ON A CLASS OF CAUCHY EXPONENTIAL SERIES

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This paper was received before the synoptic introduction became a requirement.

1. Introduction. Let $Q(z)$ be a meromorphic function with poles $z_1, z_2, z_3, \ldots$, the notation being so chosen that $|z_1| \leq |z_2| \leq |z_3| \leq \cdots$. If $f \in L(0, 1)$, define

$$c_v e^{\tau x} = \text{res}_Q Q(z) \int_0^1 f(t) e^{\tau (x-t)} dt.$$ 

Then, the series $\sum c_v e^{\tau x}$ is called the Cauchy Exponential Series (CES) of $f$ with respect to $Q(z)$. If $z_v$ is of multiplicity $m$, then $c_v$ is a polynomial in $x$ of degree at most $m - 1$; if the poles are all simple, with residue $\lambda_v$ at $z_v$, we may write

$$(1) \quad c_v = \lambda_v \int_0^1 f(t) e^{-\tau y} dt$$

and $\{c_v\}$, independent of $x$, are called the CE constants.

Let $C_p: |z| = r_p$ be an expanding sequence of contours, none of which passes through a pole of $Q(z)$. Suppose $C_p$ contains $n_p$ poles of $Q(z)$. Then,

$$\sum_{v=1}^{n_p} c_v e^{\tau x} = \frac{1}{2\pi i} \int_{C_p} Q(z) dz \int_0^1 f(t) e^{\tau (x-t)} dt,$$

$$= I_p, \text{ say}.$$ 

Denote by $C^+_p$, $C^-_p$ the parts of $C_p$ lying in the right, left half-planes respectively. If $Q(z)$ is approximately unity on $C^+_p$, and is small on $C^-_p$, in the sense that

$$(2) \quad \int_{C^+_p} (Q(z) - 1) dz \int_0^1 f(t) e^{\tau (x-t)} dt = o(1)$$

$$(3) \quad \int_{C^-_p} Q(z) dz \int_0^1 f(t) e^{\tau (x-t)} dt = o(1)$$

as $p \to \infty$, uniformly for $x \in [0, 1]$, then

$$I_p = \frac{1}{2\pi i} \int_{C_p} dz \int_0^1 f(t) e^{\tau (x-t)} dt + o(1)$$

$$= \frac{1}{\pi} \int_0^1 \frac{f(t) \sin r_p(x-t)}{x-t} dt + o(1)$$

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uniformly in \([0,1]\), and so the sums \(I_p\) behave somewhat like the partial sums of a Fourier series (F.s.). Indeed, when

\[ Q(z) = \frac{e^z}{e^z - 1} \]

the CES is the F.s. of \(f\).

In this paper, we shall suppose that

\[ Q(z) = \frac{e^z a(z)}{e^z a(z) + b(z)} = \frac{e^z a(z)}{G(z)} \tag{4} \]

where \(a(z), b(z)\) are relatively prime polynomials of degree \(n\), and that all the poles are simple. This case was investigated first by Fullerton ([1], 1–34), using a less convenient notation.

The large zeros of \(G(z)\) approximate to those of \(e^z - c\), where

\[ c = -\lim_{|z| \to \infty} \frac{b(z)}{a(z)} \tag{5} \]

i.e. to the points \(\{\zeta + 2\pi pi\}\), \(\zeta\) being the principal value of \(\log c\). Hence there is a \(\delta, 0 < \delta < 2\pi\), such that if \(r_p = 2p\pi + \delta\), each point of \(C_p\) is at a distance greater than a positive constant from the zeros of \(G(z)\) and of \(e^z - c\). This enables us to prove

**Theorem 1.** Let \(f \in L(0,1)\). Then, as \(p \to \infty\),

\[ \sum_{\nu=1}^{n_p} c_{\nu} e^{\psi_{\nu} z} - e^{z} s_p(x) \to 0 \]

uniformly for \(x \in [0,1]\), where \(s_p(x)\) is the \(p\)th partial sum of the F.s. of \(f(t)e^{-t}\).

We next show that there are \(n\) relations connecting the CE constants.

