Pacific Journal of Mathematics

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Vol. 15, No. 2 October 1965

PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

# ON A CLASS OF CAUCHY EXPONENTIAL SERIES

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**This paper was received before the synoptic introduction became a requirement.**

**1. Introduction.** Let $Q(z)$ be a meromorphic function with poles $z_1, z_2, z_3, \cdots$, the notation being so chosen that $|z_1| \leqq |z_2| \leqq |z_3| \leqq \cdots$. If $f \in L(0, 1)$, define

$$c_\nu e^{z_\nu x} = \mathrm{res}_{z_\nu} Q(z) \int_0^1 f(t) e^{z(x-t)} dt .$$

Then, the series $\Sigma c_\nu e^{z_\nu x}$ is called the Cauchy Exponential Series (CES) of $f$ with respect to $Q(z)$. If $z_\nu$ is of multiplicity $m$, then $c_\nu$ is a polynomial in $x$ of degree at most $m-1$; if the poles are all simple, with residue $\lambda_\nu$ at $z_\nu$, we may write

$$c_\nu = \lambda_\nu \int_0^1 f(t) e^{-z_\nu t} dt \tag{1}$$

and $\{c_\nu\}$, independent of $x$, are called the CE constants.

Let $C_p: |z| = r_p$ be an expanding sequence of contours, none of which passes through a pole of $Q(z)$. Suppose $C_p$ contains $n_p$ poles of $Q(z)$. Then,

$$\begin{aligned} \sum_{\nu=1}^{n_p} c_\nu e^{z_\nu x} &= \frac{1}{2\pi i} \int_{C_p} Q(z) dz \int_0^1 f(t) e^{z(x-t)} dt , \\ &= I_p , \quad \text{say} . \end{aligned}$$

Denote by $C_p^+, C_p^-$ the parts of $C_p$ lying in the right, left half-planes respectively. If $Q(z)$ is approximately unity on $C_p^+$, and is small on $C_p^-$, in the sense that

$$\int_{C_p^+} (Q(z) - 1) dz \int_0^1 f(t) e^{z(x-t)} dt = o(1) \tag{2}$$

$$\int_{C_p^-} Q(z) dz \int_0^1 f(t) e^{z(x-t)} dt = o(1) \tag{3}$$

as $p \to \infty$, uniformly for $x \in [0, 1]$, then

$$\begin{aligned} I_p &= \frac{1}{2\pi i} \int_{C_p^+} dz \int_0^1 f(t) e^{z(x-t)} dt + o(1) \\ &= \frac{1}{\pi} \int_0^1 \frac{f(t) \sin r_p(x-t)}{x-t} dt + o(1) \end{aligned}$$

Received March 4, 1963.

uniformly in $[0, 1]$, and so the sums $I_p$ behave somewhat like the partial sums of a Fourier series (F.s.). Indeed, when

$$Q(z) = e^z/e^z - 1$$

the CES is the F.s. of $f$.

In this paper, we shall suppose that

$$Q(z) = \frac{e^z a(z)}{e^z a(z) + b(z)} = \frac{e^z a(z)}{G(z)} \tag{4}$$

where $a(z), b(z)$ are relatively prime polynomials of degree $n$, and that all the poles are simple. This case was investigated first by Fullerton ([1], 1–34), using a less convenient notation.

The large zeros of $G(z)$ approximate to those of $e^z - c$, where

$$c = -\lim_{|z|\to\infty} b(z)/a(z) \tag{5}$$

i.e. to the points $\{\zeta + 2\pi p i\}$, $\zeta$ being the principal value of $\log c$. Hence there is a $\delta$, $0 < \delta < 2\pi$, such that if $r_p = 2p\pi + \delta$, each point of $C_p$ is at a distance greater than a positive constant from the zeros of $G(z)$ and of $e^z - c$. This enables us to prove

THEOREM 1. *Let $f \in L(0, 1)$. Then, as $p \to \infty$,*

$$\sum_{\nu=1}^{n_p} c_\nu e^{z_\nu x} - e^{\zeta x} s_p(x) \to 0$$

*uniformly for $x \in [0, 1]$, where $s_p(x)$ is the* $p$th *partial sum of the F.s. of $f(t)e^{-\zeta t}$.*

We next show that there are $n$ relations connecting the CE constants.

