

# Pacific Journal of Mathematics

## **HOPF ALGEBRAS OVER DEDEKIND DOMAINS AND TORSION IN $H$ -SPACES**

ALLAN CLARK

# HOPF ALGEBRAS OVER DEDEKIND DOMAINS AND TORSION IN $H$ -SPACES

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**The main purpose of this note is to show that if the loop space  $\mathcal{Q}X$  of a finite dimensional  $H$ -space is free of torsion, then  $X$  itself can have  $p$ -torsion of at most order  $p$ .**

§ 1 is devoted to proving a generalization to Dedekind domains of the decomposition theorems Hopf-Leray, and Borel, and § 2 is devoted to recalling the structure of quasimonogenic Hopf algebras over the integers as described by Halpern. § 3 gives the proof of the main theorem which relies somewhat on the statement and proof of Theorem 4.1 of [4].

Theorem 1.5 was included in the author's dissertation (Princeton University, 1961) done under the direction of Professor John Moore.

**1. Hopf algebras over Dedekind domains.** Unless further specified  $K$  will denote an arbitrary integral domain. By *standard field associated* with  $K$  we shall mean any residue class field of  $K$ . A  $K$ -algebra will be called *monogenic* if it is generated by a single element.

In this section we prove a generalization (Theorem 1.5) for torsion-free algebras over a Dedekind domain with perfect quotient field of the following well known theorem:

**THEOREM 1.1.** (*Hopf-Leray-Borel*). *If  $B$  is a connected, commutative, and associative Hopf algebra of finite type over a perfect field  $K$ , then  $B$  is isomorphic as a  $K$ -algebra to a tensor product of monogenic Hopf algebras over  $K$ .*

Proof of the separate cases  $\text{char } K = 0$  (Hopf-Leray) and  $\text{char } K \neq 0$  (Borel) may be found in Milnor and Moore [6].

**DEFINITIONS.** A *closed submodule* of a  $K$ -module  $B$  is a submodule such that for all  $x \in B$  and all  $k \in K$ ,  $kx \in A$  implies  $x \in A$  or  $k = 0$ . If  $A$  is any submodule of  $B$ , then  $\bar{A}$ , the *closure of  $A$  in  $B$* , is given by

$$\bar{A} = \{x \in B \mid kx \in A \text{ for some } k \in K, k \neq 0\}$$

**REMARKS.**  $A$  is closed in  $B$  is equivalent to  $\bar{A} = A$  and to  $B/A$  is torsion-free. Note that the intersection of closed submodules is closed

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Received March 17, 1964. The author was partially supported by a National Science Foundation Grant.

and  $\bar{A}$  is the minimal closed submodule of  $B$  which contains  $A$ . If  $Q$  denotes the field of fractions of  $K$  and  $j : B \rightarrow B \otimes Q$  is the map given by  $j(b) = b \otimes 1$ , then  $\bar{A} = j^{-1}[j(A)]$  where  $[j(A)]$  denotes the  $Q$ -submodule generated by  $j(A)$ .

**PROPOSITION 1.2.** If  $B$  is a  $K$ -algebra and  $A$  is a subalgebra of  $B$ , then the closure of  $A$  in  $B$  is a subalgebra of  $B$ . If  $B$  is a torsion-free  $K$ -coalgebra and  $A$  is a subcoalgebra of  $B$ , then the closure of  $A$  is a subcoalgebra of  $B$ .

*Proof.* The first statement is obvious. For the second let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  and  $0 \rightarrow \bar{A} \rightarrow B \rightarrow C' \rightarrow 0$  be exact sequences of  $K$ -modules defining  $C$  and  $C'$ . Let  $\tilde{X}$  denote  $X \otimes Q$  where  $Q$  is the field of fractions of  $K$ . Let  $j : B \rightarrow B \otimes Q = \tilde{B}$  as above. Then  $j$  is a monomorphism since  $B$  is torsion-free. Furthermore  $j(\bar{A}) = [j(A)] \approx \tilde{A}$  and  $C' = B/\bar{A}$  is torsion-free so that  $C' \rightarrow \tilde{C}'$  is a monomorphism. Consider the commutative diagram

