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**ON ANTI-COMMUTATIVE ALGEBRAS AND GENERAL LIE  
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## ON ANTI-COMMUTATIVE ALGEBRAS AND GENERAL LIE TRIPLE SYSTEMS

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A general Lie triple system as defined by K. Yamaguti, is considered as an anti-commutative algebra  $A$  with a trilinear operation  $[x, y, z]$  in which (among other things) the mappings  $D(x, y) : z \rightarrow [x, y, z]$  are derivations of  $A$ . It is shown that if the trilinear operation is homogeneous, and  $A$  is irreducible as a general L. t. s. or irreducible relative to the Lie algebra  $I(A)$  generated by the  $D(x, y)$ 's, then  $A$  is a simple algebra. The main result is the following. If  $A$  is a simple finite-dimensional anti-commutative algebra over a field of characteristic zero which is a general L. t. s. with a homogeneous trilinear operation  $[x, y, z]$ , then  $A$  is (1) a Lie algebra; or (2) a Malcev algebra; or (3) an algebra satisfying  $J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy)$  where  $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$ . Furthermore in all three cases  $I(A)$  is the derivation algebra of  $A$  and  $I(A)$  is completely reducible in  $A$ .

1. A general Lie triple system (general L. t. s.) has been defined in [6] to be a vector space  $V$  over a field  $F$  which is closed with respect to a trilinear operation  $[x, y, z]$  and a bilinear operation  $xy$  so that

$$(1.1) \quad [x, y, z] = 0,$$

$$(1.2) \quad x^2 = 0,$$

$$(1.3) \quad [x, y, z] + [y, z, x] + [z, x, y] - (xy)z - (yz)x - (zx)y = 0,$$

$$(1.4) \quad [wx, y, z] + [xy, w, z] + [yw, x, z] = 0,$$

$$(1.5) \quad [[u, v, w], x, y] + [[v, u, x], w, y] \\
 + [v, u, [w, x, y]] + [w, x, [u, v, y]] = 0,$$

$$(1.6) \quad [w, x, yz] + z[w, x, y] + y[x, w, z] = 0.$$

A general L. t. s. is an extension of a Lie triple system used in differential geometry and Jordan algebras. Next we note that if  $V$  is a Lie algebra with multiplication  $xy$ , then  $V$  becomes a general L. t. s. by setting  $[x, y, z] = (xy)z$ . As an extension of this it was shown in [7] that if  $V$  is a Malcev algebra [2] with multiplication  $xy$ , then  $V$  becomes a general L. t. s. by setting  $[x, y, z] = -(xy)z + (yz)x + (zx)y$ .

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In this paper we shall take the point of view that a general L. t. s. is an algebra  $A$  with multiplication  $xy$  satisfying (1.1) – (1.6); that is,  $A$  is an anti-commutative algebra with linear transformations

$$D(x, y): A \rightarrow A : z \rightarrow zD(x, y) \equiv [x, y, z]$$

for all  $x, y \in A$  satisfying (1.1)–(1.6). Thus from (1.6) we see that each  $D(x, y)$  is a derivation of  $A$  satisfying various identities. Motivated by the above examples we shall assume that the trilinear product  $[x, y, z]$  is a linear homogeneous expression in the products of  $x, y$  and  $z$ . Thus using anti-commutativity we assume there exist fixed  $\alpha, \beta, \gamma \in F$  so that

$$(1.7) \quad [x, y, z] = \alpha xy \cdot z + \beta yz \cdot x + \gamma zx \cdot y.$$

With (1.7) as the form for the trilinear operation, we next consider irreducibility conditions on  $A$ . First suppose  $A$  is irreducible as a general L. t. s.; that is,  $A$  has no proper general L. t. s. subspace  $B$  so that  $[B, A, A] \subset B$ , then  $A$  is a simple algebra. For if  $B$  is a proper ideal of  $A$ , then from (1.7)  $B$  is general L. t. s. subspace so that  $[B, A, A] \subset B$ . Next let  $I(A)$  be the subspace of the derivation Lie algebra  $D(A)$  generated by all derivations of the form  $D(x, y)$ , then from (1.5)  $I(A)$  is a Lie subalgebra of  $D(A)$  under commutation. If  $A$  is  $I(A)$ -irreducible, then by (1.7)  $A$  is a simple algebra. Motivated by these remarks the main result is the following