THEOREM 2. *Let $f \in L(0, 1)$. If $c_\nu$ is defined by* (1), *for $\nu = 1, 2, \cdots$, then*

$$\sum_{\nu=1}^{\infty} \frac{c_\nu z_\nu^r}{\lambda_\nu F'(z_\nu)} = 0 \tag{6}$$

$(r = 0, 1, \cdots, n-1)$, *where $F(z) = e^{-z}G(z)$.*

This naturally leads to the following question: if a sequence of numbers $\{\beta_\nu\}$ satisfies $\sum_{\nu=1}^{\infty} c_\nu \beta_\nu = 0$, what is the nature of the $\beta_\nu$? The answer is given by

THEOREM 3. *Let $\{\beta_\nu\}$ be a sequence of numbers such that $\sum_{\nu=1}^{\infty} c_\nu \beta_\nu = 0$ for every* CES *$\Sigma c_\nu e^{z_\nu x}$. Then, there are constants*

$\alpha_0, \cdots, \alpha_{n-1}$ *such that*

$$\beta_\nu = \sum_{r=0}^{n-1} \frac{\alpha_r z_\nu^r}{\lambda_\nu F'(z_\nu)} .$$

Because of the relations (6), we cannot expect that, given a sequence $\{c_\nu\}$ with $\sum_{\nu=1}^{\infty} |c_\nu|^2 < \infty$, there is a function $f \in L^2(0, 1)$ such that (1) is true for each $\nu$. However, we can prove

THEOREM 4. *If $\{c_\nu\}$, $\nu > n$, is a sequence with $\sum_{\nu>n} |c_\nu|^2 < \infty$, there is a function $f \in L^2(0, 1)$ such that (1) is true for each $\nu > n$, and upon defining $c_1, \cdots, c_n$ by (1), such that $\sum_{\nu=1}^{\infty} c_\nu e^{z_\nu x}$ converges in mean to $f$.*

Alternatively, we can alter every $c_\nu$ and so obtain a Riesz-Fischer analogue. We have

THEOREM 5. *Let $\{c_\nu\}$ be a sequence with $\sum_{\nu=1}^{\infty} |c_\nu|^2 < \infty$. Then, there are constants $\gamma_0, \cdots, \gamma_{n-1}$ such that if*

$$d_\nu = c_\nu + \sum_{r=0}^{n-1} \frac{\gamma_r z_\nu^r}{G'(z_\nu)} ,$$

*the numbers $d_\nu$ are the* CE *constants of a function $f \in L^2(0, 1)$.*

We next investigate the problem of the uniqueness of CES. We prove

THEOREM 6. *If $\sum_{\nu=1}^{\infty} d_\nu e^{z_\nu x} = f(x)$ almost everywhere in $[0, 1]$, then there are constants $\sigma_0, \cdots, \sigma_{n-1}$ such that*

$$d_\nu = \lambda_\nu \int_0^1 f(t) e^{-z_\nu t} dt + \sum_{r=0}^{n-1} \frac{\sigma_r z_\nu^r}{G'(z_\nu)} \tag{7}$$

Finally, the question arises whether it is possible to generalise the function $Q(z)$ given by (4), so that the CES of $f$ is uniformly equi-convergent with a F.s. The functions

$$P(z) = \frac{e^z \alpha(z) + \beta(z)}{e^z a(z) + b(z)}$$

where $\alpha(z)$, $\beta(z)$ are polynomials of degree $n$, are obvious generalisations. As $\boldsymbol{Re}\, z \to \infty$, $P(z)$ tends to a number $\omega_1 \neq 0$; as $\boldsymbol{Re}\, z \to -\infty$, to $\omega_2 \neq 0$. Suppose $\omega_1 \neq \omega_2$, and define

$$Q_1(z) = \frac{1}{\omega_1 - \omega_2} \{P(z) - \omega_2\} ;$$

then $Q_1(z)$ satisfies (2), (3). If the CES of $f$ with respect to $Q_1(z)$ is uniformly equiconvergent in $[0, 1]$ with $e^{\zeta x}$ multiplied by the F.s. of $f(t)e^{-\zeta t}$, for each $f \in L(0, 1)$, then

$$\alpha(z) = \omega_1 a(z) \qquad \text{and} \qquad \beta(z) = \omega_2 b(z) ,$$

so that $P(z) = (\omega_1 - \omega_2)Q(z) + \omega_2$. We omit the proof.