$$\begin{CD} \bar{A} @>\alpha>> B \otimes B @>\beta>> (B \otimes C') \oplus (C' \otimes B) \\ @Vj_1VV @Vj_2VV @Vj_3VV \\ \tilde{A} @>\gamma>> \tilde{B} \otimes \tilde{B} @>\delta>> (\tilde{B} \otimes \tilde{C}') \oplus (\tilde{C}' \otimes \tilde{B}) \end{CD}$$

where  $\alpha$  and  $\gamma$  are induced by the coproduct of  $B$ . Obviously  $j_1, j_2$ , and  $j_3$  are monomorphisms.

$\delta\gamma = 0$ :  $A$  is a subcoalgebra of  $B$ , and tensoring with  $Q$  the exact sequence  $0 \rightarrow \bar{A} \rightarrow B \rightarrow C' \rightarrow 0$  we obtain an exact sequence  $0 \rightarrow \tilde{A} \rightarrow \tilde{B} \rightarrow \tilde{C}' \rightarrow 0$ .

Then  $j_3\beta\alpha = \delta\gamma j_1 = 0$ .  $j_3$  is a monomorphism and thus  $\beta\alpha = 0$ ,  $\text{Im } \alpha \subset \text{Ker } \beta = \bar{A} \otimes \bar{A}$ , and  $\alpha(\bar{A}) \subset \bar{A} \times \bar{A}$ .

**COROLLARY 1.3.** If  $A$  is a sub Hopf algebra of a torsion-free Hopf algebra over  $K$ , then the closure of  $A$  is a sub Hopf algebra of  $B$ .

**PROPOSITION 1.4.** Let  $A$  be a sub Hopf algebra of a torsion-free Hopf algebra over a Dedekind domain  $K$ . Then  $A$  is closed in  $B$  if and only if  $B$  is a flat  $A$ -module.

*Proof.* The following statements are equivalent.

- (a)  $A$  is closed in  $B$ .
- (b)  $B/A$  is a torsion-free  $K$ -module.
- (c)  $0 \rightarrow A \otimes L \rightarrow B \otimes L$  is exact for every standard field  $L$  associated with  $K$ .
- (d)  $0 \rightarrow P(A \otimes L) \rightarrow P(B \otimes L)$  is exact for every standard field

$L$  associated with  $K$ .

(e)  $B$  is a flat  $A$ -module.

It is obvious that (a), (b), and (c) are equivalent. (c) and (d) are equivalent by Proposition 3.8 of Milnor and Moore [6]. That (d) and (e) are equivalent is Proposition 6 of Moore [7].

**PROPOSITION 1.5.** Let  $B$  be a torsion-free Hopf algebra over a Dedekind ring  $K$ , let  $A$  be a sub Hopf algebra of  $B$ , and suppose that  $C$  is a normal closed sub Hopf algebra of  $A$  and  $B$ . Then  $A$  is closed in  $B$  if and only if  $A//C$  is closed in  $B//C$ .

*Proof.* In view of Proposition 1.4 it is sufficient to show that  $\text{Tor}_n^A(X, B) \approx \text{Tor}_n^{A//C}(X, B//C)$  for any  $A//C$ -module  $X$ .

Proposition 1.4 implies that  $A$  and  $B$  are flat  $C$ -modules and  $\text{Tor}_n^C(K, A) = \text{Tor}_n^C(K, B) = 0$  for all  $n > 0$ . A change of rings  $C \rightarrow A$  yields  $\text{Tor}_n^A(A//C, B) \approx \text{Tor}_n^C(K, B) = 0$  since  $A//C = A \otimes_C K$ . (Cf. Cartan and Eilenberg [3, p. 117].) A second change of rings  $A \rightarrow A//C$  yields  $\text{Tor}_n^A(X, B) \approx \text{Tor}_n^{A//C}(X, A//C \otimes_A B)$ . Since  $B//C \approx A//C \otimes_A B$  as an  $A//C$ -module, the proof is complete.

