Pacific Journal of Mathematics

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Vol. 15, No. 2 October 1965

PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

SOME CONTAINMENT RELATIONS BETWEEN CLASSES OF IDEALS OF A COMMUTATIVE RING

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The first section of this paper is devoted to proving the following theorem. Let D be an integral domain with identity. Let $\mathscr{P}$ be the set of prime powers of D, $\mathscr{V}$ the set of valuation ideals of D, and let k be the quotient field of D. $\mathscr{V} \subseteqq \mathscr{P}$ if and only if the following conditions hold: (i) Each prime ideal P of D defines a P-adic valuation in the sense of van der Waerden, and (ii) every valuation of k finite on D is isomorphic to a P-adic valuation for some P.

The second section considers three additional sets of ideals; the set $\mathscr{Q}$ of primary ideals, the set $\mathscr{S}$ of semi-primary ideals, and the set $\mathscr{A}$ of ideals A such that the complement of some prime ideal is prime to A.

Commutative rings in which various containment relations exist between the sets $\mathscr{V}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{A}$, and $\mathscr{S}$ are also considered. Most of the results of this section represent applications of previous results of the author.

Let D be an integral domain with identity having quotient field K. An ideal A of D is said to be a *valuation ideal* provided there exists a valuation ring D_v with $D \subseteqq D_v \subseteqq K$ such that $AD_v \cap D = A$. More specifically, if D_v is the valuation ring of the valuation v of K, we may say A is a *v-ideal*. We denote by $\mathscr{T}(D)$ the set of valuation ideals of the domain D and by $\mathscr{Q}(D)$ the set of primary ideals of D. Where no ambiguity exists we may speak of $\mathscr{V}$ and $\mathscr{Q}$.

This paper is closely related to a paper of Gilmer and Ohm [5], and frequent reference is made to their results. In [5] the relations $\mathscr{V} \subseteqq \mathscr{Q}$, $\mathscr{V} = \mathscr{Q}$, and $\mathscr{Q} \subseteqq \mathscr{V}$ were investigated. That paper arose as a result of the following observation in [8, p. 341]:

If D is a Dedekind domain, then $\mathscr{V} = \mathscr{Q}$. But if D is Dedekind, the sets $\mathscr{P}(D)$ of prime powers of D and $\mathscr{Q}(D)$ coincide. Hence if D is Dedekind $\mathscr{V} = \mathscr{P}$. In § 2 necessary and sufficient conditions are given on a domain D in order that $\mathscr{V} \subseteqq \mathscr{P}$. In particular it is shown that $\mathscr{V} \subseteqq \mathscr{P}$ implies $\mathscr{V} = \mathscr{P}$.

In § 3 we consider the set $\mathscr{A}(R)$ consisting of all ideals A of the commutative ring R such that $R - P$ is prime to A for some prime ideal P of R. It is always true that $\mathscr{Q}(R) \subseteqq \mathscr{A}(R)$ and if R is an integral domain with identity, we also have $\mathscr{V}(R) \subseteqq \mathscr{A}(R)$. The

Received April 14, 1964.

relations $\mathscr{A}(R) \subseteqq \mathscr{Q}(R)$, $\mathscr{A}(R) \subseteqq \mathscr{P}(R)$ are investigated in §3. In particular, if R is an integral domain with identity then $\mathscr{A} \subseteqq \mathscr{V}$ if and only if R is a Prüfer domain[1] and $\mathscr{A} \subseteqq \mathscr{P}$ if and only if R is almost Dedekind[1]. The latter is a natural conjecture which is false if $\mathscr{A}$ is replaced by $\mathscr{V}$.

2. Valuation ideals and prime powers.

In [8; p. 341], it is observed that if D is a Dedekind domain, then $\mathscr{V} = \mathscr{Q}$. The converse is clearly false. In fact, it is proved in [5; Th. 3.1, Th. 3.8] that the domain D with identity has the property $\mathscr{V} = \mathscr{Q}$ if and only if D is a one-dimensional Prüfer domain.

Because an ideal of a Dedekind domain is primary if and only if it is a prime power, we also have $\mathscr{V}(D) = \mathscr{P}(D)$, the set of prime powers of D, if D is Dedekind. Theorem 1 gives necessary and sufficient conditions on a domain with identity in order that $\mathscr{V} \subseteqq \mathscr{P}$. In particular, an example in this section shows that such a domain need not be Dedekind.