**THEOREM.** *If  $A$  is a simple finite dimensional anti-commutative algebra over a field of characteristic zero with a nonzero trilinear operation  $[x, y, z]$  satisfying (1.1)–(1.7), then*

- (1)  *$A$  is a simple Lie algebra with  $\beta = \gamma, \alpha - \beta = 1$ ; or*
- (2)  *$A$  is a simple Malcev algebra [4] with  $\alpha = -1, \beta = \gamma = 1$ ; or*
- (3)  *$A$  is a simple algebra satisfying  $J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy)$  with  $\alpha = 1/2, \beta = \gamma = 1/4$  and  $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$ .*

*Furthermore in all three cases  $I(A)$  is the derivation algebra of  $A$  and  $I(A)$  is completely reducible in  $A$ .*

It should be noted that since the trilinear operation  $[x, y, z]$  given by (1.7) is homogeneous, any nonzero scalar multiple would also be an admissible trilinear operation. Therefore all superfluous nonzero scalars will be eliminated to obtain the final above normalizations for  $\alpha, \beta$  and  $\gamma$ .

**2. Identities.** We investigate the identities (1.1)–(1.7) with the assumption that  $A$  is a simple finite dimensional anti-commutative

algebra over a field  $F$  of characteristic zero and with multiplication denoted by  $xy$ . From (1.1),

$$\begin{aligned} 0 &= [x, x, y] = \alpha xx \cdot y + \beta xy \cdot x + \gamma yx \cdot x \\ &= (\beta - \gamma) xy \cdot x. \end{aligned}$$

Thus  $\gamma = \beta$  or  $xy \cdot x = 0$  for all  $x, y \in A$ . Suppose this last equation holds then we must have

$$xy \cdot z + x \cdot yz = 0 \quad x, y, z \in A.$$

Now let  $0 \neq b \in A$  and consider  $B = bA$ .  $B$  is an ideal of  $A$ ; for if  $y, z \in A$ , then  $by \cdot z = -b \cdot yz$ . Thus  $B = 0$  or  $B = A$ .  $B = 0$  implies  $bF$  is an ideal of  $A$  and therefore  $A = bF$ . This implies  $A^2 = 0$ , a contradiction to the simplicity of  $A$ . But if  $R_x$  or  $R(x)$  denotes the mapping  $a \rightarrow ax$ , then  $B = A$  implies  $R_b$  is surjective and since  $A$  is finite dimensional,  $R_b$  is injective. This contradicts  $bR_b = 0$  with  $b \neq 0$ . Thus we must have  $\gamma = \beta$ .

From (1.2),  $xy = -yx$  which is just the statement that  $A$  is anti-commutative.

From (1.3),  $\gamma = \beta$  and setting  $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$  we have

$$\begin{aligned} J(x, y, z) &\doteq \alpha(xy \cdot z + yz \cdot x + zx \cdot y) \\ &\quad + \beta(yz \cdot x + zx \cdot y + xy \cdot z) \\ &\quad + \gamma(zx \cdot y + xy \cdot z + yz \cdot x) \\ &= (\alpha + 2\beta) J(x, y, z). \end{aligned}$$

Thus if  $A$  is not a Lie algebra we have

$$(2.1) \quad 1 = \alpha + 2\beta.$$

In case  $A$  is a Lie algebra, then  $\beta = \gamma$  and  $D(x, y) = (\beta - \alpha) R(xy)$ . The remaining identities give no more information and therefore the first part of the main theorem is proved by setting  $\beta - \alpha = -1$ . Henceforth  $A$  is assumed to be a non-Lie algebra.