**DEFINITIONS.** An integral domain  $K$  is *quasiperfect* if the field of fractions  $Q$  is perfect. A torsion-free Hopf algebra  $B$  over  $K$  is *quasimonogenic* if  $B \otimes Q$  is monogenic.

**THEOREM 1.6.** Suppose  $B$  is a torsion-free, associative, and commutative Hopf algebra of finite type over a quasiperfect Dedekind domain  $K$ . Then as an algebra  $B$  is isomorphic to a tensor product of quasimonogenic Hopf algebras.

*Proof.* Since  $B$  is torsion-free, the map  $j: B \rightarrow B \otimes Q$  is a monomorphism.  $B = B \otimes Q$  satisfies the hypotheses of Theorem 1.1 and we write  $B \approx \bigotimes_{i \in I} B_i$  where  $x_i$  generates  $B_i$  and the indexing is arranged so that  $\text{deg } x_i \leq \text{deg } x_{i+1}$ .

There are nonzero elements  $k_i \in K$  such that  $k_i x_i \in \text{Im } j$  and we let  $\tilde{B}_i$  denote the closure of the subalgebra generated by  $j^{-1}(k_i x_i)$ . Let  $f: \bigotimes_{i \in I} B_i \rightarrow B$  be the map induced by the injections  $B_i \rightarrow B$ .  $f$  is a monomorphism since the diagram

$$\begin{array}{ccc} \bigotimes_{i \in I} B_i & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \bigotimes_{i \in I} \tilde{B}_i & \longrightarrow & \tilde{B} \end{array}$$

is commutative and the other maps are monomorphisms.

Let  $B^n = \bigotimes_{i=1}^n B_i$ . We show by induction that  $B^n$  is a closed sub Hopf algebra of  $B$ . By Corollary 1.3  $B^1 = B_1$  is a closed sub Hopf algebra of  $B$ . Suppose  $B^{n-1}$  is a closed sub Hopf algebra of  $B$ . We see directly from the definition of  $B_n$  that  $B_n \approx B^n // B^{n-1}$  is closed in  $B // B^{n-1}$  and by Proposition 1.5  $B^n$  is closed in  $B$ .

Consequently  $\bigotimes_{i \in I} B_i$  is a closed sub Hopf algebra of  $B$  and Coker  $f$  is torsion-free. But  $(\bigotimes_{i \in I} B_i) \otimes Q \approx B \otimes Q$  so that  $(\text{Coker } f) \times Q = 0$ . Consequently  $\text{Coker } f = 0$  and  $f$  is an epimorphism.

2. Quasimonogenic Hopf algebras. Here we recall for the special case  $K = Z$  the description due to Halpern [5] of the quasimonogenic Hopf algebras which appear as factors in the splitting guaranteed by Theorem 1.6. We include some homological computation useful in applications of the main theorem.

**THEOREM 2.1.** (Halpern). *Every quasimonogenic Hopf algebra  $B$  over the integers has one of the following forms:*

(a)  $B = E(x, 2m - 1)$ , the exterior algebra on a generator of odd degree.  $x$  is called a quasigenerator as well as a generator.

(b)  $B$  has a series of generators  $x_1, \dots, x_n, \dots$  of even degree where  $\deg x_n = n$  degree  $x_1$  which satisfy relations of the form  $x_1 x_{n-1} = \beta_n x_n$  where the  $\beta_n$  are positive integers. Furthermore the  $\beta_n$  satisfy the conditions (i)–(iii).

(i)  $\beta_1 = 1$

(ii) There are integers  $\beta_n^*$  for each  $n$  such that  $\beta_n \beta_n^* = n$ .

(iii)  $\beta_n \mid \beta_{kn}$  and  $\beta_n^* \mid \beta_{kn}^*$  for all positive integers  $k$  and  $n$ .

$B$  will be called a partially divided polynomial algebra. Any sequence of integers satisfying (i)–(iii) will be called a fundamental sequence.  $x_1$  is called the quasigenerator of  $B$ .