**Theorem 2.** Let \(f \in L(0,1)\). If \(c_{\nu}\) is defined by (1), for \(\nu = 1, 2, \cdots\), then

\[ \sum_{\nu=1}^{n} \frac{c_{\nu} z_{\nu}^{r}}{\lambda_{\nu} F'(z_{\nu})} = 0 \tag{6} \]

\((r = 0, 1, \cdots, n - 1)\), where \(F(z) = e^{-z} G(z)\).

This naturally leads to the following question: if a sequence of numbers \(\{\beta_{\nu}\}\) satisfies \(\sum_{\nu=1}^{\infty} c_{\nu} \beta_{\nu} = 0\), what is the nature of the \(\beta_{\nu}\) ? The answer is given by

**Theorem 3.** Let \(\{\beta_{\nu}\}\) be a sequence of numbers such that \(\sum_{\nu=1}^{\infty} c_{\nu} \beta_{\nu} = 0\) for every CES \(\Sigma c_{\nu} e^{\psi_{\nu} z}\). Then, there are constants
\[ \alpha_0, \ldots, \alpha_{n-1} \text{ such that} \]
\[ \beta_n = \sum_{r=0}^{n-1} \frac{\alpha_r z_r}{\lambda_r F'(z_r)}. \]

Because of the relations (6), we cannot expect that, given a sequence \( \{c_\nu\} \) with \( \sum_{\nu=n}^{\infty} |c_\nu|^2 < \infty \), there is a function \( f \in L^2(0, 1) \) such that (1) is true for each \( \nu \). However, we can prove

**Theorem 4.** If \( \{c_\nu\}, \nu > n, \) is a sequence with \( \sum_{\nu=n}^{\infty} |c_\nu|^2 < \infty \), there is a function \( f \in L^2(0, 1) \) such that (1) is true for each \( \nu > n \), and upon defining \( c_1, \ldots, c_n \) by (1), such that \( \sum_{\nu=1}^{\infty} c_\nu e^{\nu x} \) converges in mean to \( f \).

Alternatively, we can alter every \( c_\nu \), and so obtain a Riesz-Fischer analogue. We have

**Theorem 5.** Let \( \{c_\nu\} \) be a sequence with \( \sum_{\nu=n}^{\infty} |c_\nu|^2 < \infty \). Then, there are constants \( \gamma_0, \ldots, \gamma_{n-1} \) such that
\[
\begin{align*}
d_\nu &= c_\nu + \sum_{r=0}^{n-1} \frac{\gamma_r z_r^2}{G'(z_r)},
\end{align*}
\]
the numbers \( d_\nu \) are the CE constants of a function \( f \in L^2(0, 1) \).

We next investigate the problem of the uniqueness of CES. We prove

**Theorem 6.** If \( \sum_{\nu=1}^{\infty} d_\nu e^{\nu x} = f(x) \) almost everywhere in \([0, 1]\), then there are constants \( \sigma_0, \ldots, \sigma_{n-1} \) such that
\[
(7) \quad d_\nu = \lambda_\nu \int_0^1 f(t)e^{-\nu t}dt + \sum_{r=0}^{n-1} \frac{\sigma_r z_r^2}{G'(z_r)}
\]

Finally, the question arises whether it is possible to generalise the function \( Q(z) \) given by (4), so that the CES of \( f \) is uniformly equi-convergent with a F.s. The functions
\[
P(z) = \frac{e^z \alpha(z) + \beta(z)}{e^z a(z) + b(z)}
\]
where \( \alpha(z), \beta(z) \) are polynomials of degree \( n \), are obvious generalisations. As \( \text{Re} z \to \infty \), \( P(z) \) tends to a number \( \omega_1 \neq 0 \), as \( \text{Re} z \to -\infty \), to \( \omega_2 \neq 0 \). Suppose \( \omega_1 \neq \omega_2 \), and define
\[
Q_1(z) = \frac{1}{\omega_1 - \omega_2} \{P(z) - \omega_2\};
\]
then \( Q(z) \) satisfies (2), (3). If the CES of \( f \) with respect to \( Q(z) \) is uniformly equiconvergent in \([0, 1]\) with \( e^{cz} \) multiplied by the F.s. of \( f(t)e^{-zt} \), for each \( f \in L(0, 1) \), then

\[
\alpha(z) = \omega_1 a(z) \quad \text{and} \quad \beta(z) = \omega_2 b(z),
\]

so that \( P(z) = (\alpha - \beta) Q(z) + \omega_3 \). We omit the proof.