From (1.4) we obtain

$$\begin{aligned} 0 &= \alpha[(wx \cdot y)z + (xy \cdot w)z + (yw \cdot x)z] \\ &\quad + \beta[yz \cdot wx + wz \cdot xy + xz \cdot yw \\ &\quad + (z \cdot wx)y + (z \cdot xy)w + (z \cdot yw)x] \\ &= \alpha J(w, x, y)z \\ &\quad + \beta [yz \cdot wx + (z \cdot wx)y + (wx \cdot y)z - (wx \cdot y)z \\ &\quad + wz \cdot xy + (z \cdot xy)w + (xy \cdot w)z - (xy \cdot w)z \\ &\quad + xz \cdot yw + (z \cdot yw)x + (yw \cdot x)z - (yw \cdot x)z] \end{aligned}$$

$$= \alpha J(w, x, y)z - \beta J(w, x, y)z \\ + \beta [J(y, z, wx) + J(w, z, xy) + J(x, z, yw)] .$$

Thus

$$(2.2) \quad (\alpha - \beta) J(w, x, y)z = \beta [J(z, w, xy) + J(z, x, yw) + J(z, y, wx)] .$$

From (2.2) we see that  $\beta \neq 0$ . For suppose  $\beta = 0$ , then from (2.1),  $\alpha = 1$  and from (2.2),  $J(w, x, y)z = 0$ . Now if  $J(A, A, A)$  denotes the subspace spanned by the elements  $J(w, x, y)$  for all  $w, x, y \in A$  we see that  $J(A, A, A)$  is a nonzero ideal of  $A$  and so  $A = J(A, A, A)$ . But  $J(w, x, y)z = 0$  implies  $A^3 = 0$ , a contradiction.

We rewrite (1.5) in terms of the derivations  $D(u, v)$  by operating on  $y$  in (1.5) to obtain

$$(2.3) \quad [D(w, x), D(v, u)] = D(wD(v, u), x) + D(w, xD(v, u)) ,$$

where for linear transformations  $S$  and  $T$ ,  $[S, T] = ST - TS$ . We shall not use this identity since a straightforward computation, as suggested by the referee, shows (1.6) and (1.7) imply (1.5).

Next using  $[x, y, z] = (\alpha - \beta)xy \cdot z + \beta J(x, y, z)$  we obtain from (1.6),

$$0 = (\alpha - \beta) wx \cdot yz + \beta J(w, x, yz) \\ + (\alpha - \beta) z(wx \cdot y) + \beta z J(w, x, y) \\ + (\alpha - \beta) y(xw \cdot z) + \beta y J(x, w, z)$$

and therefore

$$(2.4) \quad (\alpha - \beta) J(wx, y, z) - \beta J(yz, w, x) \\ = \beta [J(w, x, z)y - J(w, x, y)z] .$$

**3. Proof of main theorem.** We shall investigate first the restrictions imposed by (2.2) and (2.4). In (2.4) set  $w = y$  and  $z = x$  to obtain

$$(\alpha - 2\beta) J(xy, x, y) = \beta [J(y, x, x)y - J(y, x, y)x] \\ = 0 .$$

Thus we have

$$\text{CASE I.} \quad J(xy, x, y) = 0, \quad \text{or}$$

$$\text{CASE II.} \quad \alpha = 2\beta .$$

We shall show that in Case I,  $A$  must be a non-Lie Malcev algebra (since we are assuming  $A$  is not a Lie algebra) and that Case II yields an anti-commutative algebra satisfying  $J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy)$ .

For Case I we linearize  $J(xy, x, y) = 0$  to obtain

$$(3.1) \quad J(wx, y, z) + J(yz, w, x) = J(wy, z, x) + J(zx, w, y),$$

$$(3.2) \quad wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = 3[J(wx, y, z) + J(yz, w, x)].$$

Using (3.2) and (2.2) we have

$$\begin{aligned} & 3(\alpha - \beta)[J(wx, y, z) + J(yz, w, x)] \\ &= (\alpha - \beta)[-J(x, y, z)w + J(y, z, w)x - J(z, w, x)y + J(w, x, y)z] \\ &= \beta[-J(w, x, yz) - J(w, y, zx) - J(w, z, xy) \\ &\quad + J(x, y, zw) + J(x, z, wy) + J(x, w, yz) \\ &\quad - J(y, z, wx) - J(y, w, xz) - J(y, x, zw) \\ &\quad + J(z, w, xy) + J(z, x, yw) + J(z, y, wx)] \\ &= \beta[2J(xy, z, w) + 2J(zw, x, y) \\ &\quad - 2J(wx, y, z) - 2J(yz, w, x) \\ &\quad + 2J(wy, x, z) + 2J(xz, w, y)] \\ &= -6\beta[J(wx, y, z) + J(yz, w, x)], \quad \text{using (3.1)}. \end{aligned}$$