**2. Proof of Theorem 1.** In (4), write

$$Q(z) = \frac{e^z}{e^z - c} + R(z) ;$$

then

$$R(z) = \frac{-e^z\{ca(z) + b(z)\}}{(e^z - c)G(z)} .$$

By the choice of $C_p$, there is a positive constant $A$ such that, on $C_p$,

$$|e^z - c| > A \max(|e^z|, 1)$$
$$|G(z)| > A \max(|e^z z^n|, |z^n|) .$$

Further, by (5),

$$ca(z) + b(z) = O(|z^{n-1}|)$$

as $|z| \to \infty$. Hence,

$$\begin{aligned}\int_{C_p^+} R(z)dz \int_0^1 f(t)e^{z(x-t)}dt &= O\Big(\int_{C_p^+}\Big|\frac{e^{z(x-1)}}{z}dz\int_0^1 f(t)e^{-zt}dt\Big|\Big) \\ &= o\Big(\int_{C_p^+}\Big|\frac{e^{z(x-1)}}{z}dz\Big|\Big) \\ &= o(1)\end{aligned}$$

as $p \to \infty$, uniformly for $x \leqq 1$. Similarly,

$$\begin{aligned}\int_{C_p^-} R(z)dz \int_0^1 f(t)e^{z(x-t)}dt &= O\Big(\int_{C_p^-}\Big|\frac{e^{zx}}{z}dz\int_0^1 f(t)e^{z(1-t)}dt\Big|\Big) \\ &= o\Big(\int_{C_p^-}\Big|\frac{e^{zx}}{z}dz\Big|\Big) \\ &= o(1)\end{aligned}$$

as $p \to \infty$, uniformly for $x \geqq 0$.

Since, for large $p$, the number of zeros of $e^z - c$ inside $C_p$ differs from $2p + 1$ by at most 1 and

$$\int_0^1 f(t)e^{(\zeta+2p\pi i)(x-t)}dt = o(1) ,$$

it follows that

$$\begin{aligned}\sum_{\nu=1}^{n_p} c_\nu e^{z_\nu x} &= \frac{1}{2\pi i}\int_{C_p}\frac{e^z}{e^z-c}dz\int_0^1 f(t)e^{z(x-t)}dt + o(1)\\ &= \sum_{\nu=-p}^{p}\int_0^1 f(t)e^{(\zeta+2\pi i\nu)(x-t)}dt + o(1)\\ &= e^{\zeta x}s_p(x) + o(1)\end{aligned}$$

as $p \to \infty$, uniformly in $[0, 1]$, and this completes the proof.

3. The proof of Theorem 2 will depend upon

LEMMA 1. *For* $r = 0, 1, \cdots, n-1$,

$$\int_{C_p}\frac{z^r e^{-zt}}{F(z)}dz = o(1)$$

*as* $p \to \infty$, *boundedly for* $0 < t < 1$.

*Proof.* Define $C_p^+$, $C_p^-$ as in § 1; then, for $r = 0, 1, \cdots, n-1$,

$$\begin{aligned}\int_{C_p^+}\frac{z^r e^{-zt}}{F(z)}dz &= O\Big(\int_{C_p^+}|z^{r-n}e^{-zt}dz|\Big)\\ &= O\Big(\int_{-\pi/2}^{\pi/2}\exp(-t\rho\cos\theta)d\theta\Big) \quad (\rho = r_p)\\ &= O\Big(\int_0^{\pi/2}\exp(-t\rho\sin\theta)d\theta\Big)\\ &= O\Big(\int_0^{\pi/2}\exp\Big(-\frac{2t\rho\theta}{\pi}\Big)d\theta\Big)\end{aligned}$$

which is $o(1)$ as $p \to \infty$, boundedly for $t > 0$. Similarly,

$$\int_{C_p^-}\frac{z^r e^{-zt}}{F(z)}dz = o(1)$$

boundedly for $t < 1$. Hence the result.