Halpern shows that every sequence of positive integers  $\{\beta_n\}$  satisfying (i)–(iii) gives rise to a quasimonogenic Hopf algebra whose dual Hopf algebra comes from  $\{\beta_n^*\}$ . When  $\beta_n = 1$  for all  $n$ , the resulting Hopf algebra is the polynomial algebra on the primitive generator  $x_1$ . If  $\beta_n = n$  for all  $n$ , the Hopf algebra associated with the sequence  $\{\beta_n\}$  is the divided polynomial algebra on the primitive generator  $x_1$ . If we set

$$\beta_m = \beta_m \cdots \beta_1 \text{ and } \beta_{m,n} = \beta_{m+n} / \beta_m \beta_n,$$

then the product and coproduct in the algebra associated with  $\{\beta_n\}$  are given by the rules

$$x_m x_n = \beta_{m,n} x_{m+n}$$

$$\Delta(x_n) = \sum_{i+j=n} \beta_{i,j}^* x_i x_j$$

where  $\beta_{m,n}^* = \beta_{m+n}^* / \beta_m^* \beta_n^*$ . Note that  $\beta_n^* \beta_n^* = n!$  and therefore  $\beta_{m,n} \beta_{m,n}^* = [m, n]$  the binomial coefficient.

To illustrate how one might construct a fundamental sequence we observe that if  $(m, n) = 1$  ( $m$  and  $n$  are relatively prime) it follows that  $\beta_{mn} = \beta_m \beta_n$ . To see this note that (ii) implies that  $(\beta_m, \beta_n) = 1 = (\beta_m^*, \beta_n^*)$ . By (iii)  $\beta_m \beta_n \mid \beta_{mn}$  and  $\beta_m^* \beta_n^* \mid \beta_{mn}^*$ . But  $\beta_m \beta_m^* \beta_n \beta_n^* = mn = \beta_{mn} \beta_{mn}^*$  consequently  $\beta_m \beta_n = \beta_{mn}$ . As a result if  $m = m_1 \cdots m_g$  is the decomposition of  $m$  into primary factors, we can write  $\beta_m = \beta_{m_1} \cdots \beta_{m_g}$ .

**LEMMA 2.2.** *Let  $\alpha_m$  denote the greatest common divisor of the quasibinomial coefficients  $\beta_{k, m-k}$ ,  $0 < k < m$ . Then  $\alpha_m = 1$  unless  $m = p^n$  and  $\beta_m = p\beta_{m/p}$  in which case  $\alpha_m = p$ .*

*Proof.*  $\beta_{k, m-k}$  divides  $\binom{m}{k}$  and therefore  $\alpha_m$  divides  $q_m$ , the greatest common divisor of the binomial coefficients  $\binom{m}{k}$  for  $0 < k < m$ . By Lucas's theorem (Adem [1, Theorem 25.1]) we see that if  $m \neq p^n$  for a given prime  $p$ , then  $\binom{m}{k} \not\equiv 0 \pmod p$  for some  $k$ ,  $0 < k < m$ . Consequently if  $m \neq p^n$  for any prime  $p$ , then  $q_m = 1$  and  $\alpha_m = 1$ .

Let  $\varepsilon(n)$  denote the number of factors of  $p$  in  $p^n!$ . Then a simple counting argument shows that  $\varepsilon(n) = p\varepsilon(n-1) + 1$ . We know that  $q_{p^n} = p^r$  for  $0 < r$  by the argument above. Writing

$$\left(\frac{p^n!}{(p^{n-1}!)^p}\right) = \binom{p^n}{p^{n-1}} \binom{(p-1)p^{n-1}}{p^{n-1}} \cdots \binom{2p^{n-1}}{p^{n-1}}$$

and nothing that Lucas's theorem implies  $\binom{ap^{n-1}}{p^{n-1}} \equiv \binom{a}{1} \not\equiv 0 \pmod p$ , we see that  $\binom{p^n}{p^{n-1}}$  has as many factors of  $p$  as  $(p^n!)/(p^{n-1}!)^p$ , namely just one. Consequently  $q_{p^n} = p$  and  $\alpha_{p^n} = 1$  or  $\alpha_{p^n} = p$ .