Thus  $\alpha - \beta = -2\beta \neq 0$  or

$$(3.3) \quad J(wx, y, z) + J(yz, w, x) = 0.$$

Now in case  $\alpha - \beta = -2\beta$  we have from (2.2),

$$2zJ(w, x, y) = J(z, w, xy) + J(z, x, yw) + J(z, y, wx)$$

and using this identity with (3.1) we have

$$\begin{aligned} 2wJ(w, x, y) &= J(w, x, yw) + J(w, y, wx) \\ &= 2J(w, x, yw). \end{aligned}$$

Thus  $A$  is a Malcev algebra. Using the results of [3] we may assume  $\alpha = -1, \beta = \gamma = 1$ . Also from [3] the derivation algebra equals  $I(A)$  and  $I(A)$  is completely reducible in  $A$ .

So we next assume  $A$  satisfies (3.3), then using (2.4) we obtain

$$\alpha J(wx, y, z) = \beta[J(w, x, z)y - J(w, x, y)z],$$

and therefore

$$\begin{aligned} \alpha(\alpha - \beta) J(wx, y, z) &= \beta(\alpha - \beta)[J(w, x, z)y - J(w, x, y)z] \\ &= \beta^2[J(y, w, xz) + J(y, x, zw) \\ &\quad + J(y, z, wx) - J(z, w, xy) \\ &\quad - J(z, x, yw) - J(z, y, wx)], \quad \text{using (2.2)} \\ &= 2\beta^2 J(wx, y, z), \quad \text{using (3.3)}. \end{aligned}$$

Thus  $J(wx, y, z) = 0,$  or  
 $\alpha(\alpha - \beta) = 2\beta^2.$

In the first case,  $A$  is a Lie algebra which is a contradiction and in the second case  $\alpha = -\beta$  or  $2\beta$ . Thus as subcases we have

Case A.  $\alpha = -\beta$  and therefore  $\alpha = -1, \beta = \gamma = 1;$

Case B.  $\alpha = 2\beta$  and therefore  $\alpha = 1/2, \beta = \gamma = 1/4.$

First consider Case A, then from (2.2) and (3.3) we have

$$\begin{aligned} 2wJ(w, x, y) &= J(w, w, xy) + J(w, x, yw) + J(w, y, wx) \\ &= 2J(w, x, yw) \end{aligned}$$

and therefore  $A$  is a Malcev algebra. We shall next show that this Malcev algebra of Case A actually does not exist. First for any anti-commutative algebra  $A$  define the linear transformation  $\Delta(x, y)$  by

$$z \Delta(x, y) = J(x, y, z)$$

and let  $\Delta(A, A)$  be the linear space of transformations spanned by these  $\Delta(x, y)$ 's for all  $x, y \in A$ . Using (3.3) we have

$$\begin{aligned} 0 &= J(wx, y, z) + J(w, x, yz) \\ &= w(R_x \Delta(y, z) + \Delta(x, yz)) \end{aligned}$$

and therefore

$$(3.4) \quad R_x \Delta(y, z) = -\Delta(x, yz) \in \Delta(A, A).$$

From identities (2.32) and (2.34) of [2] we also have

$$2 \Delta(y, z)R_x = 2R_x \Delta(y, z) - 2R(J(x, y, z)) - 4 \Delta(yz, x) \in \Delta(A, A),$$

using also the preceding identity. Thus we see from these identities that  $\Delta(A, A)$  is an ideal in the transformation algebra  $T(A)$  which is generated by  $R(A) = \{R_x : x \in A\}$ . But since  $A$  is simple,  $T(A)$  is a simple associative algebra and therefore  $\Delta(A, A) = 0$  or  $T(A) = \Delta(A, A)$ . But  $\Delta(A, A) = 0$  implies  $A$  is a Lie algebra and therefore  $\Delta(A, A)$  is a simple associative algebra. But we shall next show that  $\Delta(x, y)^2 = 0$  and therefore conclude that  $\Delta(A, A)$  have a basis consisting of nilpotent elements. Thus  $\Delta(A, A)$  must be a nilpotent associative algebra, a contradiction to the simplicity of  $\Delta(A, A)$ . Hence Case A does not exist. So to show  $\Delta(x, y)^2 = 0$  we have