4. **Proof of Theorem 2.** Since the zeros of $F(z)$ are simple,

$$\operatorname{res}_{z_\nu}\frac{z^r e^{-zt}}{F(z)} = \frac{z_\nu^r e^{-z_\nu t}}{F'(z_\nu)};$$

hence, by Lemma 1, for $r = 0, 1, \cdots, n-1$,

$$\sum_{\nu=1}^{n_p}\frac{z_\nu^r e^{-z_\nu t}}{F'(z_\nu)} = o(1)$$

as $p \to \infty$, boundedly for $0 < t < 1$. By the choice of $C_p$, $n_{p+1} - n_p = 2$

for large $p$, and so, since the terms are $o(1)$ as $\nu \to \infty$, we may replace $n_p$ by $p$ in the above summation. If we multiply by $f(t)$ and integrate over $[0, 1]$, we have (6).

5. We now prove

LEMMA 2. *Let* $a(z) = \sum_{k=0}^{n} a_k z^k$, and $b(z) = \sum_{k=0}^{n} b_k z^k$. *Then,*

$$(8) \qquad \sum_{r=0}^{n-1} z_\mu^r \sum_{k=r+1}^{n} (b_k + a_k e^{z_\nu}) z_\nu^{k-r-1} + a(z_\mu) e^{z_\mu} \int_0^1 e^{(z_\nu - z_\mu)t} dt = \begin{cases} 0 & \nu \neq \mu \\ G'(z_\mu) & \nu = \mu \end{cases}$$

*Proof.* Write the left-hand side of (8) as

$$(9) \qquad \mathscr{L} + \mathscr{M};$$

then,

$$\mathscr{L} = \sum_{k=1}^{n} (b_k + a_k e^{z_\nu}) \sum_{r=0}^{k-1} z_\mu^r z_\nu^{k-r-1} .$$

If $\nu \neq \mu$,

$$\mathscr{L} = \frac{b(z_\nu) - b(z_\mu) + e^{z_\nu}\{a(z_\nu) - a(z_\mu)\}}{z_\nu - z_\mu} .$$

$$\mathscr{M} = a(z_\mu) \frac{e^{z_\nu} - e^{z_\mu}}{z_\nu - z_\mu} ;$$

since $G(z_\nu) = G(z_\mu) = 0$, (9) is zero. If $\nu = \mu$, (9) is

$$\begin{aligned} \sum_{k=1}^{n} k(b_k + a_k e^{z_\mu}) z_\mu^{k-1} + a(z_\mu) e^{z_\mu} & \\ &= b'(z_\mu) + e^{z_\mu}(a'(z_\mu) + a(z_\mu)) \\ &= G'(z_\mu) . \end{aligned}$$

This proves the lemma.

6. **Proof of Theorem 3.** We have $\sum_{\nu=1}^{\infty} c_\nu \beta_\nu = 0$ for every sequence $\{c_\nu\}$ of CE constants, i.e.

$$\sum_{\nu=1}^{\infty} \beta_\nu \lambda_\nu \int_0^1 f(t) e^{-z_\nu t} dt = 0$$

for every $f \in L(0, 1)$. Hence, by a well-known theorem ([2], § 279),

$$(10) \qquad \int_{1-x}^{1} \sum_{\nu=1}^{p} \beta_\nu \lambda_\nu e^{-z_\nu t} dt \to 0$$

as $p \to \infty$, boundedly for $x \in [0, 1]$. We recall (8); if we multiply by $\beta_\nu \lambda_\nu e^{-z_\nu}$ and sum from $\nu = 1$ to $\nu = p$, where $p$ is greater than an assigned integer $\mu$, we obtain

$$\begin{aligned}\beta_\mu\lambda_\mu e^{-z_\mu}G'(z_\mu) &= \sum_{r=0}^{n-1} z_\mu^r \sum_{\nu=1}^{p} \beta_\nu\lambda_\nu \sum_{k=r+1}^{n} (b_k e^{-z_\nu} + a_k) z_\nu^{k-r-1} \\ &\quad + a(z_\mu)e^{z_\mu}\int_0^1 e^{-z_\mu t}\sum_{\nu=1}^{p}\beta_\nu\lambda_\nu e^{z_\nu(t-1)}dt \\ &= \sum_{r=0}^{n-1} L_{r,p} z_\mu^r + \mathscr{N}_p\,, \qquad \text{say.}\end{aligned}$$