2. Proof of Theorem 1. In (4), write

\[
Q(z) = \frac{e^t}{e^t - c} + R(z);
\]

then

\[
R(z) = \frac{-e^t [ca(z) + b(z)]}{(e^t - c) G(z)}.
\]

By the choice of \( C \), there is a positive constant \( A \) such that, on \( C \),

\[
|e^t - c| > A \max(|e^t|, 1)
\]

\[
|G(z)| > A \max(|e^t|, |z^n|).
\]

Further, by (5),

\[
ca(z) + b(z) = O(|z^n|)
\]

as \( |z| \to \infty \). Hence,

\[
\int_{\partial \nu} R(z) dz \int_0^1 f(t)e^{t(z-t)} dt = O\left( \int_{\partial \nu} \left| \frac{e^{t(z-1)}}{z} \right| \int_0^1 f(t)e^{-zt} dt \right)
\]

\[
= o\left( \int_{\partial \nu} \left| \frac{e^{t(z-1)}}{z} \right| \right)
\]

\[
= o(1)
\]

as \( p \to \infty \), uniformly for \( e^{x} \leq 1 \). Similarly,

\[
\int_{\partial \nu} R(z) dz \int_0^1 f(t)e^{t(z-t)} dt = O\left( \int_{\partial \nu} \left| \frac{e^{tz}}{z} \right| \int_0^1 f(t)e^{-zt} dt \right)
\]

\[
= o\left( \int_{\partial \nu} \left| \frac{e^{tz}}{z} \right| \right)
\]

\[
= o(1)
\]

as \( p \to \infty \), uniformly for \( e^{x} \geq 0 \).

Since, for large \( p \), the number of zeros of \( e^t - c \) inside \( C \) differs from \( 2p + 1 \) by at most 1 and

\[
\int_0^1 f(t)e^{(\xi-2p\pi i)(z-t)} dt = o(1),
\]
it follows that
\[
\sum_{\nu=1}^{n_p} c_{\nu} e^{\nu x} = \frac{1}{2\pi i} \int_{c_p} \frac{e^{z}}{e^{z} - c} \int_{0}^{1} f(t)e^{izt} dt + o(1)
\]
\[
= \sum_{\nu=p}^{n} \int_{0}^{1} f(t)e^{\nu(2\pi + i \nu)}|z-t| dt + o(1)
\]
\[
= e^{\nu} s_p(x) + o(1)
\]
as \(p \to \infty\), uniformly in \([0, 1]\), and this completes the proof.

3. The proof of Theorem 2 will depend upon

**Lemma 1.** For \(r = 0, 1, \ldots, n - 1\),
\[
\int_{\sigma^+} \frac{z^r e^{-zt}}{F(z)} dz = o(1)
\]
as \(p \to \infty\), boundedly for \(0 < t < 1\).

*Proof.* Define \(C^+, C^-\) as in § 1; then, for \(r = 0, 1, \ldots, n - 1\),
\[
\int_{\sigma^+} \frac{z^r e^{-zt}}{F(z)} dz = O\left(\int_{\sigma^+} |z^r e^{-zt} dz|\right)
\]
\[
= O\left(\int_{-\pi/2}^{\pi/2} \exp(-t\rho \cos \theta) d\theta\right) \quad (\rho = \sigma_p)
\]
\[
= O\left(\int_{0}^{\pi/2} \exp(-t\rho \sin \theta) d\theta\right)
\]
\[
= O\left(\int_{0}^{\pi/2} \exp\left(-\frac{2t\rho \theta}{\pi}\right) d\theta\right)
\]
which is \(o(1)\) as \(p \to \infty\), boundedly for \(t > 0\). Similarly,
\[
\int_{\sigma^-} \frac{z^r e^{-zt}}{F(z)} dz = o(1)
\]
boundedly for \(t < 1\). Hence the result.

