ON THE GENERALIZED F. AND M. RIESZ THEOREM

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Let $X$ be a compact Hausdorff space, $C(X)$ the algebra of all continuous complex valued functions on $X$, and let $A$ be a sup-norm algebra on $X$, that is, $A$ is a uniformly closed algebra of continuous complex valued functions on $X$ that contains the constants and separates the points. If $\phi$ is a complex homomorphism of $A$ then let $M(\phi)$ be the set of all positive, regular, Borel measures on $X$ that represent $\phi$. If $\mu$ is a finite, (complex), regular, Borel measure on $X$ then we write $\mu \perp A$ if $\int f d\mu = 0$ for all $f \in A$. Let $\phi$ be a complex homomorphism of $A$ and $m \in M(\phi)$, then we say that $m$ satisfies the Riesz theorem if whenever $\mu$ is a finite, (complex), regular, Borel measure on $X$ and $\mu \perp A$ then $\mu_a \perp A$ and $\mu_s \perp A$ where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of $\mu$ with respect to $m$. It is quite easy to see that if $m \in M(\phi)$ and $m$ satisfies the Riesz theorem then for all $\rho \in M(\phi)$ we have $\rho$ is absolutely continuous with respect to $m$. We will show that this condition is also sufficient. This is done by means of a theorem which says that if $F \subseteq X$ is a compact $G_\delta$ such that $m(F) = 0$ for all $m \in M(\phi)$ then there exists a sequence $f_n$ in $A$ such that $|f_n| \leq 1$ on $X$, $\phi(f_n) \to 1$, and $f_n \to 0$ uniformly on $F$.

The proof given is not a generalization of the modern proof of the F. and M. Riesz theorem as given in [4], for instance, but is closer in form to the original proof of F. and M. Riesz. If $X = S_2 \cup S_2$ is the decomposition of $X$ corresponding to the decomposition $\mu = \mu_a + \mu_s$, then by means of Theorem 1 we find a bounded sequence in $A$ that converges to the characteristic function of $S_1$ almost everywhere with respect to the total variation of the measure $\mu$. It is known (see Hoffman [4] and Lumer [5]) that if $M(\phi) = \{m\}$ then the Riesz theorem holds for the measure $m$. It is known that $M(\phi)$ is not empty [4].

It what follows, all measures are assumed to be finite, regular, Borel measure, and $\phi$ is a fixed complex homomorphism of $A$.

**Lemma 1.** Let $\{\nu_n\}$ be a sequence of positive measures on $X$ having the measure $m$ as a weak-star accumulation point. Suppose $F \subseteq Y$
is compact and that \( \nu_n(F) \geq \varepsilon_0 > 0 \) for all \( n \). Then \( m(F) \geq \varepsilon_0 \).

**Proof.** There exists a decreasing sequence of open sets \( \mathcal{O}_n \supseteq F \) such that \( m(\mathcal{O}_n - F) \to 0 \). There exists a sequence \( u_n \) of continuous real valued functions such that \( u_n = 1 \) on \( F \), \( u_n = 0 \) on \( X - \mathcal{O}_n \) and \( 0 \leq u_n \leq 1 \) elsewhere. From the construction, \( u_n \to \chi_F \) a.e. (\( m \)), where \( \chi_F \) is the characteristic function of \( F \). So we have,

\[
m(F) = \int (\chi_F - u_k) dm + \int u_k dm - \int u_k dm - d\nu_n.
\]

Note that \( \int u_k dm \geq \nu_k(F) \geq \varepsilon_0 \) for all \( n \) and \( k \). Now, \( \int (\chi_F - u_k) dm \) can be made small by choosing \( k \) large, and once \( k \) is fixed \( \int u_k dm - d\nu_n \) can be made small by proper choice of \( n \). This proves the lemma.

The proof of the next lemma can be found in [1], Theorem 3.b.

**LEMMA 2.** Let \( u \in C(X) \) be real valued and suppose

\[
\sup \{ \text{Re} \phi(g) \mid \text{Re} g \leq u, g \in A \} \\
\leq \gamma \leq \inf \{ \text{Re} \phi(g) \mid \text{Re} g \geq u, g \in A \}
\]

then there exists \( \rho \in M(\phi) \) such that \( \int ud\rho = \gamma \). In particular, there exists \( \rho_n \in M(\phi) \) such that

\[
\sup \{ \text{Re} \phi(g) \mid \text{Re} g \leq u, g \in A \} = \int ud\rho_n.
\]

**THEOREM 1.** Let \( F \subseteq X \) be a compact \( G_\delta \) such that \( m(F) = 0 \) for all \( m \in M(\phi) \), then there exists a sequence \( f_n \in A \) such that

1. \( |f_n| \leq 1 \) on \( X \).
2. \( \phi(f_n) \geq e^{-2/n} \).
3. \( |f_n| \leq e^{-n} \) on \( F \).