THEOREM 1. *Let D be an integral domain with identity. Let $\mathscr{P}$ be the set of prime powers of D, $\mathscr{V}$ the set of valuation ideals of D, and let k be the quotient field of D. $\mathscr{V} \subseteqq \mathscr{P}$ if and only if the following conditions hold:*

(i) *If P is a nonzero proper prime ideal of D, $\bigcap_{n=0}^{\infty} P^n = (0)$ and the function $v_p\colon D - \{0\} \to Z$ defined by $v_p(x) = i$ if $x \in P^i - P^{i+1}$ can be extended to a valuation of k.*

(ii) *Every valuation of k finite on D is isomorphic to some v_p.*

Proof. We first show that D is one-dimensional. Thus suppose P_1, P_2 are prime ideals of D such that $(0) \subset P_1 \subset P_2 \subset D$. There exists a valuation ring D' containing prime ideals M_1, M_2 such that $M_i \cap D = P_i$ [6; p. 37]. There is no loss of generality in assuming $M_1 = \sqrt{dD'} = \sqrt{P_1 D'}$ for some element d of P_1. This implies $M_1 = \sqrt{d^k D'}$ for any k. Now $d^2D' \cap D \subset dD' \cap D$ and $\sqrt{d^2 D'} \cap D = P_1$. Because $\mathscr{V} \subseteqq \mathscr{P}$, $d^2D' \cap D = P_1^r \subset dD' \cap D = P_1^s$ for some r, s with $s < r$. Hence, $P_1^r D' \neq P_1 D'$ and in particular, $P_1 \subsetneqq P_1^2 D'$. We choose $p \in P_1 - P_1^2 D'$. Then $P_1^2 \subseteqq P_1^2 D' \cap D \subset pD' \cap D \subseteqq P_1 D' \cup D$. This implies $pD' \cap D = P_1$ and consequently $P_1 D' = pD'$. Now if $r \in P_2 - P_1$ we have $rD' \supset pD'$. Hence $P_1 D' = pD' \supset rpD' \supset p^2 D' = P_1^2 D'$. It follows that $P_1 \supset rpD' \cap D \supset p^2 D' \cap D \supseteqq P_1^2$. This contradicts the assumption that $\mathscr{V} \subseteqq \mathscr{P}$. Hence D is one-dimensional.

[1]An integral domain J with identity is said to be a *Prüfer domain* if J_P is a valuation ring for each prime ideal P of J. J is *almost Dedekind* if J_P is a valuation ring for each prime P of J.

Now let P be a nonzero proper prime ideal of D and let v be a valuation of k finite on D and having center P on D. If D_v is the valuation ring of v and if $P_v = \sqrt{PD_v}$, then by passage to $(D_v)_{P_v}$ we may assume v is of rank one. If p is a nonzero element of P, then $p^2D_v \cap D = P^s \subset P$ for some integer s. Thus $P^sD_v \subset PD_v$. This implies the powers of PD_v properly descend, for if $P^tD_v = P^{t+1}D_v$, then P^tD_v is an idempotent ideal of a valuation ring. Hence P^tD_v is prime, [5; Lemma 2.10], $P^tD_v = PD_v$, and $PD_v = P^sD_v$ — a contradiction.

We next show that $\mathscr{P} \subseteqq \mathscr{V}$. In fact, we will show by induction that P^n is a v-ideal for all n. Thus if P^r is a v-ideal and if $t \in P^{r+1}D_v - P^{r+2}D_v$, then $P^r = P^rD_v \cap D \supset P^{r+1}D_v \cap D \supseteqq tD_v \cap D \supset P^{r+2}D_v \cap D \supseteqq P^{r+2}$. Hence, since $\mathscr{V} \subseteqq \mathscr{P}$, $tD_v \cap D$ must equal P^{r+1} so that P^{r+1} is a v-ideal. We have shown in the process of the proof that if $x \in P^t - P^{t+1}$, $y \in P^m - P^{m+1}$, then $xD_v = P^tD_v$, $yD_v = P^mD_v$ so that $xyD_v = P^{m+t}D_v \supset P^{m+t+1}$. Whence $xy \in P^{m+t} - P^{m+t+1}$. Hence (i) holds.

We proceed to show $D_{v_p} = D_v$. Since D_v has rank one, it suffices to show $D_v \subseteqq D_{v_p}$. Thus let $x/y \in D_v$ where $y \in P^t - P^{t+1}$. Then $x = (x/y)y \in yD_v = P^tD_v$. Hence $v_p(x) \geqq t = v_p(y)$ so that $x/y \in D_{v_p}$. Therefore $D_{v_p} = D_v$.