4. Proof of Theorem 2. Since the zeros of \(F(z)\) are simple,
\[
\text{res}_{z_\nu} \frac{z^r e^{-zt}}{F(z)} = \frac{z^r e^{-z \nu \iota}}{F'(z_\nu)}
\]
hence, by Lemma 1, for \(r = 0, 1, \ldots, n - 1\),
\[
\sum_{\nu=1}^{n_p} \frac{z^r e^{-z \nu \iota}}{F'(z_\nu)} = o(1)
\]
as \(p \to \infty\), boundedly for \(0 < t < 1\). By the choice of \(C_p\), \(n_{p+1} - n_p = 2\)
for large $p$, and so, since the terms are $o(1)$ as $\nu \to \infty$, we may replace $n_p$ by $p$ in the above summation. If we multiply by $f(t)$ and integrate over $[0, 1]$, we have (6).

5. We now prove

**Lemma 2.** Let $a(z) = \sum_{k=0}^{n} a_k z^k$, and $b(z) = \sum_{k=0}^{n} b_k z^k$. Then,

\[
\sum_{r=0}^{n-1} \sum_{k=r+1}^{n} (b_k + a_k e^{r\mu}) z_{\nu}^{k-r-1} + a(z_{\mu}) e^{r\mu} \int_{0}^{1} e^{(\mu \nu - z_{\mu})t} dt = \begin{cases} 0 & \nu \neq \mu \\ G'(z_{\mu}) & \nu = \mu \end{cases}
\]

**Proof.** Write the left-hand side of (8) as

\[
\mathscr{L} + \mathcal{M};
\]

then,

\[
\mathscr{L} = \sum_{k=1}^{n} (b_k + a_k e^{r\nu}) \sum_{r=0}^{k-1} z_{\mu}^{k-r-1}.
\]

If $\nu \neq \mu$,

\[
\mathcal{M} = a(z_{\mu}) e^{r\nu} - e^{r\mu}.
\]

since $G(z_{\nu}) = G(z_{\mu}) = 0$, (9) is zero. If $\nu = \mu$, (9) is

\[
\sum_{k=1}^{n} k(b_k + a_k e^{r\nu}) z_{\mu}^{k-1} + a(z_{\mu}) e^{r\mu} = b'(z_{\mu}) + e^{r\mu}(a'(z_{\mu}) + a(z_{\mu})) = G'(z_{\mu}).
\]

This proves the lemma.

6. **Proof of Theorem 3.** We have $\sum_{\nu=1}^{\nu=p} c_{\nu} \beta_{\nu} = 0$ for every sequence $\{c_{\nu}\}$ of CE constants, i.e.

\[
\sum_{\nu=1}^{\nu=p} \beta_{\nu} c_{\nu} \int_{0}^{1} f(t) e^{-\nu t} dt = 0
\]

for every $f \in L(0, 1)$. Hence, by a well-known theorem ([2], § 279),

\[
\int_{0}^{1} \sum_{\nu=1}^{\nu=p} \beta_{\nu} c_{\nu} e^{-\nu t} dt \to 0
\]

as $p \to \infty$, boundedly for $x \in [0, 1]$. We recall (8); if we multiply by $\beta_{\nu} e^{-\nu t}$ and sum from $\nu = 1$ to $\nu = p$, where $p$ is greater than an
assigned integer \( \mu \), we obtain
\[
\beta_\mu \lambda_\mu e^{-\mu t} G'(z_\mu) = \sum_{r=0}^{n-1} z_\mu^r \sum_{v=1}^{p} \beta_v \lambda_v \sum_{k=r+1}^{n} (b_k e^{-z_v} + a_k) z_v^{k-r-1}
\]
\[
+ a(z_\mu) e^{\mu} \int_0^1 e^{-\mu t} \sum_{v=1}^{p} \beta_v \lambda_v e^{\rho v(t-1)} dt
\]
\[
= \sum_{r=0}^{n-1} L_{r,p} z_\mu^r + \mathcal{N}_p, \quad \text{say.}
\]