**Proof.** Since \( F \) is a compact \( G_\delta \), there is a sequence of open sets \( \{\mathcal{O}_n\} \) such that \( \mathcal{O}_{n+1} \subseteq \mathcal{O}_n \) and \( \bigcap_n \mathcal{O}_n = F \). Let \( \varepsilon > 0 \) be given, then there exists an integer \( N \) such that for all \( n \geq N \), \( \rho(\mathcal{O}_n) < \varepsilon \) for all \( \rho \in M(\phi) \). For suppose this were not true, then there would exist \( \varepsilon_0 > 0 \) and sequences \( \rho_k \in M(\phi) \) and \( \mathcal{O}_{n_k} \) such that \( \rho_k(\mathcal{O}_{n_k}) \geq \varepsilon_0 \). Let \( U_k = \mathcal{O}_{n_k} \) then we have \( \rho_k(U_k) \geq \varepsilon_0 > 0 \) and \( \bar{U}_{k+1} \subseteq U_k \). The sequence \( \rho_k \) has a weak-star limit point \( \rho \), and it is well known that \( \rho \in M(\phi) \) hence \( \rho(F) = 0 \). Fix \( k \), then \( \rho(U_k) \geq \rho(\bar{U}_{k+1}) \), now \( \rho_n(\bar{U}_{k+1}) \geq \rho_n(U_{k+1}) \geq \rho_n(U_n) \geq \varepsilon_0 > 0 \) for all \( n \geq k + 1 \). Therefore by Lemma 1 we have \( \rho(U_k) \geq \varepsilon_0 > 0 \) for all \( k \). But this contradicts the fact that \( \rho(F) \neq 0 \). Hence by proper choice of subsequence we may assume that \( \rho(\mathcal{O}_n) < (1/n^2) \)
for all \( \rho \in M(\phi) \). Now for each \( n \) there exists \( u_n \in C(X) \) such that \( u_n = -n \) on \( F \), \( u_n = 0 \) on \( X - \mathcal{O}_n \) and \( -n \leq u_n \leq 0 \) elsewhere. By Lemma 2, there exists \( \rho_n \in M(\phi) \) such that
\[
\sup \{ \text{Re } \phi(g) \mid \text{Re } g \leq u_n, \ g \in A \} = \int u_n d\rho_n ,
\]
and hence for each \( n \) there exists \( g_n \in A \) such that \( \text{Re } g_n \leq u_n \) and
\[
\int \text{Re } g_n dm \geq \int u_n d\rho_n - \frac{1}{n} \geq -n\rho_n(\mathcal{O}_n) - \frac{1}{n} \geq -\frac{2}{n} .
\]
We may also assume that \( \int \text{Im } g_n dm = 0 \). If we now define \( f_n = e^{\phi_n} \) it follows that
\[
(1) \ |f_n| = e^{\text{Re } \phi_n} \leq e^{u_n} \leq 1 \\
(2) \ \int f_n dm = \exp \left( \int g_n dm \right) = \exp \left( \int \text{Re } g_n dm \right) \geq e^{-\frac{2}{n}} \\
(3) \ |f_n| = e^{\text{Re } \phi_n} \leq e^{u_n} \text{ on } F .
\]
The sequence \( \{f_n\} \) of Theorem 1 is bounded in norm by 1, yet \( \phi(f_n) \to 1 \). We show that this implies that \( \psi(f_n) \to 1 \) for all \( \psi \) in the same part as \( \phi \). For definition of part see [4]. For this we use a result of Bishop [2]: if \( \phi, \psi \) are in the same part and \( m_\phi \) is a representing measure for \( \phi \), then there exists a representing measure \( m_\psi \) for \( \psi \) such that \( m_\phi \leq A m_\psi \) for some constant \( A \).

**Corollary 1.** If \( \{f_n\} \) is the sequence of Theorem 1 and \( \psi \) is in the same part as \( \phi \), then \( \psi(f_n) \to 1 \).

**Proof.** Let \( m \) be a representing measure for \( \psi \), and \( \rho \) be a representing measure for \( \phi \) such that \( m \leq A \rho \) for some constant \( A \). Then we have \( m = g \rho \) where \( g \) is bounded. Since \( \psi(f_n) \to 1 \) we have \( \int f_n dm \to 1 \). This, together with the fact that \( |f_n| \leq 1 \) implies that \( f_n \to 1 \) in measure, with respect to the measure \( \rho \). Since \( g \) is bounded it follows that \( f_n g \to g \) in measure with respect to the measure \( \rho \). The fact that \( |f_n g| \leq |g| \) now implies that \( \psi(f_n) = \int f_n g dm \to \int g dm = \int dm = 1 \).

**Corollary 2.** Suppose there is a measure \( m \in M(\phi) \) such that \( \rho \leq m \) for all \( \rho \in M(\phi) \), and suppose \( F \subseteq X \) is compact and \( m(F) = 0 \). Then there exists a sequence \( f_n \in A \) satisfying (1), (2), (3) of Theorem 1.