$$\begin{aligned} -2z \Delta(x, y) R(xy) &= 2xy \cdot J(x, y, z) \\ &= J(xy, x, yz) + J(xy, y, zx) \\ &\quad + J(xy, z, xy) \end{aligned}$$

$$\begin{aligned}
 &= - J(x \cdot yz, x, y) - J(y \cdot zx, x, y) \\
 &\quad - J(z \cdot xy, x, y), \quad \text{using (3.3)} \\
 &= J(J(x, y, z), x, y) \\
 &= z \Delta(x, y)^2,
 \end{aligned}$$

that is,  $\Delta(x, y)^2 = -2 \Delta(x, y) R(xy)$ . But from identity (2.33) of [2],  $\Delta(x, y)^2 = -3 \Delta(x, y) R(xy)$  and therefore  $\Delta(x, y)^2 = 0$ .

Next we derive more identities for Case B and use methods similar to those used in Case A to show Case B does not exist either. Using the notation from Case A we obtain again from (3.3) the identity (3.4). Also with  $\alpha = 1/2, \beta = 1/4$  in (2.2) we obtain

$$(3.5) \quad J(w, x, y)z = J(z, w, xy) + J(z, x, yw) + J(z, y, wx) .$$

From (3.5) and (3.4) we have

$$\Delta(x, y)R_z = -\Delta(z, xy) - R_y \Delta(z, x) + R_x \Delta(z, y) \in \Delta(A, A) .$$

Thus as in Case A we see  $\Delta(A, A) = T(A)$  is a simple associative algebra. Next from (3.5) we also have

$$(3.6) \quad R(J(w, x, y)) = \Delta(wx, y) + \Delta(xy, w) + \Delta(yw, x) ,$$

and using (3.4) we obtain

$$\begin{aligned}
 \Delta(x, y)^2 &= [R_x, R_y] \Delta(x, y) - R(xy) \Delta(x, y) \\
 &= R_x R_y \Delta(x, y) - R_y R_x \Delta(x, y) + \Delta(xy, xy) \\
 &= \Delta(x, y \cdot xy) - \Delta(y, x \cdot xy) + \Delta(xy, xy) \\
 &= -\Delta(xy \cdot x, y) - \Delta(xy, xy) - \Delta(y \cdot xy, x) \\
 &= -R(J(xy, x, y)) \\
 &= 0 ,
 \end{aligned}$$

where the last equality follows from (3.3). Thus as in Case A, we conclude that Case B does not exist, this completes Case I.

Next consider Case II where  $\alpha = 2\beta \neq 0$ . From (2.2) and (2.4) we see that  $A$  satisfies (3.5) and

$$(3.7) \quad \begin{aligned} &J(wx, y, z) - J(yz, w, x) \\ &= J(w, x, z)y - J(w, x, y)z . \end{aligned}$$

Next we rewrite (3.5) and (3.7) in terms of right multiplications to obtain (3.6) and

$$\Delta(z, wx) - R_z \Delta(w, x) = -R(J(w, x, z)) - \Delta(w, x)R_z ,$$

by operation on  $y$  in (3.7). Using this and (3.6) we have

$$\Delta(w, x)R_z - R_z \Delta(w, x) = -\Delta(wx, z) - \Delta(xz, w)$$



$$\begin{aligned} & - \Delta(zw, x) - \Delta(z, wx) \\ & = \Delta(zx, w) - \Delta(zw, x) ; \end{aligned}$$

(3.8)  $[\Delta(w, x), R_z] = \Delta(zx, w) - \Delta(zw, x) .$

Now using (3.8) and the Jacobi identity we have

$$\begin{aligned} [\Delta(w, x), \Delta(u, v)] &= [\Delta(w, x), [R_u, R_v]] - [\Delta(w, x), R(uv)] \\ &= - [[R_u, \Delta(w, x)], R_v] - [R_u, [R_v, \Delta(w, x)]] \\ &\quad - [\Delta(w, x), R(uv)] \in \Delta(A, A) . \end{aligned}$$

Thus defining the *Lie transformation algebra of A*, denoted by  $L(A)$ , to be the Lie algebra generated by  $R(A) = \{R_x : x \in A\}$  [5], the above calculations prove

LEMMA 3.9.  $\Delta(A, A)$  is a Lie algebra and  $L(A) = \Delta(A, A)$ .