Let
\[
\phi_\mu(t) = \sum_{v=1}^{p} \beta_v \lambda_v e^{\rho v(t-1)} ,
\]
\[
\Phi_\mu(x) = \int_0^x \phi_\mu(t) dt = \sum_{v=1}^{p} \beta_v \lambda_v e^{-\mu v} dt .
\]

By (10), \( \Phi_\mu(x) \to 0 \) as \( p \to \infty \), boundedly for \( x \in [0, 1] \). Thus,
\[
\mathcal{N}_p = \sum_{v=1}^{p} \beta_v \lambda_v e^{\rho v} \int_0^1 e^{-\mu v} \phi_\mu(t) dt
\]
\[
= a(z_\mu) e^{\mu} \left\{ \Phi_\mu(1) e^{-\mu} + z_\mu \int_0^1 e^{-\mu v} \phi_\mu(t) dt \right\}
\]
\[
= o(1) \quad \text{as} \quad p \to \infty .
\]

Hence, since \( e^{-\mu} G(z) = F(z) \),
\[ (11) \sum_{r=0}^{n-1} L_{r,p} z_\mu^r = \beta_\mu \lambda_\mu F'(z_\mu) + \varepsilon_\mu \]
where the numbers \( \{L_{r,p}\} \) are independent of \( \mu \), and \( \varepsilon_\mu \to 0 \) as \( p \to \infty \).

Giving \( \mu \) distinct values \( \mu_1, \ldots, \mu_n \), (11) yields a regular system of \( n \) linear equations for \( L_{0,p}, \ldots, L_{n-1,p} \). The solution is
\[
L_{r,p} = \frac{\sum_{i=1}^{n} \{ \beta_{\mu_i} \lambda_{\mu_i} F'(z_{\mu_i}) + \varepsilon_{\mu_i} \} \Delta_i^{(r)}}{\det (z_{\mu_i}^{r-1})}
\]
where \( \Delta_i^{(r)} \) are cofactors of elements in the \((r+1)\)th column of the matrix \( (z_{\mu_i}^{r-1}) \), \( (i, j = 1, 2, \ldots, n) \). The only nonconstant terms in this expression for \( L_{r,p} \) are \( \varepsilon_{\mu_i} \), which are \( o(1) \) as \( p \to \infty \). Hence, for \( r = 0, 1, \ldots, n - 1 \), \( \{L_{r,p}\} \) converges, to \( \alpha_r \), say. Letting \( p \to \infty \) in (11), we have the result.

7. To prove Theorem 4, we require three lemmas.

**Lemma 3.** If \( p > n \), there are numbers \( d_1, \ldots, d_n \) such that
\[
e^{\mu d_\mu} + \sum_{k=1}^{n} d_k e^{\rho k \tau}
\]
is its own CES.
Proof. We shall show that there are numbers $d_1, \cdots, d_n$ such that, if

$$S(x) = e^{xv} + \sum_{k=1}^{n} d_k e^{xv},$$

then, for $\mu \notin \{1, \cdots, n, p\}$,

$$\left(12\right) \int_0^1 S(x)e^{-z^\mu}dx = 0.$$

Since the functions $e^{xv}, \cdots, e^{xv}$ are linearly independent, and by Theorem 1, the CES of $S(x)$ converges everywhere in $(0, 1)$ to $S(x)$, it will then follow that $S(x)$ is its own CES.

For $\mu \neq k$,

$$\int_0^1 e^{(x_k-z_k)\mu}dx = \frac{e^{-z_k\mu}}{z_k-z_\mu} \left\{ e^{z_k} - e^{z_\mu} \right\}$$

$$= \frac{e^{-z_k\mu}\{a(z_k)b(z_\mu) - a(z_\mu)b(z_k)\}}{a(z_k)a(z_\mu)(z_k-z_\mu)}$$

$$= \frac{e^{-z_k\mu}a(z_k, z_\mu)}{a(z_k)a(z_\mu)}, \quad \text{say.}$$

Thus, if $\mu \notin \{1, \cdots, n, p\}$, and $d_1, \cdots, d_n$ are any $n$ numbers, the left-hand side of (12) is