Writing  $\beta[a, b]$  for  $\beta_{a,b}$  we find that, for  $(a-1)p^{n-1} \leq k < ap^{n-1}$  ( $0 < a < p$ ),

$$\beta[k, p^n - k] = \beta[k, ap^{n-1} - k] \beta[ap^{n-1}, (p-a)p^{n-1}] / \beta[ap^{n-1} - k, (p-a)p^{n-1}].$$

Then  $\beta[k, ap^{n-1} - k]$  divides  $\binom{ap^{n-1}}{k} \not\equiv 0 \pmod p$  (by Lucas's theorem) and similarly  $\beta[ap^{n-1} - k, (p-a)p^{n-1}] \not\equiv 0$ , so that  $p$  divides  $\beta[k, p^n - k]$  if and only if  $p$  divides  $\beta[ap^{n-1}, (p-a)p^{n-1}]$ . Taking  $k = p^{n-1}$ , it follows that  $\alpha_m = p$  if and only if  $\beta[p^{n-1}, (p-1)p^{n-1}] \equiv 0 \pmod p$ . However

$$\beta[p^{n-1}, (p-1)p^{n-1}] = (\beta_{p^n} / \beta_{p^{n-1}}) \beta[p^{n-1} - 1, (p-1)p^{n-1}]$$

and  $\beta[p^{n-1} - 1, (p-1)p^{n-1}] \not\equiv 0 \pmod p$  by the usual argument using the theorem of Lucas cited above. Thus  $\alpha_m = p$  if and only if  $p$  divides  $\beta_{p^n} / \beta_{p^{n-1}}$ , or in other words, if and only if  $\beta_{p^n} = p\beta_{p^{n-1}}$ .

**THEOREM 2.3.** *Let  $B$  denote a torsion-free, even dimensional, quasimonogenic Hopf algebra over the integers with generators  $\{x_n\}$  and with fundamental sequence  $\{\beta_n\}$ . Let  $S_p = \{p^n \mid \beta_{p^n} = p\beta_{p^{n-1}}\}$  for a given prime  $p$ . Then:*

- (1) *Among the generators  $\{x_n\}$  the indecomposable ones are  $x_1$  and the  $x_m$ 's for which  $m \in S_p$  for some prime  $p$ .*
- (2) *The relations among these indecomposable generators may all be derived from the relations of the form*

$$(\beta_n?)x_n = (\beta_m?)x_m^{n/m}$$

where  $m$  and  $n$  are consecutive elements of some  $S_p$  or else  $m = 1$  and  $n$  is the smallest element of some  $S_p$ .

*Proof.* (1) If  $m > 1$  and  $m \in S_p$  for any prime  $p$ , then by 2.2,  $\alpha_m = 1$ . Consequently there exist integers  $\lambda_k$  for  $0 < k < m$  such that  $\sum_{k=1}^{m-1} \lambda_k \beta_{k, m-k} = 1$ , and therefore

$$x_m = \left( \sum_{k=1}^{m-1} \lambda_k \beta_{k, m-k} \right) x_m = \sum_{k=1}^{m-1} \lambda_k x_k x_{m-k}$$

and  $x_m$  is decomposable.

(2) Clearly the relations given do hold between the generators involved. On the other hand from the relations given we may easily obtain the relations  $(\beta_n?)x_n = (x_1)^n$  for  $n \in S_p$  for some prime  $p$ . From these we may obtain the relations  $(\beta_n?)x_n = (x_1)^n$  for any integer  $n$ ,  $x_n$  being written in terms of indecomposable generators. Finally we can obtain the relations  $x_1 x_{m-1} = \beta_m x_m$  which characterize the algebra  $B$ .

**REMARK 2.4.** The relations in  $B$  are all derived from relations of the form  $x^{p^q} = pRy$  where  $x$  and  $y$  are indecomposable,  $p$  is prime, and  $R \neq 0$  since for  $m$  and  $n$  consecutive elements of  $S_p$  we have that  $p$  divides  $(\beta_n?)/(\beta_m?)^{n/m}$  precisely once which is easily proved by an argument similar to that of 2.2.