Let

$$\begin{aligned}\phi_p(t) &= \sum_{\nu=1}^{p}\beta_\nu\lambda_\nu e^{z_\nu(t-1)}\,, \\ \Phi_p(x) &= \int_0^x \phi_p(t)dt = \int_{1-x}^1 \sum_{\nu=1}^{p}\beta_\nu\lambda_\nu e^{-z_\nu t}dt\,.\end{aligned}$$

By (10), $\Phi_p(x) \to 0$ as $p \to \infty$, boundedly for $x \in [0, 1]$. Thus,

$$\begin{aligned}\mathscr{N}_p &= a(z_\mu)e^{z_\mu}\int_0^1 e^{-z_\mu t}\phi_p(t)dt \\ &= a(z_\mu)e^{z_\mu}\left\{\Phi_p(1)e^{-z_\mu} + z_\mu\int_0^1 e^{-z_\mu t}\Phi_p(t)dt\right\} \\ &= o(1) \qquad \text{as } p \to \infty\,.\end{aligned}$$

Hence, since $e^{-z}G(z) = F(z)$,

$$\sum_{r=0}^{n-1} L_{r,p}\, z_\mu^r = \beta_\mu\lambda_\mu F'(z_\mu) + \varepsilon_\mu \tag{11}$$

where the numbers $\{L_{r,p}\}$ are independent of $\mu$, and $\varepsilon_\mu \to 0$ as $p \to \infty$.

Giving $\mu$ distinct values $\mu_1, \cdots, \mu_n$, (11) yields a regular system of $n$ linear equations for $L_{0,p}, \cdots, L_{n-1,p}$. The solution is

$$L_{r,p} = \frac{\sum_{i=1}^{n}\{\beta_{\mu_i}\lambda_{\mu_i}F'(z_{\mu_i}) + \varepsilon_{\mu_i}\}\Delta_i^{(r)}}{\det(z_{\mu_i}^{j-1})}$$

where $\Delta_i^{(r)}$ are cofactors of elements in the $(r+1)$th column of the matrix $(z_{\mu_i}^{j-1})$, $(i, j = 1, 2, \cdots, n)$. The only nonconstant terms in this expression for $L_{r,p}$ are $\varepsilon_{\mu_i}$, which are $o(1)$ as $p \to \infty$. Hence, for $r = 0, 1, \cdots, n-1$, $\{L_{r,p}\}$ converges, to $\alpha_r$ say. Letting $p \to \infty$ in (11), we have the result.

7. To prove Theorem 4, we require three lemmas.

LEMMA 3. *If $p > n$, there are numbers $d_1, \cdots, d_n$ such that*

$$e^{z_p x} + \sum_{k=1}^{n} d_k e^{z_k x}$$

*is its own* CES.

*Proof.* We shall show that there are numbers $d_1, \cdots, d_n$ such that, if

$$S(x) = e^{z_p x} + \sum_{k=1}^{n} d_k e^{z_k x} ,$$

then, for $\mu \notin \{1, \cdots, n, p\}$,

$$\int_0^1 S(x) e^{-z_\mu x} dx = 0 . \tag{12}$$

Since the functions $e^{z_1 x}, \cdots, e^{z_n x}, e^{z_p x}$ are linearly independent, and by Theorem 1, the CES of $S(x)$ converges everywhere in $(0, 1)$ to $S(x)$, it will then follow that $S(x)$ is its own CES.

For $\mu \neq k$,

$$\begin{aligned}\int_0^1 e^{(z_k - z_\mu)x} dx &= \frac{e^{-z_\mu}}{z_k - z_\mu}\{e^{z_k} - e^{s}_{\mu}\} \\ &= \frac{e^{-z_\mu}\{a(z_k)b(z_\mu) - a(z_\mu)b(z_k)\}}{a(z_k)a(z_\mu)(z_k - z_\mu)} \\ &= \frac{e^{-z_\mu}\sigma(z_k, z_\mu)}{a(z_k)a(z_\mu)} , \qquad \text{say.}\end{aligned}$$

Thus, if $\mu \notin \{1, \cdots, n, p\}$, and $d_1, \cdots, d_n$ are any $n$ numbers, the left-hand side of (12) is

$$\begin{aligned}&\frac{e^{-z_\mu}}{a(z_\mu)}\left\{\frac{\sigma(z_p, z_\mu)}{a(z_p)} + \sum_{k=1}^{n} \frac{d_k \sigma(z_k, z_\mu)}{a(z_k)}\right\} \\ &\qquad = \frac{e^{-z_\mu}}{a(z_\mu)a(z_p)}\left\{\sigma(z_p, z_\mu) + \sum_{k=1}^{n} \delta_k \sigma(z_k, z_\mu)\right\} \\ &\qquad = I_\mu \qquad \text{say, where } \delta_k = \frac{a(z_p)d_k}{a(z_k)}.\end{aligned}$$