$$\int_0^1 e^{(x_k-z_k)\mu}dx = \frac{e^{-z_k\mu}}{a(z_\mu)} \left\{ \sigma(z_\mu, z_\mu) + \sum_{k=1}^{n} d_k \sigma(z_k, z_\mu) \right\}$$

$$= \frac{e^{-z_k\mu}}{a(z_\mu)a(z_\mu)} \left\{ \sigma(z_\mu, z_\mu) + \sum_{k=1}^{n} \delta_k \sigma(z_k, z_\mu) \right\}$$

$$= I_\mu \quad \text{say, where } \delta_k = \frac{a(z_\mu)d_k}{a(z_k)}.$$ 

The symmetric polynomial

$$\sigma(x, y) = \frac{a(x)b(y) - a(y)b(x)}{x-y}$$

can be expressed in the form

$$\sum_{r=0}^{n-1} P_r(x)y^r$$

where $P_r(x)$ is a polynomial in $x$ of degree at most $n - 1$. Then,

$$I_\mu = \frac{e^{-z_k\mu}}{a(z_\mu)a(z_\mu)} \sum_{r=0}^{n-1} z_\mu^r \left\{ P_r(z_\mu) + \sum_{k=1}^{n} \delta_k P_r(z_k) \right\}.$$ 

This is zero for each $\mu \notin \{1, \cdots, n, p\}$ if
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\[ P_r(z_p) + \sum_{k=1}^{n} \delta_k P_r(z_k) = 0 \quad (r = 0, 1, \cdots, n - 1), \]

which happens if

\[ z_r^* + \sum_{k=1}^{n} \delta_k z_k^* = 0 \quad (r = 0, 1, \cdots, n - 1). \]

Since this system of \( n \) linear equations for the unknowns \( \delta_1, \cdots, \delta_n \) is regular, the lemma follows.

**COROLLARY.** Given the constants \( c_{n+1}, \cdots, c_p \) of Theorem 4, there are numbers \( c_1^{(p)}, \cdots, c_n^{(p)} \) such that

\[ T_p(x) = \sum_{k=1}^{n} c_k^{(p)} e^{z_k x} + \sum_{y=n+1}^{p} c_y e^{z_y x} \]

is its own CES.

**Lemma 4.** The numbers \( c_1^{(p)}, \cdots, c_n^{(p)} \) are unique and, for \( k = 1, 2, \cdots, n \), the sequence \( \{c_k^{(p)}\} \) converges.

**Proof.** By Theorem 2, the numbers \( c_1^{(p)}, \cdots, c_n^{(p)} \) satisfy the regular system of linear equations

\[ \frac{c_1^{(p)} z_1^*}{\lambda_1 F'(z_1)} + \cdots + \frac{c_n^{(p)} z_n^*}{\lambda_n F'(z_n)} = - \sum_{y=n+1}^{p} \frac{c_y z_y^*}{\lambda_y F'(z_y)} \]

\((r=0, 1, \cdots, n-1)\), and so are determined uniquely. Since \( \sum_{y>n} |c_y|^2 < \infty \), and

\[ |\lambda_r F'(z_r)| > K |z_r^*| \]

where \( K \) is a constant,

\[ \sum_{y=n+1}^{p} \frac{c_y z_y^*}{\lambda_y F'(z_y)} \]

converges, for \( r = 0, 1, \cdots, n - 1 \). Hence, by an argument used in the proof of Theorem 3, \( \{c_k^{(p)}\} \) converges, for \( k = 1, 2, \cdots, n \).

**Lemma 5.** There is a positive constant \( A \) such that if \( \{a_v\} \) is any finite set of numbers, then

\[ \int_0^1 |\Sigma a_v e^{z_v x}|^2 dx \leq A\Sigma |a_v|^2. \]

This may be proved by an argument similar to that of Lemma 3 of [3].
8. Proof of Theorem 4. Let \( p, q \) be integers such that \( q > p > n \). Then,

\[
T_q(x) - T_p(x) = \sum_{k=1}^{n} (c_k^{(q)} - c_k^{(p)})e^{y_k^q} + \sum_{y=p+1}^{q} c_y e^{y^q}.
\]

By Lemma 5, there is a constant \( A > 0 \) such that

\[
\int_0^1 |T_q(x) - T_p(x)|^2 \, dx \leq A\left\{ \sum_{k=1}^{n} |c_k^{(q)} - c_k^{(p)}|^2 + \sum_{y=p+1}^{q} |c_y|^2 \right\}.
\]

Hence, by Lemma 4, \( \{T_p(x)\} \) converges in mean to a function \( f \in L^2(0,1) \).

