{ij}$. Thus we can define $2^{\aleph_0}$ different summation methods for the least admissible $S$ which contains $R$ by setting $\varphi(r)$ to be 0 or $g$ arbitrarily for each $r \in R$, and then extending $\varphi$ to all of $S$ by linearity (over $T$).

On the other hand, in a nontrivial group no summation method
can assign a sum to all the sequences of elements of the group.

**Theorem 2.** Let $G \neq 0$ be a $T$-group and $g_i \in G$, $(i = 1, \ldots, n)$ such that $\sum_{i=1}^{n} g_i \neq 0$. Then there exists no summation method defined for

$$s = (g_1, g_2, \ldots, g_n, g_1, g_2, \ldots, g_n, g_1, \ldots).$$

**Proof.** $s^{(n+1)} - s = (g_1, g_2, \ldots, g_n, 0, 0, \ldots)$ would lead to

$$\varphi(s^{(n+1)}) - \varphi(s) = \varphi(s) - \varphi(s) = 0 = g_1 + g_2 + \cdots + g_n,$$

a contradiction.

**Theorem 3.** If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are summation methods on $G$ with domain $S$, and $e_1, e_2, \ldots, e_n$ are $T$-endomorphisms of $G$ so that $e_1 + e_2 + \cdots + e_n = 1$, then $e_1 \varphi_1 + e_2 \varphi_2 + \cdots + e_n \varphi_n$ is a summation method on $G$ with domain $S$.

**Proof.** Let $\varphi = e_1 \varphi_1 + e_2 \varphi_2 + \cdots + e_n \varphi_n$. Then $\varphi$ is obviously a $T$-homomorphism $S \rightarrow G$. Since $\varphi_i(s') = \varphi_i(s)$, the same is true for $\varphi$, and for a $g \in G$ we have $\varphi(g, 0, 0, \ldots) = g$.

**Theorem 4.** Let $[S_1, \varphi_1], [S_2, \varphi_2]$ be two summation methods on $G$ which agree on $D_1 = S_1 \cap S_2$. Then there is a summation method $\varphi$ on $G$ with domain $S = S_1 + S_2$, such that $\varphi|S_i = \varphi_i$ for $i = 1, 2$.

**Proof.** The group $S$ is evidently admissible. Denote $D_i = (S_i \setminus D_0) \cup \{0\}$, $i = 1, 2$. Then any $s \in S$ can be written (not necessarily uniquely)

$$(7) \quad s = d_0 + d_1 + d_2, \quad d_i \in D_i, \quad i = 0, 1, 2.$$  

Define $\varphi$ by

$$\varphi(s) = \varphi_1(d_0) + \varphi_2(d_1) + \varphi_2(d_2).$$

This definition is independent of the representation (7), since if $s = d_0 + d_1 + d_2$ with $d_i \in D_i$, then $A = \varphi_1(d_0 + d_1 + d_2) - \varphi_1(d_0 + d_1 + d_2) = \varphi_1(d_0 - d_0) + \varphi_1(d_1 - d_1) + \varphi_2(d_2 - d_2)$. The element $d_2 - d_2$ is in $S_2$, but since $d_2 - d_2 = d_0 - d_0 + d_1 - d_1$, it is in $D_0$, and therefore $\varphi_2(d_2 - d_2) = \varphi_2(d_0 - d_0 + d_1 - d_1)$. Hence $A = \varphi_1(d_0 - d_0) + \varphi_1(d_1 - d_1) + \varphi_2(d_0 - d_0 + d_1 - d_1) = 0$. A similar reasoning is needed in order to show that $\varphi(s + \bar{s}) = \varphi(s) + \varphi(\bar{s})$ for $s, \bar{s} \in S$, since the sum of two representations of type (7) is generally not of the same type. Property (3) of $\varphi$ is obvious, since $\Gamma \subset D_0$, and (4) follows easily, since (7)
implies \( s' = d'_0 + d'_1 + d'_2 \), where \( d'_i \in D_i, \ i = 0, 1, 2 \). Since the decomposition (7) can be extended to \( ts \), \( \varphi \) is a \( T \)-homomorphism, which finishes the proof.

