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SOME CONTAINMENT RELATIONS BETWEEN CLASSES OF IDEALS OF A COMMUTATIVE RING

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The first section of this paper is devoted to proving the following theorem. Let D be an integral domain with identity. Let $\mathscr{P}$ be the set of prime powers of D, $\mathscr{V}$ the set of valuation ideals of D, and let k be the quotient field of D. $\mathscr{V} \subseteqq \mathscr{P}$ if and only if the following conditions hold: (i) Each prime ideal P of D defines a P-adic valuation in the sense of van der Waerden, and (ii) every valuation of k finite on D is isomorphic to a P-adic valuation for some P.

The second section considers three additional sets of ideals; the set $\mathscr{Q}$ of primary ideals, the set $\mathscr{S}$ of semi-primary ideals, and the set $\mathscr{A}$ of ideals A such that the complement of some prime ideal is prime to A.

Commutative rings in which various containment relations exist between the sets $\mathscr{V}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{A}$, and $\mathscr{S}$ are also considered. Most of the results of this section represent applications of previous results of the author.

Let D be an integral domain with identity having quotient field K. An ideal A of D is said to be a *valuation ideal* provided there exists a valuation ring D_v with $D \subseteqq D_v \subseteqq K$ such that $AD_v \cap D = A$. More specifically, if D_v is the valuation ring of the valuation v of K, we may say A is a *v-ideal*. We denote by $\mathscr{T}(D)$ the set of valuation ideals of the domain D and by $\mathscr{Q}(D)$ the set of primary ideals of D. Where no ambiguity exists we may speak of $\mathscr{V}$ and $\mathscr{Q}$.

This paper is closely related to a paper of Gilmer and Ohm [5], and frequent reference is made to their results. In [5] the relations $\mathscr{V} \subseteqq \mathscr{Q}$, $\mathscr{V} = \mathscr{Q}$, and $\mathscr{Q} \subseteqq \mathscr{V}$ were investigated. That paper arose as a result of the following observation in [8, p. 341]:

If D is a Dedekind domain, then $\mathscr{V} = \mathscr{Q}$. But if D is Dedekind, the sets $\mathscr{P}(D)$ of prime powers of D and $\mathscr{Q}(D)$ coincide. Hence if D is Dedekind $\mathscr{V} = \mathscr{P}$. In § 2 necessary and sufficient conditions are given on a domain D in order that $\mathscr{V} \subseteqq \mathscr{P}$. In particular it is shown that $\mathscr{V} \subseteqq \mathscr{P}$ implies $\mathscr{V} = \mathscr{P}$.

In § 3 we consider the set $\mathscr{A}(R)$ consisting of all ideals A of the commutative ring R such that $R - P$ is prime to A for some prime ideal P of R. It is always true that $\mathscr{Q}(R) \subseteqq \mathscr{A}(R)$ and if R is an integral domain with identity, we also have $\mathscr{V}(R) \subseteqq \mathscr{A}(R)$. The

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relations $\mathscr{A}(R) \subseteqq \mathscr{Q}(R)$, $\mathscr{A}(R) \subseteqq \mathscr{P}(R)$ are investigated in §3. In particular, if R is an integral domain with identity then $\mathscr{A} \subseteqq \mathscr{V}$ if and only if R is a Prüfer domain[1] and $\mathscr{A} \subseteqq \mathscr{P}$ if and only if R is almost Dedekind[1]. The latter is a natural conjecture which is false if $\mathscr{A}$ is replaced by $\mathscr{V}$.

2. Valuation ideals and prime powers. In [8; p. 341], it is observed that if D is a Dedekind domain, then $\mathscr{V} = \mathscr{Q}$. The converse is clearly false. In fact, it is proved in [5; Th. 3.1, Th. 3.8] that the domain D with identity has the property $\mathscr{V} = \mathscr{Q}$ if and only if D is a one-dimensional Prüfer domain.

Because an ideal of a Dedekind domain is primary if and only if it is a prime power, we also have $\mathscr{V}(D) = \mathscr{P}(D)$, the set of prime powers of D, if D is Dedekind. Theorem 1 gives necessary and sufficient conditions on a domain with identity in order that $\mathscr{V} \subseteqq \mathscr{P}$. In particular, an example in this section shows that such a domain need not be Dedekind.