For clearly  $\Delta(A, A) \subset L(A) = \Sigma_i M_i$  where  $M_1 = R(A)$  and  $M_i = [M_{i-1}, M_1]$  for  $i > 1$ . But also since  $A = J(A, A, A)$  we have from (3.6) that  $M_1 \subset \Delta(A, A)$  and since  $\Delta(A, A)$  is a Lie algebra,  $M_i \subset \Delta(A, A)$ ; thus  $L(A) \subset \Delta(A, A)$ .

Next we consider the center  $C$  of  $L(A) = \Delta(A, A)$ . Since  $A$  is simple,  $\Delta(A, A)$  is an irreducible Lie algebra over a field of characteristic zero and therefore from [1, Th. 1],  $\Delta(A, A) = C \oplus \Delta'$  where  $\Delta'$  is a semi-simple Lie algebra and  $C = \{\Delta \in \Delta(A, A) : [\Delta, T] = 0 \text{ for all } T \in \Delta(A, A)\}$ .

LEMMA 3.10.  $C = 0$  and therefore  $\Delta(A, A)$  is a semi-simple Lie algebra.

*Proof.* Let  $\Delta = \Sigma_i \Delta(x_i, y_i) \in C$ , then for any  $u, v \in A$  we have

$$\begin{aligned} (uv)\Delta &= -vR_u \Delta \\ &= -v\Delta R_u, \text{ since } R_u \in \Delta(A, A) \\ &= -v\Delta \cdot u \\ &= u \cdot v\Delta . \end{aligned}$$

But next we have

$$\begin{aligned} (vu)\Delta &= \Sigma_i vu\Delta(x_i, y_i) \\ &= \Sigma_i J(vu, x_i, y_i) \\ &= \Sigma_i [J(x_i y_i, v, u) - J(x_i, y_i, u)v + J(x_i, y_i, v)u] , \\ &\hspace{15em} \text{using (3.7)} \\ &= J(z, v, u) - \Sigma_i u\Delta(x_i, y_i)v + \Sigma_i v\Delta(x_i, y_i)u, \\ &\hspace{15em} \text{where } z = \Sigma x_i y_i \end{aligned}$$

$$\begin{aligned} &= J(z, v, u) - u\Delta \cdot v + v\Delta \cdot u \\ &= J(z, v, u) + v \cdot u\Delta - u \cdot v\Delta \\ &= J(z, v, u) + (vu)\Delta - (uv)\Delta. \end{aligned}$$

Therefore  $(uv)\Delta = J(z, v, u)$  where  $z = \sum x_i y_i$ .

*Case 1.*  $z = \sum x_i y_i = 0$ . Then  $(uv)\Delta = 0$  and since  $A = A^2$  we have  $\Delta = 0$ .

*Case 2.*  $z = \sum x_i y_i \neq 0$ . Then there exists  $w \in A$  with  $a = zw \neq 0$ ; otherwise  $zF$  would be a nonzero ideal in  $A$ . Therefore

$$a\Delta = (zw)\Delta = J(z, w, z) = 0$$

and  $K = \ker \Delta \neq 0$ . But  $K$  is an ideal of  $A$ . For if  $x \in K, y \in A$ , then  $(yx)\Delta = y \cdot x\Delta = 0$  and therefore  $KA \subset K$ . Thus  $K = A$  which means  $\Delta = 0$ . From both of these cases we conclude  $C = 0$ .

Next as for Malcev algebras we have the following

DEFINITION. The set  $N = \{n \in A : J(n, A, A) = 0\}$  is called the *J-nucleus* of  $A$ .