**Remark 2.** On the other hand, if \( [S_1, \varphi_1] \) and \( [S_2, \varphi_2] \) are summation methods which do not agree on \( S_1 \cap S_2 \), then there need not exist a summation method for the admissible subgroup \( S_1 + S_2 \). Take \( S, \varphi_1 \) as \( S \) and \( \varphi \) in Theorem 1, and define \( s_2 = (g^n_{\alpha})_{\alpha=1}^\infty \) by

\[
g^n_{\alpha} = \begin{cases} 0 & \text{if } n = 2^k \\ g & \text{otherwise} \end{cases}
\]

Again, if \( \bar{g} \) is any element of \( G \) such that \( tg = 0 \) implies \( t\bar{g} = 0 \) for any \( t \in T \), then \( \varphi_2(s_2) = \bar{g} \) is a valid definition that can be extended to a summation method on the minimal admissible subgroup \( S_2 \) containing \( s_2 \). But \( S_1 + S_2 \) cannot be the domain of any summation method, since it contains the element \((g, g, g, \cdots)\), in contradiction to the construction in Theorem 2.

**Remark 3.** Let \((G_a)_{a \in A}\), where \( A \) is a set of indices, be a family of abelian groups with operators \( T \); assume that \( S_a \) is an admissible subgroup of \( G_a \) and that \( \varphi_a \) is a summation method on \( G_a \) with domain \( S_a \) for each \( a \in A \). Consider the (weak or strong) direct sum \( G = \bigoplus_{a \in A} G_a \). Then it is easily shown that \( S = \bigoplus_{a \in A} S_a \) is admissible for \( G \), and that \( \varphi = (\varphi_a)_{a \in A} \) is a summation method with domain \( S \) on \( G \). It is clear that \([S, \varphi]\) is nontrivial if and only if at least one of the summation methods \([S_a, \varphi_a]\) is nontrivial.

2. Subgroups and ideals. To each subgroup \( H \) of \( G \) we associate the (left) annihilator ideal \( T_H \) of \( T \) consisting of all \( t \in T \) such that \( tH = 0 \). If \( H \) is a \( T \)-subgroup of \( G \), then \( T_H \) is a two-sided ideal, since \( 0 = t_H(tH) = (t_Ht)H \) for every \( t_H \in T_H \) and \( t \in T \). Clearly \( T_H^* = T_H \).

Let \([S, \varphi]\) be a summation method on \( G \), and let \( H \) be a \( T \)-subgroup of \( G \). Then \( \varphi(S \cap H^*) = H \), is a \( T \)-subgroup of \( G \) which contains \( H \). We call this group the \([S, \varphi]\)-extension of \( H \). It is easy to see that if \( H_1 \) is an \([S, \varphi]\)-extension of \( H \), then \( T_{H_1} = T_H \); since \( H_1 \supset H \), we obviously have \( T_{H_1} \subset T_H \). On the other hand, \( T_{H_1} \supset T_{H_1}^* = T_H \). From this, it follows:

**Theorem 5.** If \( H \) is a maximal \( T \)-subgroup for the annihilator ideal \( T_H \), then \( H \) has no proper \([S, \varphi]\)-extensions.

**Theorem 6.** Let \( H_1 \) be a denumerable \( T \)-subgroup of \( G \), and
$H_2 > H_1$ a $T$-subgroup of $G$ of cardinality not greater than $2^{\aleph_0}$ such that $T_{H_1} = T_{H_2}$. Then there is a summation method $[S, \varphi]$ on $G$ so that $H_2$ is the $[S, \varphi]$-extension of $H_1$.