Finally, we show $\{v_p\}$ is the set of nontrivial valuations of k finite on D. Thus suppose D_w is the valuation ring of a valuation w of k having center $P \subset D$ on D. As shown previously, if $P_w = \sqrt{PD_w}$, P_w is minimal in D_w and $(D_w)_{P_w} = D_{v_p}$. Consequently, $P_w = M_{v_p}$, the maximal ideal of D_{v_p}. We show that the assumption $D_w \subset D_{v_p}$ leads to a contradiction. Thus if M_w is the maximal ideal of D_w, then $M_w \supset M_{v_p}$. Hence there exists $\xi = a/b \in D_w$ such that ξ is a unit of D_{v_p}, but not of D_w. This implies there exists $r > 0$ such that $a, b \in P^r - P^{r+1}$ and a^2D_w 56 $baD_w \subset b^2D_w \subseteqq P^{2r}D_w$. To complete the proof we notice $a^2D_w \supseteqq P^{2r+1}D_w$. This follows from a more general result: For any k, $P^kD_w \cap D = P^k$ since $P^kD_w \cap D \subseteqq P^kD_{v_p} \cap D = P^k$. Hence $P^{2r+1} = P^{2r+1}D_w \cap D \subseteqq a^2D_w \cap D$ 56 $baD_w \cap D \subset b^2D_w \cap D \subseteqq P^{2r}$. This contradiction to the assumption $\mathscr{V} \subseteqq \mathscr{P}$ shows $D_w = D_{v_p}$ so that w and v_p are isomorphic.

This shows (i) and (ii) are necessary in order that $\mathscr{V} \subseteqq \mathscr{P}$. Obviously (i) and (ii) are sufficient.

COROLLARY 1. *Using the notation of Theorem* 1, *if* $\mathscr{V} \subseteqq \mathscr{P}$, *then* $\mathscr{V} = \mathscr{P}$ *and* D *is one-dimensional.*

The following example shows that $\mathscr{V} \subseteqq \mathscr{P}$ does not imply D is Dedekind. In fact, D need not be *almost Dedekind* in the sense of [3].

Let R be a rank one discrete valuation ring with maximal ideal

M. Suppose also the $R = K + M$ where K is a proper algebraic extension field over the subfield k (we may take $R\,4\,(K[X])_{(X)}$, for example). If $D = k + M$, then D is a one-dimensional quasi-local domain with maximal ideal M, but D is not a valuation ring [5; Prop. 5.1]. Clearly (i) holds in D. Because K is algebraic over k, R is the integral closure of D. Since R has rank one, R is the only nontrivial valuation ring containing D and contained in the quotient field of D. Hence (ii) holds. But $R = D_{v_M} \cap D$.

By a slight modification of the example just given we see that (ii) is independent of (i). For if we take $K = F(Y)$ where F is a field and Y is an indeterminate over F, then $F + M$ satisfies (i) but not (ii).

3. A certain set of ideals containing $\mathscr{V}$.

The first example of §2 shows that a domain in which $\mathscr{V} \subseteqq \mathscr{P}$ need not be almost Dedekind. Also, numerous examples shows that $\mathscr{Q} \subseteqq \mathscr{V}$ does not imply D is Prüfer. But by considering a certain set, to be denoted by $\mathscr{A}$, which contains both $\mathscr{V}$ and $\mathscr{Q}$, we obtain both these results by replacing $\mathscr{V}$ by $\mathscr{A}$ and $\mathscr{Q}$ by $\mathscr{A}$, respectively. The set $\mathscr{A}$ to which we refer consists of all ideals A such that the complement of P is prime to A for some prime ideal P[2]. We shall consistently use the fact that if A and P are ideals of the commutative ring R such that $A \subseteqq P$ and P is prime, then the smallest ideal B of R such that B contains A and such that $R - P$ is prime to B is $B = A_P = \{x \mid x \in R,\ xm \in A$ for some $m \notin P\}$. More to the point as far as we are concerned, $R - P$ is prime to the ideal A if and only if $AD_P \cap D = A$ (D a domain).

The following theorem gives the relationship between the sets $\mathscr{A}$ and $\mathscr{V}$.

THEOREM 2. *Let D be an integral domain with identity. Then $\mathscr{V} \subseteqq \mathscr{A}$. $\mathscr{V} = \mathscr{A}$ if and only if D is a Prüfer domain.*

Proof. It is easy to see that if A is a v-ideal, the complement of the center of v on D is prime to A. Hence $\mathscr{V} \subseteqq \mathscr{A}$.