### 3. Application to torsion in $H$ -spaces.

**THEOREM 3.1.** *Let  $X$  be a pathwise connected and simply connected  $H$ -space of finite homological type and dimension whose loop space is torsion-free. Then  $X$  can have  $p$ -torsion of at most order  $p$ .*

*Proof.* The hypotheses imply that  $H_*(\Omega X; \mathbb{Z})$  is a torsion-free Hopf algebra of finite type, and furthermore  $H_*(\Omega X; \mathbb{Z})$  has no elements of odd degree since  $H_*(\Omega X; \mathbb{Q})$  is even dimensional by [8, § 7,

Theorem III]. Therefore 1.6 and 1.7 imply that  $H_*(\Omega X; Z)$  is a tensor product of partially divided polynomial Hopf algebras. In each factor the relations among generators are of the form  $x^{p^q} = pRy$  where  $p$  is prime and  $R \not\equiv 0 \pmod p$ , as remarked in 2.4. It follows easily that  $H_*(\Omega X; Z_p)$  is a tensor product of polynomial and truncated polynomial algebras—either from a direct computation using the structure of  $H_*(\Omega X; Z)$  or applying the decomposition theorem of Borel to  $H_*(\Omega X; Z_p)$  which has no elements of odd degree by the remark above.

The classifying space  $B_{\Omega X}$  has the same homotopy type as  $X$  by Corollary 9.2 of [9], and there is a homotopy equivalence of chain complexes  $B(C(\Omega X)) \rightarrow C(B_{\Omega X})$  from the bar construction on the chains of  $\Omega X$  to the chains of the classifying space  $B_{\Omega X}$ . Therefore the homology of  $X$  may be computed as the homology of the bar construction on the chains of  $\Omega X$ .

Let  $x \in H_{2m}(\Omega X; Z)$  and suppose that  $x^{p^q} = pRy$  is one of the relations in  $H_*(\Omega X; Z)$ ,  $x$  and  $y$  being indecomposable elements. Let  $a$  and  $b$  denote chains of  $\Omega X$  which represent  $x$  and  $y$  respectively. Then  $a^{p^q} = pRb + \partial c$ . Then the element  $[a^{p^{q-1}} | a] + [c]$  is a cycle mod  $p$  and its homology class  $z$  is called the transpotence of  $x'$ , where  $x'$  denotes the reduction of  $x \pmod p$ . Clearly  $\beta z = Rs_*y'$ , where  $\beta$  denotes the Bockstein, and  $s_*$  denotes the suspension homomorphism in homology mod  $p$ , and  $y'$  denotes the reduction of  $y \pmod p$ .

From the proof of Theorem 4.1 of [4] it follows clearly that every primitive generator of even degree in  $H_*(X; Z_p)$  is such a transpotence  $z$  of some generator  $x'$  in  $H_*(\Omega X; Z_p)$ .<sup>1</sup> See also, S. Gitler, Nota Sobre La Transpotencia de Cartan, Bol. Soc. Mex. 1963, 85–91. Furthermore by Theorem 4.1 of [4] it follows that  $s_*y' \neq 0$  since  $\deg y' \equiv 0 \pmod p$ . (For  $p = 2$  it is true because  $\deg y' \equiv 0 \pmod 4$ ). Therefore the primitive generators of even degree in  $H_*(\Omega X; Z_p)$  have nonzero Bockstein while those of odd degree have zero Bockstein since  $\Omega X$  is torsion free and suspension commutes with the Bockstein.

The remainder of the proof is a simple exercise in spectral sequences of Hopf algebras showing that in the Bockstein spectral sequence,  $E^2$  is an exterior algebra on generators of odd degree, and that therefore  $E^2 = E^\infty$ . Since the higher differentials in the Bockstein spectral sequence may be identified with the higher order Bockstein operations, it is immediate that  $X$  has  $p$ -torsion of order  $p$  at most.

Bott [2] shows that a compact, connected, and simply connected Lie group satisfies the hypotheses of 3.1.

<sup>1</sup> See also, S. Gitler, Nota Sobre La Transpotencia de Cartan, Bol. Soc. Mex. 1963, 85–91.



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