**Proof.** Let $\{h_1, h_2, \ldots\}$ be an enumeration of $H_1$, and let $M$ be an increasing sequence of integers. Define sequences $s_{M,i} = (g_{n,i})_{n=1}^\infty$ by

$$g_{s,i} = \begin{cases} h_i & \text{if } n = 2^n, \ m \in M, \ p_i = \text{ith prime} \\ 0 & \text{otherwise} \end{cases}.$$ 

It is easy to find (see Remark 1) a set $\mathcal{M}$ of $2^{\aleph_0}$ sequences $M$ such that any relation of the form $\sum_{i=1}^\infty t_{i,j}g_{M,i}^{(j)} \in \Gamma$ implies $t_{i,j}g_{M,i}^{(j)} = 0$ for all $r$ and $j$, which in turn implies that $t_{i,j} \in T_{H_1}$. Now, let $\{h_\alpha^{(2)}\}_{\alpha \in A}$ be a minimal system of generators of $H_1$, that is $\sum_\alpha t_{\alpha}h_{\alpha}^{(3)} = 0$ (finite sum) if and only if $t_{\alpha}h_{\alpha} = 0$ for all $\alpha$. For any choice of the subsystem $M_\alpha$ of $\mathcal{M}$ the definition $(s_{M,\alpha}) = h_{\alpha}^{(2)}$ for $\alpha \in A$ yields a summation method on the minimal admissible subgroup $S$ of $G^\omega$ containing all the $s_{M,\alpha}$.

**Remark 4.** The restrictions on the cardinalities of $H_1$ and $H_2$ can be removed if we allow summation methods using, instead of $G^\omega$, the strong direct sum $G^\xi$, where $\xi$ is an arbitrary infinite ordinal.

**Example 1.** Let $G$ be a finite abelian group, and $T$ the ring of integers modulo the minimal annihilator $N$ of $G$. To each subgroup $H$ of $G$ corresponds the ideal generated by its minimal annihilator. Clearly, to every divisor $D$ of $N$, there corresponds a unique maximal subgroup $H_D$ of $G$ with minimal annihilator $D$. Each subgroup of $G$ can be $[S, \varphi]$-extended to exactly one $H_D$.

**Example 2.** If $G$ is the additive group of a ring $R$ considered as the ring of operators $T$ on $G$, then $T$-subgroups of $G$ are the left ideals of $R$. Given a subset $M \subset R$, it determines a left annihilator ideal $T_M$ of $M$. Any finitely generated left ideal containing $M$ whose annihilator is $T_M$ can be represented as an $[S, \varphi]$-extension of the left ideal generated by $M$.

**3. Ordered groups.** Let $G$ be an abelian group with a partial ordering relation $\geq$ satisfying: (1) there is a semigroup $H \subset G$ containing the zero element and at least one element $\neq 0$, in which the binary reflexive and transitive relation $\geq$ is defined; (2) if $h, h_1 \in H$ and $h > 0$, then $h_1 + h > h_1$; (3) the archimedean axiom: if $h_1, h_2 \in H$, $h_1 > 0$ and $h_2 > 0$, then there is a positive integer $n$ such that $nh_1 > h_2$.

**Definition 3.** Let $G$ be a partially ordered abelian group. $s = (g_1, g_2, \ldots, g_\alpha, \ldots) \in G^\omega$ will be called positive if $g_\alpha \in H$ and $g_\alpha \geq 0$ for
n = 1, 2, \ldots, \text{ and if } g_{n_0} > 0 \text{ for at least one index } n_0. \ A \text{ summation method } [S, \varphi] \text{ will be called positive if } s \in S \text{ and } s \text{ positive imply } \varphi(s) > 0.

The positive elements of } G^\omega \text{ or of } S \text{ evidently form a semigroup. Furthermore, if } s \text{ is positive, so is its translate } s'.

**Theorem 7.** Let } G \text{ be a partially ordered abelian group, and } [S, \varphi] \text{ a positive summation method on } G. \ If } s = (g_1, g_2, \ldots, g_n, \ldots) \in G \text{ is such that } g_{k_n} \geq g > 0 \text{ for infinitely many indices } k_n, \text{ then } s \in S.

**Proof.** The hypothesis implies that } s \text{ is a positive element. Assume } s \in S \text{ and } \varphi(s) = \gamma, \text{ then } 0 > \gamma = \varphi(s) = \varphi(g_1, g_2, \ldots, g_{k_1}, 0, 0, \ldots) + \varphi(0, \ldots, 0, g_{k_1}, g_{k_1+1}, \ldots) = \sum_{i=1}^{k_1} g_i + \varphi(0, \ldots, 0, g_{k_1}, g_{k_1+1}, \ldots) > ng \text{ for each positive integer } n. \text{ This contradicts the archimedean axiom.}

**Corollary 7.1.** There is no positive nontrivial summation method for the group of integers with their natural ordering.

