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A CHARACTERIZATION OF FREE PROJECTIVE PLANES

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A characterization of free projective planes is given that is more symmetrical than the original definition of M. Hall. It tends to very simple proofs of two fundamental theorems due to M. Hall and L. I. Kopejkina—one being the result that every subplane of a free plane is free.

In a fundamental paper Marshall Hall defines a free plane to be a projective plane which either is degenerate or is generated as follows from a 'basis configuration,' π_0 , consisting of at least two points on a line together with two isolated points. For each pair of points not already joined in π_0 create a distinct line that joins them and add it to π_0 . In the resulting configuration, π_1 , consider pairs of lines that do not intersect, and for each create a distinct intersection point and add it to π_1 , thus forming π_2 . Continuing, construct $\pi_3, \pi_4, \pi_5, \pi_6$, etc. adding alternately lines and points as indicated above. Then $\pi = \bigcup_n \pi_n$ (with the obvious incidence relation) is a projective plane. It is by definition a free plane. Hall proved that a free plane contains no confined configuration, that is, no finite configuration that, like the Desargues configuration, has ≥ 3 points on each line and ≥ 3 lines through each point. Further, using a complicated argument, he showed that, if a finitely generated plane contains no confined configuration, it is free. It follows that any finitely generated subplane of a free plane is free.

L. I. Kopejkina [2] proved, shortly after, that an arbitrary subplane of a free plane is free. (Of interest is the analogy with free groups.) An exposition of Kopejkina's theorem appears in [3].

Because it suggests a more symmetrical definition of free plane that leads to very direct proofs of the above theorems, we introduce the notion of an extension process.

DEFINITION 1. An *extension process* is a well ordered nested sequence of partial planes $\pi_0 \subset \pi_1 \subset \dots \subset \pi_n \subset \dots$ (the subscripts $0, 1, \dots, n, \dots$ belonging to a well ordered set) such that if a point p and a line l appear in a term $\pi_n, n > 0$, and in no earlier term then p is not incident with l —in other words, the *new* elements in π_n may be incident in π_n with elements appearing in earlier terms, but have no incidences among themselves. From this point we adopt as definition the characterization of free plane we aim to justify.

DEFINITION 2. A *free plane* is a (possibly degenerate) projective plane, π , for which there exists an extension process $\pi_0 \subset \pi_1 \subset \dots \subset \pi_n \subset \dots$ with $\pi_0 = \emptyset$ (the empty plane) and $\pi = \bigcup_n \pi_n$ such that every new point [or line] in a term π_n is incident in π_n with at most two lines [points].

Immediately we can prove

THEOREM I (Kopejkina). *Every subplane of a free plane is free.*

Proof. If π is a free plane and $\pi_0 \subset \pi_1 \subset \dots \subset \pi_n \subset \dots$ is an extension process as above for π , then, given any subplane π' , the sequence $\pi_0 \cap \pi' \subset \pi_1 \cap \pi' \subset \dots \subset \pi_n \cap \pi' \subset \dots$ is visibly such an extension process for π' . So π' is free. ■

Some definitions and notations are collected in § 2. In § 3 the result of Hall is proved. Our definition of a free plane is apparently broader than M. Hall's. In § 4 we prove that the definitions are equivalent.

2. A set of elements consisting of points and lines, together with a relation of incidence between points and lines is said to form a *partial plane* (or *configuration*) if every two distinct lines [points] are together incident with at most one point [respectively line], (which when it exists we call their *join*). If every two distinct lines [points] are together incident with exactly one point [respectively line] the system becomes a (projective) *plane*. A plane is said to be *non-degenerate* if it contains 4 points no 3 of which are incident with the same line, and otherwise is said to be *degenerate*.

If ρ and σ are subpartial planes of the partial plane π , then $\rho + \sigma$ (or $\rho \cup \sigma$), $\rho \cap \sigma$, and $\rho - \sigma$ are subpartial planes of π defined in the obvious way.

A configuration ρ in a plane π is said to *generate* the least subplane containing ρ . This subplane is denoted by $[\rho]_\pi$ (or $[\rho]$) and is called the *completion* of ρ in π . The plane π is *finitely generated* if, for some finite configuration $\rho \subset \pi$, $[\rho] = \pi$.