The symmetric polynomial

$$\sigma(x, y) = \frac{a(x)b(y) - a(y)b(x)}{x - y}$$

can be expressed in the form

$$\sum_{r=0}^{n-1} P_r(x) y^r$$

where $P_r(x)$ is a polynomial in $x$ of degree at most $n - 1$. Then,

$$I_\mu = \frac{e^{-z_\mu}}{a(z_\mu)a(z_p)} \sum_{r=0}^{n-1} z_\mu^r \left\{ P_r(z_p) + \sum_{k=1}^{n} \delta_k P_r(z_k)\right\} .$$

This is zero for each $\mu \notin \{1, \cdots, n, p\}$ if

$$P_r(z_p)+\sum_{k=1}^{n}\delta_k P_r(z_k)=0 \qquad (r=0,1,\cdots,n-1),$$

which happens if

$$z_p^r+\sum_{k=1}^{n}\delta_k z_k^r=0 \qquad (r=0,1,\cdots,n-1).$$

Since this system of $n$ linear equations for the unknowns $\delta_1,\cdots,\delta_n$ is regular, the lemma follows.

COROLLARY. *Given the constants $c_{n+1},\cdots,c_p$ of Theorem 4, there are numbers $c_1^{(p)},\cdots,c_n^{(p)}$ such that*

$$T_p(x)=\sum_{k=1}^{n}c_k^{(p)}e^{z_k x}+\sum_{\nu=n+1}^{p}c_\nu e^{z_\nu x}$$

*is its own* CES.

LEMMA 4. *The numbers $c_1^{(p)},\cdots,c_n^{(p)}$ are unique and, for $k=1,2,\cdots,n$, the sequence $\{c_k^{(p)}\}$ converges.*

*Proof.* By Theorem 2, the numbers $c_1^{(p)},\cdots,c_n^{(p)}$ satisfy the regular system of linear equations

$$\frac{c_1^{(p)}z_1^r}{\lambda_1F'(z_1)}+\cdots+\frac{c_n^{(p)}z_n^r}{\lambda_nF'(z_n)}=-\sum_{\nu=n+1}^{p}\frac{c_\nu z_\nu^r}{\lambda_\nu F'(z_\nu)}$$

$(r=0,1,\cdots,n-1)$, and so are determined uniquely. Since $\sum_{\nu>n}|c_\nu|^2<\infty$, and

$$|\lambda_\nu F'(z_\nu)|>K|z_\nu^n|$$

where $K$ is a constant,

$$\sum_{\nu=n+1}^{p}\frac{c_\nu z_\nu^r}{\lambda_\nu F'(z_\nu)}$$

converges, for $r=0,1,\cdots,n-1$. Hence, by an argument used in the proof of Theorem 3, $\{c_k^{(p)}\}$ converges, for $k=1,2,\cdots,n$.

LEMMA 5. *There is a positive constant $A$ such that if $\{a_\nu\}$ is any finite set of numbers, then*

$$\int_0^1|\Sigma a_\nu e^{z_\nu x}|^2\,dx\leqq A\Sigma|a_\nu|^2.$$

*This may be proved by an argument similar to that of Lemma 3 of* [3].

**8. Proof of Theorem 4.** Let $p$, $q$ be integers such that $q > p > n$. Then,

$$T_q(x) - T_p(x) = \sum_{k=1}^{n} (c_k^{(q)} - c_k^{(p)}) e^{z_k x} + \sum_{\nu=p+1}^{q} c_\nu e^{z_\nu x} .$$

By Lemma 5, there is a constant $A > 0$ such that

$$\int_0^1 | T_q(x) - T_p(x) |^2 dx \leqq A \left\{ \sum_{k=1}^{n} | c_k^{(q)} - c_k^{(p)} |^2 + \sum_{\nu=p+1}^{q} | c_\nu |^2 \right\} .$$

Hence, by Lemma 4, $\{T_p(x)\}$ converges in mean to a function $f \in L^2(0, 1)$.