**Corollary 7.2.** Let } G \text{ be an abelian group with a linear ordering, and } [S, \varphi] \text{ a positive summation method on } G. \ If } s = (g_1, g_2, \ldots, g_n, \ldots) \in S \text{ is positive, then } \text{g.l.b } g = 0 \text{ and } \varphi(s) \geq 1.\text{u.b.}_{1 \leq n < \infty} = \sum_{i=1}^{n} g_i.

From the last part of Corollary 7.2 it follows that if the partial sums of a "series" with positive terms are unbounded, then the "series" does not belong to the domain of any positive summation method.

**Theorem.** Let } G \text{ be a linearly ordered abelian group. Then there is a nontrivial positive summation method on } G \text{ if and only if } G \text{ contains an infinite sequence } g_1, g_2, \ldots, \text{ of positive elements and an element } g, \text{ such that } g_1 + \cdots + g_n \leq g \text{ for all } n.

**Proof.** The necessity follows immediately from Corollary 7.2. To prove sufficiency, set } s = (s_n)_{n=1} \text{ and define } s_n = \begin{cases} g_k & \text{for } n = 2^k \\ 0 & \text{otherwise} \end{cases}.

Then the least admissible } S \text{ which contains } s \text{ has elements which can be expressed uniquely in the form } t = \gamma + \sum_{k=0}^{n} a_k s^{(k)}.
where the $a_i$ are integers and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m, 0, 0, \ldots) \in \Gamma$. $t \geq 0$ implies $a_i \geq 0$ and $\sum_{i=1}^{n} \gamma_i > -(\sum_{i=0}^{n} a_i)g$. Thus if we define $\varphi(s) = g$ we obtain

$$\varphi(t) = \sum_{i=1}^{m} \gamma_i + \left(\sum_{i=0}^{n} a_i\right)g$$

where $\varphi(t) \geq 0$ whenever $t \geq 0$, and $\varphi$ can be extended in an obvious way to a summation method, and is nontrivial.

**Theorem 9.** Let $G$ be a linearly ordered abelian group and $[S, \varphi]$ a positive summation method such that $S$ contains all the positive elements $s = (g_i)_{i=1}^{n} \in G$ for which the “partial sums” $\sum_{i=1}^{n} g_i$ are bounded for all $n$. Then $\varphi(s) = 1.\text{u.b.}_{s \in S} \sum_{i=1}^{n} g_i$ for any positive $s \in S$.

**Proof.** By Corollary 7.2 we know that $\varphi(s) \geq \bar{g} = 1.\text{u.b.}_{n} \sum_{i=1}^{n} g_i$. Assume $\varphi(s) > \bar{g}$. Then $\varphi(0, \ldots, 0, g_N, g_{N+1}, \ldots) \geq \varphi(s) - \bar{g} > 0$ for any $N$, and $g_N + g_{N+1} + \ldots + g_{N+k} < \bar{g}$ for all $k$. It follows that there is a greatest positive integer $n_1$ such that $(2n_1)(g_1 + \ldots + g_k) < \bar{g}$ for all $k$. Determine $n_2$ as greatest positive integer such that $(2n_2)(g_s + \ldots + g_k) < \bar{g} - n_1g$ for all $k$, etc. This defines a nondecreasing sequence of positive integers $n_j$ with $n_j \rightarrow \infty$. Consider the element $\bar{s} = (n_jg_j)_{j=1}^{\infty} \in G^{\omega}$. It is obviously in $S$, since the partial sums $\sum_{j=1}^{n} n_jg_j$ are bounded for all $r$. On the other hand

$$\varphi(\bar{s}) > n_j(\varphi(s) - \bar{g})$$

for all $j$, which is in contradiction with the archimedean property of the order in $G$.

4. Limits.

**Definition 4.** Let $[S, \varphi]$ be a summation method on the abelian group $G$. The sequence $\{g_1, g_2, \ldots, g_n, \ldots\}$ of elements of $G$ will be called $[S, \varphi]$-convergent to $g$ (notation: $g = \lim_{[S, \varphi]} g_n$, or $g_n \rightarrow g$) if

1. $s = (g_n - g_{n-1})_{n=1}^{\infty} \in S$, and
2. $\varphi(s) = g$. (Here $g_0 = 0$.)