**Proof.** There exists a sequence \( \{\mathcal{O}_n\} \) of open sets such that \( F \subseteq \mathcal{O}_{n+1} \subseteq \mathcal{O}_n \) and \( m(\mathcal{O}_n) \to 0 \). For each \( n \), there exists a set \( F_n \) which is a compact \( \mathcal{G}_s \) such that \( F \subseteq F_n \subseteq \mathcal{O}_n \). Let \( F_1 = \bigcap_n F_n \), then \( F \subseteq F_1 \),
$F_1$ is a compact $G_δ$ and $m(F_1) = 0$. It follows that $\rho(F_1) = 0$ for all $\rho \in M(\phi)$. Now apply Theorem 1 to the set $F_1$.

**Theorem 2.** Suppose there exists $m \in M(\phi)$ such that $\rho \ll m$ for all $\rho \in M(\phi)$. Let $\mu \perp A$ and let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of $\mu$ with respect to $m$. Then $\mu_a \perp A$ and $\mu_s \perp A$.

**Proof.** Let $S$ be a Borel set that carries $\mu_s$ and $m(S) = 0$. Then there exists an increasing sequence $F_n \subseteq S$ of compact sets such that $|\mu_s| (F_n) \to |\mu_s| (S)$, where $|\mu_s|$ denotes the total variation of $\mu_s$. For each $F_n$ we have a sequence $f_{n,k} \in A$ such that

1. $|f_{n,k}| \leq 1$.
2. $\int f_{n,k} dm \geq e^{-\frac{1}{n}}$.
3. $|f_{n,k}| \leq e^{-k}$ on $F_n$.

Define $h_n = f_{n,k}$ then we have:

1'. $|h_n| = |f_{n,k}| \leq 1$.
2'. $\int h_n dm = \int f_{n,k} dm \geq e^{-\frac{1}{n}}$.
3'. $|h_n| = |f_{n,k}| \leq e^{-n}$ on $F_n$.

From 1' and 2' it follows that $h_n \to 1$ in measure with respect to $m$ and hence we have a subsequence $h_{n_k} \to 1$ a.e. ($m$). From 3' we have $h_{n_k} \to 0$ a.e. ($|\mu_s|$). Hence $g_k = h_{n_k} \to f_{X-s}$ a.e. ($|\mu|$). So if $f \in A$ then for each $k, g_k f \in A$ and we have $0 = \int g_k f d\mu \to \int f_{X-s} d\mu = \int f d\mu_s$. This proves the theorem.

We point out that if the homomorphism $\phi$ has a representing measure $m$ such that $\rho \in M(\phi)$ implies $\rho \ll m$ then it follows easily from the result of Bishop mentioned earlier that every $\psi$ that lies in the same part as $\phi$ has a representing measure with this same property.

**References**

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Patrick Robert Ahern, *On the generalized F. and M. Riesz theorem* .......... 373
A. A. Albert, *On exceptional Jordan division algebras* ......................... 377
J. A. Anderson and G. H. Fullerton, *On a class of Cauchy exponential series* ................................................. 405
Allan Clark, *Hopf algebras over Dedekind domains and torsion in H-spaces* ............................................... 419
John Dauns and D. V. Widder, *Convolution transforms whose inversion functions have complex roots* ...................... 427
Ronald George Douglas, *Contractive projections on an $L_1$ space* ........ 443
Robert E. Edwards, *Changing signs of Fourier coefficients* .................. 463
Ramesh Anand Gangolli, *Sample functions of certain differential processes on symmetric spaces* .................................... 477
Robert William Gilmer, Jr., *Some containment relations between classes of ideals of a commutative ring* ...................... 497
Basil Gordon, *A generalization of the coset decomposition of a finite group* .......................................................... 503
Teruo Ikebe, *On the phase-shift formula for the scattering operator* ....... 511
Makoto Ishida, *On algebraic homogeneous spaces* ............................... 525
Donald William Kahn, *Maps which induce the zero map on homotopy* ...... 537
Frank James Kosier, *Certain algebras of degree one* ........................... 541
Betty Kvarda, *An inequality for the number of elements in a sum of two sets of lattice points* ........................................ 545
Jonah Mann and Donald J. Newman, *The generalized Gibbs phenomenon for regular Hausdorff means* ................................. 551
Charles Alan McCarthy, *The nilpotent part of a spectral operator. II* ...... 557
Donald Steven Passman, *Isomorphic groups and group rings* ................. 561
R. N. Pederson, *Laplace's method for two parameters* ......................... 585
Tom Stephen Pitcher, *A more general property than domination for sets of probability measures* ................................. 597
Arthur Argyle Sagle, *Remarks on simple extended Lie algebras* ............. 613
Arthur Argyle Sagle, *On simple extended Lie algebras over fields of characteristic zero* ............................................. 621
Tôru Saitô, *Proper ordered inverse semigroups* ................................ 649
Oved Shisha, *Monotone approximation* ........................................... 667
Indranand Sinha, *Reduction of sets of matrices to a triangular form* ...... 673
Raymond Earl Smithson, *Some general properties of multi-valued functions* .............................................................. 681
John Stuelpnagel, *Euclidean fiberings of solvmanifolds* ....................... 705
Richard Steven Varga, *Minimal Gerschgorin sets* ............................... 719
James Juei-Chin Yeh, *Convolution in Fourier-Wiener transform* ............. 731