THEOREM 1. *Let D be an integral domain with identity. Let $\mathscr{P}$ be the set of prime powers of D, $\mathscr{V}$ the set of valuation ideals of D, and let k be the quotient field of D. $\mathscr{V} \subseteqq \mathscr{P}$ if and only if the following conditions hold:*

(i) *If P is a nonzero proper prime ideal of D, $\bigcap_{n=0}^{\infty} P^n = (0)$ and the function $v_p: D - \{0\} \to Z$ defined by $v_p(x) = i$ if $x \in P^i - P^{i+1}$ can be extended to a valuation of k.*

(ii) *Every valuation of k finite on D is isomorphic to some v_p.*

Proof. We first show that D is one-dimensional. Thus suppose P_1, P_2 are prime ideals of D such that $(0) \subset P_1 \subset P_2 \subset D$. There exists a valuation ring D' containing prime ideals M_1, M_2 such that $M_i \cap D = P_i$ [6; p. 37]. There is no loss of generality in assuming $M_1 = \sqrt{dD'} = \sqrt{P_1 D'}$ for some element d of P_1. This implies $M_1 = \sqrt{d^k D'}$ for any k. Now $d^2 D' \cap D \subset dD' \cap D$ and $\sqrt{d^2 D'} \cap D = P_1$. Because $\mathscr{V} \subseteqq \mathscr{P}$, $d^2 D' \cap D = P_1^r \subset dD' \cap D = P_1^s$ for some r, s with $s < r$. Hence, $P_1^r D' \neq P_1 D'$ and in particular, $P_1 \supsetneqq P_1^2 D'$. We choose $p \in P_1 - P_1^2 D'$. Then $P_1^2 \subseteqq P_1^2 D' \cap D \subset pD' \cap D \subseteqq P_1 D' \cup D$. This implies $pD' \cap D = P_1$ and consequently $P_1 D' = pD'$. Now if $r \in P_2 - P_1$ we have $rD' \supset pD'$. Hence $P_1 D' = pD' \supset rpD' \supset p^2 D' = P_1^2 D'$. It follows that $P_1 \supset rpD' \cap D \supset p^2 D' \cap D \supseteqq P_1^2$. This contradicts the assumption that $\mathscr{V} \subseteqq \mathscr{P}$. Hence D is one-dimensional.

[1]An integral domain J with identity is said to be a *Prüfer domain* if J_P is a valuation ring for each prime ideal P of J. J is *almost Dedekind* if J_P is a valuation ring for each prime P of J.

Now let P be a nonzero proper prime ideal of D and let v be a valuation of k finite on D and having center P on D. If D_v is the valuation ring of v and if $P_v = \sqrt{PD_v}$, then by passage to $(D_v)_{P_v}$ we may assume v is of rank one. If p is a nonzero element of P, then $p^2D_v \cap D = P^s \subset P$ for some integer s. Thus $P^sD_v \subset PD_v$. This implies the powers of PD_v properly descend, for if $P^tD_v = P^{t+1}D_v$, then P^tD_v is an idempotent ideal of a valuation ring. Hence P^tD_v is prime, [5; Lemma 2.10], $P^tD_v = PD_v$, and $PD_v = P^sD_v$ — a contradiction.

We next show that $\mathscr{P} \subseteqq \mathscr{V}$. In fact, we will show by induction that P^n is a v-ideal for all n. Thus if P^r is a v-ideal and if $t \in P^{r+1}D_v - P^{r+2}D_v$, then $P^r = P^rD_v \cap D \supset P^{r+1}D_v \cap D \supseteqq tD_v \cap D \supset P^{r+2}D_v \cap D \supseteqq P^{r+2}$. Hence, since $\mathscr{V} \subseteqq \mathscr{P}$, $tD_v \cap D$ must equal P^{r+1} so that P^{r+1} is a v-ideal. We have shown in the process of the proof that if $x \in P^t - P^{t+1}$, $y \in P^m - P^{m+1}$, then $xD_v = P^tD_v$, $yD_v = P^mD_v$ so that $xyD_v = P^{m+t}D_v \supset P^{m+t+1}$. Whence $xy \in P^{m+t} - P^{m+t+1}$. Hence (i) holds.

We proceed to show $D_{v_p} = D_v$. Since D_v has rank one, it suffices to show $D_v \subseteqq D_{v_p}$. Thus let $x/y \in D_v$ where $y \in P^t - P^{t+1}$. Then $x = (x/y)y \in yD_v = P^tD_v$. Hence $v_p(x) \geqq t = v_p(y)$ so that $x/y \in D_{v_p}$. Therefore $D_{v_p} = D_v$.