The following properties are immediate:

**Theorem 10.** (1) The sequence $\{g, g, g, \ldots\}$ is $[S, \varphi]$-convergent to $g$ for any $[S, \varphi]$. (2) If $g_n \rightarrow g$ and $\bar{g}_n \rightarrow \bar{g}$ then $g_n + \bar{g}_n \rightarrow g + \bar{g}$. (3) $\lim_{[S, \varphi]} (-g_n) = -\lim_{[S, \varphi]} g_n$. (4) If $g = \lim_{[S, \varphi]} g_n$ and $h_1, h_2, \ldots, h_k$ are arbitrary elements of $G$, then the sequence $\{h_1, h_2, \ldots, h_k, g_1, g_2, \ldots, g_n, \ldots\}$ is $[S, \varphi]$-convergent to $g$.

The last part of Theorem 10 implies that if $\lim_{[S, \varphi]} g_n = g$, then
{g_k, g_{k+1}, \cdots} is [S, \varphi]-convergent to g, too.

An arbitrary subsequence of an [S, \varphi]-convergent sequence will not always be [S, \varphi]-convergent to the same limit, even if it is [S, \varphi]-convergent.

**Example 3.** Let G be an abelian group with an element g of order > 2. Define S to be the minimal admissible subgroup of Gω containing the element

\[ s = (2g, -2g, 2g, -2g, \cdots). \]

Since \( s' + s = (2g, 0, 0, \cdots) \) we may define \( \varphi(s) = g \). Then the sequence \( \{2g, 0, 2g, 0, \cdots\} \) is [S, \varphi]-convergent to g, but the subsequence \( \{2g, 2g, \cdots\} \) is [S, \varphi]-convergent to 2g.

This example shows that it is not always possible to define a topology in G by means of [S, \varphi]-convergent sequences.

**Theorem 11.** Let G be an abelian group. A non-trivial summation method [S, \varphi] on G, with the property that every subsequence of any [S, \varphi]-convergent sequence is [S, \varphi]-convergent to the same limit, exists if and only if G is infinite.

**Proof.** Let G be finite. If a sequence of elements of G is not eventually constant, then two different elements must occur infinitely often. Hence no summation method [S, \varphi] with the required property is possible.

Assume G infinite, and distinguish among the following cases:

(a) G contains an element g of infinite order. Let S be the minimal admissible subgroup of Gω containing all the sequences \( (n_i g)_{i=1}^{\infty} \) such that \( \sum n_i \) converges p-adically to a rational integer n. Define then

\[ \varphi((n_i g)_{i=1}^{\infty}) = ng. \]

(b) There exists an element \( g \neq 0 \) of G of finite order divisible by arbitrarily high powers of some prime p. Let M be the subgroup of the additive group of rationals, containing all the sequences \( (p^{-k_i}a_n)_{n=1}^{\infty} \) where \( a_n \) and \( k_n \) are integers, such that \( \sum_{n=1}^{\infty} p^{-k_n}a_n \) converges to a number of the form \( p^{-k}a, \) \( a \) and \( k \) integers. Let S be the minimal admissible subgroup of Gω that contains the sequence \( (p^{-k_n}a_n g)_{n=1}^{\infty}, \) and define \( \varphi((p^{-k_n}a_n g)_{n=1}^{\infty}) = p^{-k}ag. \)

(c) All elements of G are finite but not of bounded order, and no element of G is infinitely divisible (by powers of some prime).
Define $G_n = n!G$; let $S$ be the minimal admissible subgroup of $G^\omega$ consisting of the sequences $(g_n)_{n=1}^{\infty}$ so that there exists a $g$ in $G$ with $g - g_1 - g_2 - \cdots - g_n \in G_n$ for $n = 1, 2, \cdots$. Define $\varphi((g_n)_{n=1}^{\infty}) = g$.