An extension process, \mathcal{E} , is regularly presented in the form $\mathcal{E} = \{\pi_n; n \in N\}$ where $N = \{0, 1, \dots, n, \dots\}$ is a well ordered set. π_{n-} will denote $\bigcup_{m < n} \pi_m$. \mathcal{E} is said to *act on* π_0 and have the *result* $\mathcal{E}(\pi_0) = \bigcup_n \pi_n$. It should be pointed out that the partial plane $\mathcal{E}(\pi_0)$ need not be a full projective plane. The \mathcal{E} -*stage* (or simply *stage*) of an element $x \in \mathcal{E}(\pi_0)$ is the least n such that $x \in \pi_n$. If $n > 0$, x is said to *appear at* \mathcal{E} -*stage* n ; it is incident in π_n with certain elements called its \mathcal{E} -*bearers* (or *bearers*), and these must all lie in $\pi_{n-} = \bigcup_{m < n} \pi_m$. Elements in π_0 (by convention) have no bearers. Observe that if $x \notin \pi_0$

is incident with y , and y is *not* a bearer of x , then x must be a bearer of y , for they cannot both appear at the same stage.

If ρ is a subpartial plane of $\pi = \mathcal{E}(\pi_0)$, there is a naturally defined *restriction* of \mathcal{E} to ρ , denoted $\mathcal{E} \cap \rho$, viz. $\{\pi_n \cap \rho; n \in N\}$. This is apparently a process that acts on $\pi_0 \cap \rho$ with result ρ . Also there is a naturally defined *saturation* of \mathcal{E} by ρ , denoted $\mathcal{E} + \rho$, viz. $\{\pi_n + \rho; n \in N\}$. It apparently acts on $\pi_0 + \rho$ with result π . We make two simple but important observations.

(1) The bearers in $\mathcal{E} \cap \rho$ of an element $x \in \rho$ are just those \mathcal{E} -bearers that lie in ρ .

(2) The bearers in $\mathcal{E} + \rho$ of an element $x \notin \pi_0 + \rho$ include its \mathcal{E} -bearers and in addition any elements of ρ incident with x in π .

If \mathcal{E} and \mathcal{F} are extension processes, \mathcal{E} acting on π_0 and \mathcal{F} acting on $\mathcal{E}(\pi_0)$, then there is a naturally defined composition of \mathcal{F} with \mathcal{E} denoted $\mathcal{F} \circ \mathcal{E}$.

We call a process, \mathcal{E} , (a) *bound*; (b) *free*; (c) *hyperfree* if for all $n > 0$ every new element of π_n has (a) ≥ 2 ; (b) 2; (c) ≤ 2 bearers. Of course an extension process need not fall into any of these categories. A *free plane* is by definition a plane which is the result of a hyperfree process acting on the empty plane, \emptyset .

A bound process \mathcal{E} , whose result, $\mathcal{E}(\pi_0)$, is a full projective plane, is called a *completion process* for π_0 . If ρ is a configuration in a plane π , there is a canonical completion process $\mathcal{E} = \{\rho_n; n \in J^+\}$, indexed on the integers ≥ 0 , that acts on ρ with result $[\rho]_\pi$. In fact $\rho_0 = \rho$ and ρ_n is defined inductively as the subpartial plane of π whose elements are those of ρ_{n-1} together with all points [lines] of π that are joins of lines [or points] of ρ_{n-1} resp. as n is even or odd.

3. Now we prove Hall's result. Suppose π is a finite partial plane having P points, L lines, and I incidences between the points and lines. The *rank* of π as defined by Hall is

$$r(\pi) = 2(P + L) - I.$$

LEMMA 1. *Any finite partial plane, π , which contains no confined configurations is the result of a hyperfree process.*

Proof. Set $\pi_m = \pi$ where m is the number of points and lines in π . Since π_m is not confined, there exists some element $x_m \in \pi_m$ which is incident with ≤ 2 elements in π_m . Define $\pi_{m-1} \subset \pi_m$ to be π_m less the element x_m . Since π_{m-1} is not confined, the process may be repeated, and after exactly m steps we obtain $\pi_0 = \emptyset$. Then $\mathcal{F} = \{\pi_i; i = 0, 1, \dots, m\}$ is hyperfree and $\mathcal{F}(\emptyset) = \pi$.