Finally, we show $\{v_p\}$ is the set of nontrivial valuations of k finite on D. Thus suppose D_w is the valuation ring of a valuation w of k having center $P \subset D$ on D. As shown previously, if $P_w = \sqrt{PD_w}$, P_w is minimal in D_w and $(D_w)_{Pw} = D_{v_p}$. Consequently, $P_w = M_{v_p}$, the maximal ideal of D_{v_p}. We show that the assumption $D_w \subset D_{v_p}$ leads to a contradiction. Thus if M_w is the maximal ideal of D_w, then $M_w \supset M_{v_p}$. Hence there exists $\xi = a/b \in D_w$ such that ξ is a unit of D_{v_p}, but not of D_w. This implies there exists $r > 0$ such that $a, b \in P^r - P^{r+1}$ and $a^2D_w\, 56\, baD_w \subset b^2D_w \subseteqq P^{2r}D_w$. To complete the proof we notice $a^2D_w \supseteqq P^{2r+1}D_w$. This follows from a more general result: For any k, $P^kD_w \cap D = P^k$ since $P^kD_w \cap D \subseteqq P^kD_{v_p} \cap D = P^k$. Hence $P^{2r+1} = P^{2r+1}D_w \cap D \subseteqq a^2D_w \cap D\, 56\, baD_w \cap D \subset b^2D_w \cap D \subseteqq P^{2r}$. This contradiction to the assumption $\mathscr{V} \subseteqq \mathscr{P}$ shows $D_w = D_{v_p}$ so that w and v_p are isomorphic.

This shows (i) and (ii) are necessary in order that $\mathscr{V} \subseteqq \mathscr{P}$. Obviously (i) and (ii) are sufficient.

COROLLARY 1. *Using the notation of Theorem 1, if* $\mathscr{V} \subseteqq \mathscr{P}$, *then* $\mathscr{V} = \mathscr{P}$ *and D is one-dimensional.*

The following example shows that $\mathscr{V} \subseteqq \mathscr{P}$ does not imply D is Dedekind. In fact, D need not be *almost Dedekind* in the sense of [3].

Let R be a rank one discrete valuation ring with maximal ideal

M. Suppose also the $R = K + M$ where K is a proper algebraic extension field over the subfield k (we may take $R\,4\,(K[X])_{(X)}$, for example). If $D = k + M$, then D is a one-dimensional quasi-local domain with maximal ideal M, but D is not a valuation ring [5; Prop. 5.1]. Clearly (i) holds in D. Because K is algebraic over k, R is the integral closure of D. Since R has rank one, R is the only nontrivial valuation ring containing D and contained in the quotient field of D. Hence (ii) holds. But $R = D_{v_M} \cap D$.

By a slight modification of the example just given we see that (ii) is independent of (i). For if we take $K = F(Y)$ where F is a field and Y is an indeterminate over F, then $F + M$ satisfies (i) but not (ii).

3. A certain set of ideals containing $\mathscr{V}$.

The first example of §2 shows that a domain in which $\mathscr{V} \subseteq \mathscr{P}$ need not be almost Dedekind. Also, numerous examples shows that $\mathscr{Q} \subseteq \mathscr{V}$ does not imply D is Prüfer. But by considering a certain set, to be denoted by $\mathscr{A}$, which contains both $\mathscr{V}$ and $\mathscr{Q}$, we obtain both these results by replacing $\mathscr{V}$ by $\mathscr{A}$ and $\mathscr{Q}$ by $\mathscr{A}$, respectively. The set $\mathscr{A}$ to which we refer consists of all ideals A such that the complement of P is prime to A for some prime ideal P[2]. We shall consistently use the fact that if A and P are ideals of the commutative ring R such that $A \subseteq P$ and P is prime, then the smallest ideal B of R such that B contains A and such that $R - P$ is prime to B is $B = A_P = \{x \mid x \in R,\ xm \in A$ for some $m \notin P\}$. More to the point as far as we are concerned, $R - P$ is prime to the ideal A if and only if $AD_P \cap D = A$ (D a domain).

The following theorem gives the relationship between the sets $\mathscr{A}$ and $\mathscr{V}$.

THEOREM 2. *Let D be an integral domain with identity. Then $\mathscr{V} \subseteq \mathscr{A}$. $\mathscr{V} = \mathscr{A}$ if and only if D is a Prüfer domain.*

Proof. It is easy to see that if A is a v-ideal, the complement of the center of v on D is prime to A. Hence $\mathscr{V} \subseteq \mathscr{A}$.