Let $\nu > n$. Since $T_p(x)$ is its own CES,

$$c_\nu = \lambda_\nu \int_0^1 T_p(x) e^{-z_\nu x} dx \qquad (p \geqq \nu).$$

Hence,

$$\begin{aligned} c_\nu &= \lambda_\nu \lim_{p \to \infty} \int_0^1 T_p(x) e^{-z_\nu x} \, dx \\ &= \lambda_\nu \int_0^1 f(x) e^{-z_\nu x} dx . \end{aligned}$$

Define $c_1, \cdots, c_n$ by this formula; then,

$$c_k = \lim_{p \to \infty} c_k^{(p)} \qquad (k = 1, 2, \cdots, n) ,$$

and $\sum_{\nu=1}^{\infty} c_\nu e^{z_\nu x}$ converges in mean to $f$. This completes the proof.

**9. Proof of Theorem 5.** If we multiply (8) by $c_\nu$ and sum from $\nu = 1$ to $\nu = p$, where $p$ is greater than an assigned integer $\mu$, we obtain

$$\begin{aligned} c_\mu G'(z_\mu) &= \sum_{r=0}^{n-1} z_\mu^r \sum_{\nu=1}^{p} c_\nu \sum_{k=z+1}^{n} (a_k e^{r_\nu} + b_k) z_\nu^{k-r-1} \\ &\quad + a(z_\mu) e^{z_\mu} \int_0^1 e^{-z_\mu t} \sum_{\nu=1}^{p} c_\nu e^{z_\nu t} dt \qquad (13) \\ &= \mathscr{L}_p + \mathscr{M}_p , \qquad \text{say.} \end{aligned}$$

Since $\sum_{\nu=1}^{\infty} | c_\nu |^2 < \infty$, $\sum_{\nu=1}^{\infty} c_\nu e^{z_\nu t}$ converges in mean to a function $f \in L^2(0, 1)$. Hence,

$$\mathscr{M}_p \to d_\mu G'(z_\mu) \qquad \text{as } p \to \infty$$

where

$$d_\mu = \lambda_\mu \int_0^1 f(t) e^{-z_\mu t} dt ,$$

Next,

$$\mathscr{L}_p = \sum_{r=0}^{n-1} \delta_r z_\mu^r - \sum_{r=0}^{n-1} z_\mu^r \sum_{\nu=2}^{p} c_\nu \sum_{k=0}^{z} (a_k e^{r\nu} + b_k) z_\nu^{k-r-1} \tag{14}$$

where

$$\delta_r = c_1 \sum_{k=r+1}^{n} (a_k e^{r_1} + b_k) z_1^{k-r-1} .$$

Since

$$\sum_{k=0}^{r} (a_k e^{z_\nu} + b_k) z_\nu^{k-r-1} = O(\nu^{-1})$$

the summation over $\nu$ in (14) converges, as $p \to \infty$, to $\eta_r$ say. The result now follows upon writing

$$\eta_r + \delta_r = \gamma_r .$$

10. Before establishing the uniqueness theorem, we prove two lemmas.

LEMMA 6. *If* $\sum_{\nu=1}^{\infty} d_\nu e^{z_\nu x} = f(x)$ *almost everywhere in* $[0, 1]$, *and* $d_\nu = O(\nu^{-2})$, *there are constants* $\sigma_0, \cdots, \sigma_{n-1}$ *such that* (7) *is satisfied for* $\nu = 1, 2, \cdots$.

*Proof.* We have (13), with $c_\nu$ replaced by $d_\nu$. We may write this as

$$d_\mu G'(z_\mu) = \sum_{r=0}^{n-1} M_{r,p} z_\mu^r + \lambda_\mu G'(z_\mu) \int_0^1 e^{-z_\mu t} \Big\{ f(t) - \sum_{\nu=p+1}^{\infty} d_\nu e^{z_\nu t} \Big\} dt .$$

Since

$$\begin{aligned} \int_0^1 e^{-z_\mu t} \sum_{\nu=p+1}^{\infty} d_\nu e^{z_\nu t} dt &= O\Big( \sum_{\nu=p+1}^{\infty} |d_\nu| \Big) \\ &= o(1) \qquad \text{as } p \to \infty , \end{aligned}$$

and $\{M_{r,p}\}$ converges, to $\sigma_r$ say, for $r = 0, 1, \cdots, n-1$, we obtain (7).