LEMMA 3.11. *If  $a, b \in A$  are such that  $J(a, b, A) = 0$ , then  $ab \in N$ .*

*Proof.* Suppose  $J(a, b, u) = 0$  for all  $u \in A$ , then from (3.7)

$$\begin{aligned} J(ab, y, z) &= J(yz, a, b) + J(a, b, z)y - J(a, b, y)z \\ &= 0. \end{aligned}$$

COROLLARY 3.12.  *$N$  is an ideal of  $A$  which is a Lie algebra and therefore  $N = 0$ .*

COROLLARY 3.13.  *$R_x$  is a derivation of  $A$  if and only if  $x \in N = 0$ .*

Now let

$$D(x, y) = \frac{1}{4} [R_x, R_y] - \frac{1}{2} R_{xy}$$

i.e. 
$$zD(x, y) = [x, y, z] = \frac{1}{2} xy \cdot z + \frac{1}{4} (yz \cdot x + zx \cdot y).$$

Then  $D(x, y)$  is an inner derivation, that is,  $D(x, y) \in L(A) = \Delta(A, A)$ . If  $I(A)$  denotes the linear subspace spanned by all such  $D(x, y)$ 's, then we have

LEMMA 3.14. *The derivation algebra of  $A$  equals  $I(A)$  and  $I(A)$  is completely reducible in  $A$ .*

*Proof.* First we shall show that any derivation  $D$  of  $A$  in  $\mathcal{A}(A, A)$  is actually in  $I(A)$ . For let  $D = R_z + (1/4) \sum_i [R_{x_i}, R_{y_i}] \in \mathcal{A}(A, A) = L(A)$ . Then write  $D = R(z + (1/2) \sum_i x_i y_i) + \sum_i D(x_i, y_i)$ . From this equation  $R(z + (1/2) \sum_i x_i y_i)$  is a derivation and therefore by Corollary 3.13 it equals zero. Thus  $D \in I(A)$ .

Next we shall show any derivation of  $A$  is in  $I(A)$ . Since  $D$  is a derivation  $[R_x, D] = R(xD)$  and therefore using the Jacobi identity we obtain  $[\mathcal{A}(A, A), D] \subset \mathcal{A}(A, A)$ . Now the map

$$\mathcal{A}(A, A) \rightarrow \mathcal{A}(A, A): X \rightarrow [X, D]$$

is a derivation of  $\mathcal{A}(A, A)$ . But since  $\mathcal{A}(A, A)$  is a semi-simple Lie algebra all derivations are inner and therefore there exists  $T \in \mathcal{A}(A, A)$  so that

$$(3.15) \quad [X, D] = [X, T] \text{ for all } X \in \mathcal{A}(A, A).$$

Now since  $\mathcal{A}(x, y) = [R_x, R_y] - R(xy) = 4D(x, y) + R(xy)$  we have, using Corollary 3.13,

$$(3.16) \quad \mathcal{A}(A, A) = R(A) \oplus I(A)$$

as a vector space sum. Therefore let  $T = R_z + D_1$  where  $D_1 \in I(A)$  and  $z \in A$ , then for any  $x \in A$ ,

$$\begin{aligned} [R_x, R_z] &= [R_x, T - D_1] \\ &= [R_x, T] - [R_x, D_1] \\ &= [R_x, D - D_1], \text{ using (3.15)} \\ &= R(x\tilde{D}), \end{aligned}$$

where  $\tilde{D} = D - D_1$  is a derivation. Therefore

$$\begin{aligned} R(x(\tilde{D} - 2R_z)) &= R(x\tilde{D}) - 2R(xz) \\ &= [R_x, R_z] - 2R(xz) \\ &= 4D(x, z). \end{aligned}$$

Thus  $R(x(\tilde{D} - 2R_z))$  is a derivation and using Corollary 3.13 we have

$$x(\tilde{D} - 2R_z) = 0 \quad \text{for any } x \in A.$$

However this implies  $2R_z = \tilde{D}$  is a derivation and therefore  $0 = 2R_z = \tilde{D}$ . Thus from the definition of  $\tilde{D}$  we have  $D = D_1 \in I(A)$  so that every derivation of  $A$  is in  $I(A)$ .

The last part of the main theorem is proved in a manner analogous to the proof of Theorem 9 in [1]. We note from (3.16) that the completely reducible Lie algebra  $\mathcal{A}(A, A)$  is such that the subalgebra  $I(A)$  is splittable and has a complementary subspace, namely  $R(A)$ ,

which is  $I(A)$ -invariant (because  $[R_x, D] = R(xD)$ ). Thus from [1, Th. 5],  $I(A)$  is completely reducible in  $A$  and the proof of the main theorem is complete.

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