(d) All elements of $G$ have bounded order $\leq m$. Then $G$ must contain an infinite subgroup, all of whose elements have order $p$ for some fixed prime $p$. Otherwise there would be a least divisor $d$ of $m$ for which there is an infinite subgroup $G_1$ of $G$ such that $dG_1 = 0$. If $d$ is composite, then, for every prime divisor $q$ of $d$ the group $qG_1$ is finite, and hence the kernel of the homomorphism $G_1 \to qG_1$ is an infinite group $G_2$ with $qG_2 = 0$, contrary to the hypothesis.

Now, an infinite abelian group all of whose elements are of order $p$ is the direct sum of infinitely many cyclic groups of order $p$, say $Z^{(p)}_1 \oplus Z^{(p)}_2 \oplus \cdots$. Let $S$ be the minimal admissible subgroup of $G^\omega$ containing the sequences $(g_n)_{n=1}^{\infty}$ for which there exists a $g \in G$ such that $g - g_1 - \cdots - g_n \in Z^{(p)}_{n+1} \oplus Z^{(p)}_{n+2} \oplus \cdots$, and define $\varphi((g_n)_{n=1}^{\infty}) = g$. 
Donald Charles Benson, *Unimodular solutions of infinite systems of linear equations* ................................................................. 1
Richard Earl Block, *Transitive groups of collineations on certain designs* ............................... 13
Joseph Patrick Brannen, *A note on Hausdorff’s summation methods* ................................. 29
Dennison Robert Brown, *Topological semilattices on the two-cell* ........................................... 35
Peter Southcott Bullen, *Some inequalities for symmetric means* ............................................ 47
David Geoffrey Cantor, *On arithmetic properties of coefficients of rational functions* .......... 55
Luther Elic Claborn, *Dedekind domains and rings of quotients* ............................................. 59
Allan Clark, *Homotopy commutativity and the Moore spectral sequence* .............................. 65
Allen Devinatz, *The asymptotic nature of the solutions of certain linear systems of differential equations* .......................................................... 75
Robert E. Edwards, *Approximation by convolutions* ............................................................... 85
Theodore William Gamelin, *Decomposition theorems for Fredholm operators* ..................... 97
Edmond E. Granirer, *On the invariant mean on topological semigroups and on topological groups* .................................................................................. 107
Noel Justin Hicks, *Closed vector fields* ..................................................................................... 141
Charles Ray Hobby and Ronald Pyke, *Doubly stochastic operators obtained from positive operators* .............................................................................. 153
Robert Franklin Jolly, *Concerning periodic subadditive functions* .......................................... 159
Tosio Kato, *Wave operators and unitary equivalence* ............................................................... 171
Paul Katz and Ernst Gabor Straus, *Infinite sums in algebraic structures* ................................. 181
Herbert Frederick Kreimer, Jr., *On an extension of the Picard-Vessiot theory* ....................... 191
Radha Govinda Laha and Eugene Lukacs, *On a linear form whose distribution is identical with that of a monomial* ...................................................... 207
Donald A. Ludwig, *Singularities of superpositions of distributions* ........................................ 215
Albert W. Marshall and Ingram Olkin, *Norms and inequalities for condition numbers* .......... 241
Horace Yomishi Mochizuki, *Finitistic global dimension for rings* ......................................... 249
Robert Harvey Oehmke and Reuben Sandler, *The collineation groups of division ring planes. II. Jordan division rings* ......................................................... 259
George H. Orland, *On non-convex polyhedral surfaces in $E^3$* ............................................. 267
Theodore G. Ostrom, *Collineation groups of semi-translation planes* ..................................... 273
Arthur Argyle Sagle, *On anti-commutative algebras and general Lie triple systems* ................ 281
Laurent Siebenmann, *A characterization of free projective planes* .......................................... 293
Edward Silverman, *Simple areas* ............................................................................................... 299
James McLean Sloss, *Chebyshev approximation to zero* ......................................................... 305
Robert S. Strichartz, *Isometric isomorphisms of measure algebras* ........................................ 315
Richard Joseph Turyn, *Character sums and difference sets* .................................................... 319
L. E. Ward, *Concerning Koch’s theorem on the existence of arcs* ........................................... 347
Israel Zuckerman, *A new measure of a partial differential field extension* .............................. 357