COROLLARY. *The rank of a finite partial plane containing no confined configuration is nonnegative.*

Proof. In fact $r(\pi) \geq r(\pi_m) \geq \dots \geq r(\pi_0) = 0$. ■

THEOREM II (Hall). *A free plane contains no confined configuration. A finitely generated plane that contains no confined configuration is free.*

REMARK. Kopejkina [2] constructed a plane (not finitely generated) that contains no confined configuration, but is not free.

Proof. Suppose first that π is a free plane and $\mathcal{F} = \{\pi_n; n \in N\}$ is a hyperfree process such that $\mathcal{F}(\emptyset) = \pi$. If ρ is any finite configuration in π , there is an element $x \in \rho$ having maximal \mathcal{F} -stage, m . Since x is incident with at most two elements of π_m , it obviously cannot be incident with at least three elements of $\rho \subset \pi_m$. Thus ρ cannot be a confined configuration.

The proof of the second assertion depends essentially on our definition of free plane. Suppose that the plane π is generated by a finite configuration π_0 and that π contains no confined configuration. Let $\mathcal{E} = \{\pi_n; n \in J^+\}$ be the canonical completion process for π_0 . Observe that each partial plane π_n is finite and $\mathcal{E}(\pi_0) = \pi$. Since \mathcal{E} is bound $r(\pi_0) \geq r(\pi_1) \geq \dots \geq r(\pi_n)$. But by the corollary above $r(\pi_n) \geq 0$ for all n . Hence for some integer m the minimal rank be attained, and thereafter $r(\pi_n)$ will have this minimal value. But this means that in the bound process $\mathcal{S} = \{\pi_n; n \geq m\}$ every element has exactly 2 bearers i.e., \mathcal{S} is free. Now, by Lemma 1, $\pi_m = \mathcal{F}(\pi_0)$ where \mathcal{F} is hyperfree. So composing \mathcal{S} with \mathcal{F} we obtain a hyperfree process with $\mathcal{S} \circ \mathcal{F}(\emptyset) = \pi$. ■

4. This last section is devoted to proving that the adopted definition of free plane is equivalent to Hall's.

LEMMA 2. *Suppose \mathcal{F} is hyperfree and $\pi = \mathcal{F}(\pi_0)$ is a plane. Then*

(1) $\mathcal{F}_1 = \mathcal{F} \cap [\pi_0]$ is a free completion process for π_0 in π ; in fact, for $x \in [\pi_0]$, the \mathcal{F} -bearers lie in $[\pi_0]$ and coincide with those in any completion process for π_0 in π .

(2) $\mathcal{F}_2 = \mathcal{F} + [\pi_0]$ is still hyperfree; in fact, for $x \notin [\pi_0]$, the \mathcal{F}_2 -bearers coincide with the \mathcal{F} -bearers.

Proof. (1) Let \mathcal{E} be any completion process for π_0 in π . If the first assertion is false, let x be an element of least \mathcal{E} -stage, m , for

which the \mathcal{E} -bearers do not coincide with the \mathcal{F} -bearers. Clearly $m > 0$. Then at least one of the ≥ 2 \mathcal{E} -bearers of x , say y , must fail to be an \mathcal{F} -bearer. Since $x \notin \pi_0$, y must have x as an \mathcal{F} -bearer. But y doesn't have x as an \mathcal{E} -bearer and y has \mathcal{E} -stage $< m$. This is a contradiction.

(2) Supposing the second assertion false, we have some $x \notin [\pi_0]$ with an \mathcal{F}_2 -bearer y that is not an \mathcal{F} -bearer and (hence) lies in $[\pi_0] - \pi_0$. But x is incident with $y \notin \pi_0$; so $x \notin [\pi_0]$ must be an \mathcal{F} -bearer of $y \in [\pi_0]$ in contradiction to (1). ■

REMARK. This lemma has a useful generalization in which π_0 is replaced in (1) and (2) by a partial plane $\rho \subset \pi_0$ that is 'complete' in π_0 (see [1, 4.3]). Then, in the proof of (1), the possibility that $x \in \pi_0$ must be eliminated.

DEFINITION 3. A *free completion* of a partial plane, π_0 , is a plane π which is the result of a free process acting on π_0 .