Let \( \nu > n \). Since \( T_p(x) \) is its own CES,

\[
c_\nu = \lambda_\nu \int_0^1 T_p(x)e^{-y^\nu} \, dx \quad (p \geq \nu).
\]

Hence,

\[
c_\nu = \lambda_\nu \lim_{p \to \infty} \int_0^1 T_p(x)e^{-y^\nu} \, dx
\]

\[
= \lambda_\nu \int_0^1 f(x)e^{-y^\nu} \, dx.
\]

Define \( c_\nu, \ldots, c_n \) by this formula; then,

\[
c_k = \lim_{p \to \infty} c_k^{(p)} \quad (k = 1, 2, \ldots, n),
\]

and \( \sum_{\nu=1}^{n} c_\nu e^{-y^\nu} \) converges in mean to \( f \). This completes the proof.

9. Proof of Theorem 5. If we multiply (8) by \( c_\nu \) and sum from \( \nu = 1 \) to \( \nu = p \), where \( p \) is greater than an assigned integer \( \mu \), we obtain

\[
c_\mu G_\mu(z_\mu) = \sum_{r=0}^{n-1} z_\mu^r \sum_{\nu=1}^{p} c_\nu \sum_{k=1}^{n} (a_k e^{y^\nu} + b_k) z_\mu^{r-1}
\]

\[
+ a(z_\mu)e^{y^\mu} \int_0^1 e^{-y^\nu} \sum_{\nu=1}^{p} c_\nu e^{y^\nu} \, dt
\]

\[
= \mathcal{L}_p + \mathcal{M}_p, \quad \text{say.}
\]

Since \( \sum_{\nu=1}^{p} |c_\nu|^2 < \infty \), \( \sum_{\nu=1}^{\infty} c_\nu e^{y^\nu} \) converges in mean to a function \( f \in L^2(0,1) \). Hence,

\[
\mathcal{M}_p \to d_\mu G_\mu(z_\mu) \quad \text{as} \quad p \to \infty
\]

where

\[
d_\mu = \lambda_\mu \int_0^1 f(t)e^{-y^\mu} \, dt.
\]

Next,
(14) \[ \mathcal{P} = \sum_{r=0}^{n-1} \delta_r z_{\mu}^r - \sum_{r=0}^{n-1} z_{\mu}^r \sum_{\gamma=1}^{p} c_{\gamma} \sum_{k=0}^{r} (a_k e^{r} + b_k) z_{\nu}^{k-r-1} \]

where

\[ \delta_r = c_1 \sum_{k=r+1}^{n} (a_k e^{r} + b_k) z_{\nu}^{k-r-1}. \]

Since

\[ \sum_{r=0}^{r} (a_k e^{r} + b_k) z_{\nu}^{k-r-1} = O(\nu^{-1}) \]

the summation over \( \nu \) in (14) converges, as \( p \to \infty \), to \( \eta_r \) say. The result now follows upon writing

\[ \eta_r + \delta_r = \gamma_r. \]

10. Before establishing the uniqueness theorem, we prove two lemmas.

**Lemma 6.** If \( \sum_{\nu=1}^{\infty} d_{\nu} e^{r} = f(x) \) almost everywhere in \([0,1] \), and \( d_{\nu} = O(\nu^{-\gamma}) \), there are constants \( \sigma_0, \ldots, \sigma_{n-1} \) such that (7) is satisfied for \( \nu = 1, 2, \ldots \).