Obviously $\mathscr{V} = \mathscr{A}$ if D is Prüfer. Conversely, if $\mathscr{A} \subseteq \mathscr{V}$ and if P is a proper prime ideal of D, we shall show D_P is a valuation ring and hence that D is Prüfer. Thus if x, y are nonzero elements of D, we let $A = (xy)_P$. $A \in \mathscr{A}$, so $A \in \mathscr{V}$ and therefore $x^2 \in A$ or $y^2 \in A$. If, say, $x^2 \in A$, then $x^2 m = dxy$ for some $m \in D - P$, $d \in D$. Hence $x/y = d/m \in D_P$. This proves the theorem.

[2]If A is an ideal of the commutative ring R and $x \in R$, we say *x is prime to A* if $ax \in A$, $a \in R$, implies $a \in A$ [7; p. 223]. A subset N of R is prime to A if each element of N is prime to A.

Before proceeding to consider the relation $\mathscr{A} \subseteqq \mathscr{P}$ we note that this condition is meaningful in a ring with zero divisors. Also, the relation $\mathscr{A} \subseteqq \mathscr{Q}$ is meaningful for arbitrary commutative rings. We consider this case. First we need some definitions.

Suppose R is a commutative ring. R is a *primary ring*[3] if R contains at most two prime ideals [1]. A *primary domain* is a primary ring without proper divisors of zero. R is called a *u-ring* if the only ideal A of R such that $\sqrt{A} = R$ is R itself. *R satisfies Condition* (*) if $\mathscr{S}(R)$, the set of ideals of R with prime radical, is a subset of $\mathscr{Q}(R)$.

Theorem 1 of [2] states: A ring R satisfies (*) if and only if R is one of the following:

(a) a primary domain.

(b) a ring, every element of which is nilpotent.

(c) a zero-dimensional u-ring.

or (d) a one-dimensional u-ring having the property that if P and M are prime ideals of R such that $P \subset M \subset R$, then $(0)_M = P$.

From this result, it is clear that if R satisfies (*), then every ideal of R_P is primary for each prime ideal P of R. But because of the one-to-one correspondence between primary ideals of R contained in P and primary ideals of R_P, we see that $\mathscr{A} \subseteqq \mathscr{Q}$ if and only if every ideal of R_P is primary for each prime P of R. Hence, if R satisfies (*), then $\mathscr{A} \subseteqq \mathscr{Q}$. The converse is false, as can be seen by considering the ring of even integers. The converse is true, however, in a ring with identity or, more generally, in a u-ring as the following theorem shows:

THEOREM 3. *Let R be a u-ring. If $\mathscr{A} \subseteqq \mathscr{Q}$, then R satisfies* (*).

Proof. Suppose P and M are prime ideals of R such that $P \subset M \subset R$. We let $p \in P$ and $m \in M - P$. The ideal $A = (mp)_M$ is a in $\mathscr{A}$ and is therefore primary. Since $m \notin P \supseteqq \sqrt{A}$, $p \in A$. Therefore $py = rmp + kmp$ for some $y \notin M$, $r \in R$, $k \in Z$ and $p(y - rm - km) = 0$. Further $y - rm - km \equiv y \not\equiv 0 \pmod{M}$ and because P and M are arbitrary, R has dimension $\leqq 1$. That R satisfies (*) now follows.

Similarly, if $\mathscr{P}$ denotes the set of prime powers of the ring R, then because any ideal of R_P is the extension of its contraction in R [7; p. 223], every ideal of R_P is a prime power for each prime ideal P of R if $\mathscr{A} \subseteqq \mathscr{P}$.

In view of Theorem 12 and 14 of [4], we may then state

[3]For the case of a ring with identity, this definition agrees with terminology of Zariski-Samuel [7; p. 204]. But unlike the case of a ring with identity, an ideal of a primary ring need not be a primary ideal.

THEOREM 4. *Suppose R is a u-ring. The following are equivalent conditions*:

(a) $\mathscr{A} \subseteqq \mathscr{P}$,

(b) *every ideal of R with prime radical is a prime power*

and (c) *R satisfies (*) and primary ideals of S are prime powers.*

COROLLARY 2. *Let D be an integral domain with identity. $\mathscr{A} \subseteqq \mathscr{P}$ if and only if D is almost Dedekind.*

In terms of $\mathscr{S}$, the set of ideals of R having prime radical, Theorem 4 can be stated thusly:

THEOREM 5. *Suppose R is a u-ring. The following are equivalent conditions*:

(a) $\mathscr{A} \subseteqq \mathscr{P}$,

(b) $\mathscr{S} \subseteqq \mathscr{P}$,

(c) $\mathscr{A} \subseteqq \mathscr{Q} \subseteqq \mathscr{P}$.

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