Again this seems less restrictive than Hall's definition, but

THEOREM III. *Any two free completions of a partial plane π_0 are related by a unique isomorphism that fixes π_0 .*

Proof. Let \mathcal{F} be a free completion process for π_0 , and let \mathcal{E} be the canonical completion process for π_0 in $\mathcal{F}(\pi_0)$. Clearly $\mathcal{E}(\pi_0) = [\pi_0] = \mathcal{F}(\pi_0)$; and according to assertion (1) of the above lemma, \mathcal{E} is free. The theorem now follows from the fact that, if \mathcal{E} and \mathcal{E}' are two canonical free completion processes for π_0 , there is a unique isomorphism of $\mathcal{E}(\pi_0)$ onto $\mathcal{E}'(\pi_0)$ that fixes π_0 . ■

In a free extension process $\{\pi_0, \pi_1\}$ of just two terms we say that π_1 is derived from π_0 by a *free addition*, and π_0 from π_1 by a *free subtraction*. Two partial planes are *free equivalent* if the one can be derived from the other by a finite sequence of free additions and subtractions. Free equivalent partial planes evidently have isomorphic free completions.

Recall that a basis configuration consists of a number of points on a 'base' line and two isolated points.

THEOREM IV. *Every nondegenerate free plane contains a basis configuration of which it is a free completion.*

Proof. Suppose $\pi = \mathcal{F}(\emptyset)$ is a free plane, where $\mathcal{F} = \{\pi_n; n \in \mathbb{N}\}$ is a hyperfree process. We may assume without loss of generality that at each stage one element and no more is added, i.e., $\pi_n = \pi_{n-1} + x_n$, where x_n is an element that has 0, 1, or 2 bearers.

Applying Lemma 2 to \mathcal{F} for stages $\geq n$ we find that $\mathcal{F}' = \mathcal{F} + [\pi_{n-}]$ is hyperfree; applying the same lemma to \mathcal{F}' for stages $> n$ we find that $\mathcal{F}' \cap [\pi_n]$ free completes $[\pi_{n-}] + x_n$ to $[\pi_n]$.

Now let a be the first stage such that $[\pi_a]$ is nondegenerate. Then $[\pi_{a-}]$ must be a degenerate plane, and, as we have observed, $[\pi_a]$ is a free completion of $[\pi_{a-}] + x_a$. By an inspection of the various cases one shows that $[\pi_{a-}] + x_a$ is in every case free equivalent to some basis configuration π_0^α . This implies that $[\pi_a]$ is a free completion of π_0^α .

For $n > a$, one readily shows (cf. [1] or [3]) that $[\pi_{n-}] + x_n$ is free equivalent to $[\pi_{n-}]$ with a set μ_n of points adjoined to the base line, l , of π_0^α . (The set μ_n consists of 2 or 1 points if x_n is incident with 0 or 1 elements in $[\pi_{n-}]$, and of course μ_n is empty when $x_n \in [\pi_{n-}]$.) Thus $[\pi_n]$ is a free completion of $[\pi_{n-}] + \mu_n$.

We will show that π is a free completion of the basis configuration $\pi_0^\beta = \pi_0^\alpha + \bigcup_{n>a} \mu_n$. Let $\{([\pi_{n-}] + \mu_n)^k; k \in J^+\}$ be the canonical process that free completes $[\pi_{n-}] + \mu_n$ to π_n , and form the composition of all these processes to obtain $\mathcal{S} = \{([\pi_{n-}] + \mu_n)^k; (n, k) \in N \times J^+, n \geq a\}$ where $([\pi_{a-}] + \mu_a)$ is to be read as π_0^α , and $N \times J^+$ is well ordered lexicographically. This is a hyperfree process acting on π_0^α with result π . The saturation $\mathcal{S} = \mathcal{S} + \pi_0^\beta$ is the desired free completion process.

Clearly $\bar{\mathcal{S}}(\pi_0^\beta) = \pi$; and $\bar{\mathcal{S}}$ is a bound process since all elements with < 2 \mathcal{S} -bearers lie in π_0^β . To show that $\bar{\mathcal{S}}$ is actually free observe that every new element, x , of $\bar{\mathcal{S}}$ appeared in \mathcal{S} with exactly 2 bearers. If x has an extra bearer, y , in $\bar{\mathcal{S}}$, then $y \in \mu_n$ where x has $\bar{\mathcal{S}}$ -stage $(m, k) \in N \times J^+$ and $n > m$. But y is incident with both x and base line l of π_0^α , i.e., y is the join of x and l . Then $y \in [\pi_m]$ in contradiction to $y \in \mu_n$. So $\bar{\mathcal{S}}$ must be free. ■

This completes the proof that the definition of free plane we have proposed coincides with M. Hall's definition.

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