**Proof.** We have (13), with \( c_{\nu} \) replaced by \( d_{\nu} \). We may write this as

\[ d_{\nu} G(z_{\mu}) = \sum_{r=0}^{n-1} M_{r,\nu} z_{\mu}^r + \lambda_{\nu} G(z_{\mu}) \left( \int_0^1 e^{-r} G(t) - \sum_{r=p+1}^{\infty} d_{\nu} e^{r} \right) dt. \]

Since

\[ \int_0^1 e^{-r} \sum_{r=p+1}^{\infty} d_{\nu} e^{r} \, dt = O\left( \sum_{r=p+1}^{\infty} |d_{\nu}| \right) \]

\[ = o(1) \quad \text{as } p \to \infty, \]

and \( \{M_{r,\nu}\} \) converges, to \( \sigma_r \) say, for \( r = 0, 1, \ldots, n - 1 \), we obtain (7).

**Lemma 7.** If the series \( \sum_{\nu=2}^{\infty} b_{\nu} \) is convergent, then

\[ \sum_{\nu=2}^{\infty} b_{\nu} \left( \frac{\sinh z_{\nu} h}{z_{\nu} h} \right)^2 \to \sum_{\nu=2}^{\infty} b_{\nu} \]

as \( h \downarrow 0 \).

**Proof.** By a classical result, it is sufficient to show that

(i) \( \left( \frac{\sinh z_{\nu} h}{z_{\nu} h} \right)^2 \to 1 \) as \( h \downarrow 0 \), for \( \nu = 2, 3, \ldots \)
is bounded as $h \downarrow 0$. It is evident that (i) is satisfied; (ii) may be established by the method of Theorem 1 of [4].

11. Proof of Theorem 6. The hypothesis of convergence implies that $d_v = o(1)$. If we define

$$\Psi(x) = \sum_{v=2}^{\infty} \frac{d_v e^{\nu x}}{\nu^2}$$

this series is uniformly and absolutely convergent, in $[0,1]$. Now

$$\frac{\Psi(x + 2h) + \Psi(x - 2h) - 2\Psi(x)}{4h^2} = \sum_{v=2}^{\infty} \frac{d_v e^{\nu x} \left( \frac{\sinh z_v h}{z_v h} \right)^2}{\nu^2}$$

and hence, by Lemma 7, the second generalised derivative of $\Psi(x)$ equals $f(x) - d_v e^{\nu x}$ almost everywhere in $[0,1]$. It follows that

$$\Psi(x) = \int_0^x \int_0^t (f(u) - d_v e^{\nu u})du + lx + m$$

where $l, m$ are constants. Since

$$d_v/z_v = o(v^{-2})$$

we may apply Lemma 6 to the series (15). Thus, there are constants $\alpha_0, \cdots, \alpha_{n-1}$ such that

$$\frac{d_v}{z_v} = \lambda_v \int_0^1 \Psi(t)e^{-\nu t}dt + \sum_{\nu=2}^{n-1} \frac{\alpha_v z_v}{G(z_v)}$$

for $\nu = 2, 3, \cdots$.

If we integrate by parts twice, we can write (16) in the form

$$d_v = \lambda_v \int_0^1 f(t)e^{-\nu t}dt + \sum_{\nu=2}^{n+1} \sigma_v z_v e^{\nu t} G'(z_v),$$

where $\sigma_0, \cdots, \sigma_{n+1}$ are constants. Since $G'(z_v) \sim -b_n z_v^n$,

$$d_v = o(1)$$

and

$$\lambda_v \int_0^1 f(t)e^{-\nu t}dt = o(1),$$

we have

$$\sigma_n = \sigma_{n+1} = 0,$$

and for $\nu = 2, 3, \cdots$, we have (7). Finally, by Theorem 1 and Lemma 1,

$$\sum_{\nu=2}^{\infty} \left\{ \lambda_v \int_0^1 f(t)e^{-\nu t}dt + \sum_{\nu=2}^{n-1} \frac{\sigma_v z_v}{G'(z_v)} e^{\nu t} \right\} = 0.$$
is summable \((C, 1)\) almost everywhere in \([0, 1]\) to

\[
f(x) = \left\{ \lambda_1 \int_0^1 f(t)e^{-tz}dt + \sum_{r=0}^{n-1} \frac{\sigma_r z^r}{G'(z)} \right\} e^{\sigma_1 z}
\]

so that we have (7) for \(\nu = 1\), and the proof is complete.

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REFERENCES


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