LEMMA 7. *If the series* $\sum_{\nu=2}^{\infty} b_\nu$ *is convergent, then*

$$\sum_{\nu=2}^{\infty} b_\nu \left( \frac{\sinh z_\nu h}{z_\nu h} \right)^2 \to \sum_{\nu=2}^{\infty} b_\nu$$

*as* $h \downarrow 0$.

*Proof.* By a classical result, it is sufficient to show that

(i) $\left( \dfrac{\sinh z_\nu h}{z_\nu h} \right)^2 \to 1$ as $h \downarrow 0$, for $\nu = 2, 3, \cdots$

(ii) $\sum_{\nu=2}^{\infty}\left|\left(\frac{\sinh z_{\nu+1}h}{z_{\nu+1}h}\right)^2-\left(\frac{\sinh z_\nu h}{z_\nu h}\right)^2\right|$

is bounded as $h\downarrow 0$. It is evident that (i) is satisfied; (ii) may be established by the method of Theorem 1 of [4].

**11. Proof of Theorem 6.** The hypothesis of convergence implies that $d_\nu = o(1)$. If we define

$$\Psi(x)=\sum_{\nu=2}^{\infty}\frac{d_\nu e^{z_\nu x}}{z_\nu^2} \tag{15}$$

this series is uniformly and absolutely convergent, in $[0, 1]$. Now

$$\frac{\Psi(x+2h)+\Psi(x-2h)-2\Psi(x)}{4h^2}=\sum_{\nu=2}^{\infty}d_\nu e^{z_\nu x}\left(\frac{\sinh z_\nu h}{z_\nu h}\right)^2$$

and hence, by Lemma 7, the second generalised derivative of $\Psi(x)$ equals $f(x)-d_1e^{z_1x}$ almost everywhere in $[0, 1]$. It follows that

$$\Psi(x)=\int_0^x dt\int_0^t (f(u)-d_1e^{z_1u})du+lx+m$$

where $l, m$ are constants. Since

$$d_\nu/z_\nu^2=o(\nu^{-2})\ ,$$

we may apply Lemma 6 to the series (15). Thus, there are constants $\alpha_0, \cdots, \alpha_{n-1}$ such that

$$\frac{d_\nu}{z_\nu^2}=\lambda_\nu\int_0^1\Psi(t)e^{-z_\nu t}dt+\sum_{z=0}^{n-1}\frac{\alpha_r z_\nu^r}{G'(z_\nu)} \tag{16}$$

for $\nu = 2, 3, \cdots$.

If we integrate by parts twice, we can write (16) in the form

$$d_\nu=\lambda_\nu\int_0^1 f(t)e^{-z_\nu t}dt+\sum_{r=0}^{n+1}\frac{\sigma_r z_\nu^r}{G'(z_\nu)}\ ,$$

where $\sigma_0, \cdots, \sigma_{n+1}$ are constants. Since $G'(z_\nu)\sim -b_nz_\nu^n$,

$$d_\nu=o(1)\qquad\text{and}\qquad \lambda_\nu\int_0^1 f(t)e^{-z_\nu t}dt=o(1)\ ,$$

we have

$$\sigma_n=\sigma_{n+1}=0\ ,$$

and for $\nu = 2, 3, \cdots$, we have (7). Finally, by Theorem 1 and Lemma 1,

$$\sum_{\nu=1}^{\infty}\left\{\lambda_\nu\int_0^1 f(t)e^{-z_\nu t}dt+\sum_{r=0}^{n-1}\frac{\sigma_r z_\nu^r}{G'(z_\nu)}\right\}e^{z_\nu x}$$

is summable $(C, 1)$ almost everywhere in $[0, 1]$ to

$$f(x) - \left\{\lambda_1 \int_0^1 f(t)e^{-z_1 t}dt + \sum_{r=0}^{n-1} \frac{\sigma_r z_1^r}{G'(z_1)}\right\} e^{z_1 x}$$

so that we have (7) for $\nu = 1$, and the proof is complete.

In conclusion, the authors wish to express their gratitude to Professor S. Verblunsky of Belfast, for his helpful criticism and advice.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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# Pacific Journal of Mathematics

Vol. 15, No. 2 October, 1965