Obviously $\mathscr{V} = \mathscr{A}$ if D is Prüfer. Conversely, if $\mathscr{A} \subseteqq \mathscr{V}$ and if P is a proper prime ideal of D, we shall show D_P is a valuation ring and hence that D is Prüfer. Thus if x, y are nonzero elements of D, we let $A = (xy)_P$. $A \in \mathscr{A}$, so $A \in \mathscr{V}$ and therefore $x^2 \in A$ or $y^2 \in A$. If, say, $x^2 \in A$, then $x^2m = dxy$ for some $m \in D - P$, $d \in D$. Hence $x/y = d/m \in D_P$. This proves the theorem.

[2]If A is an ideal of the commutative ring R and $x \in R$, we say *x is prime to A* if $ax \in A$, $a \in R$, implies $a \in A$ [7; p. 223]. A subset N of R is prime to A if each element of N is prime to A.

Before proceeding to consider the relation $\mathscr{A} \subseteqq \mathscr{P}$ we note that this condition is meaningful in a ring with zero divisors. Also, the relation $\mathscr{A} \subseteqq \mathscr{Q}$ is meaningful for arbitrary commutative rings. We consider this case. First we need some definitions.

Suppose R is a commutative ring. R is a *primary ring*[3] if R contains at most two prime ideals [1]. A *primary domain* is a primary ring without proper divisors of zero. R is called a *u-ring* if the only ideal A of R such that $\sqrt{A} = R$ is R itself. R *satisfies Condition* (*) if $\mathscr{S}(R)$, the set of ideals of R with prime radical, is a subset of $\mathscr{Q}(R)$.

Theorem 1 of [2] states: A ring R satisfies (*) if and only if R is one of the following:

(a) a primary domain.
(b) a ring, every element of which is nilpotent.
(c) a zero-dimensional u-ring.
or (d) a one-dimensional u-ring having the property that if P and M are prime ideals of R such that $P \subset M \subset R$, then $(0)_M = P$.

From this result, it is clear that if R satisfies (*), then every ideal of R_P is primary for each prime ideal P of R. But because of the one-to-one correspondence between primary ideals of R contained in P and primary ideals of R_P, we see that $\mathscr{A} \subseteqq \mathscr{Q}$ if and only if every ideal of R_P is primary for each prime P of R. Hence, if R satisfies (*), then $\mathscr{A} \subseteqq \mathscr{Q}$. The converse is false, as can be seen by considering the ring of even integers. The converse is true, however, in a ring with identity or, more generally, in a u-ring as the following theorem shows:

THEOREM 3. *Let R be a u-ring. If $\mathscr{A} \subseteqq \mathscr{Q}$, then R satisfies* (*).

Proof. Suppose P and M are prime ideals of R such that $P \subset M \subset R$. We let $p \in P$ and $m \in M - P$. The ideal $A = (mp)_M$ is a in $\mathscr{A}$ and is therefore primary. Since $m \notin P \supseteqq \sqrt{A}$, $p \in A$. Therefore $py = rmp + kmp$ for some $y \notin M$, $r \in R$, $k \in Z$ and $p(y - rm - km) = 0$. Further $y - rm - km \equiv y \not\equiv 0 \pmod{M}$ and because P and M are arbitrary, R has dimension $\leqq 1$. That R satisfies (*) now follows.

Similarly, if $\mathscr{P}$ denotes the set of prime powers of the ring R, then because any ideal of R_P is the extension of its contraction in R [7; p. 223], every ideal of R_P is a prime power for each prime ideal P of R if $\mathscr{A} \subseteqq \mathscr{P}$.

In view of Theorem 12 and 14 of [4], we may then state

[3]For the case of a ring with identity, this definition agrees with terminology of Zariski-Samuel [7; p. 204]. But unlike the case of a ring with identity, an ideal of a primary ring need not be a primary ideal.

THEOREM 4. *Suppose R is a u-ring. The following are equivalent conditions*:

(a) $\mathscr{A} \subseteqq \mathscr{P}$,

(b) *every ideal of R with prime radical is a prime power*

and (c) *R satisfies* (*) *and primary ideals of S are prime powers.*

COROLLARY 2. *Let D be an integral domain with identity.* $\mathscr{A} \subseteqq \mathscr{P}$ *if and only if D is almost Dedekind.*

In terms of $\mathscr{S}$, the set of ideals of R having prime radical, Theorem 4 can be stated thusly:

THEOREM 5. *Suppose R is a u-ring. The following are equivalent conditions*:

(a) $\mathscr{A} \subseteqq \mathscr{P}$,

(b) $\mathscr{S} \subseteqq \mathscr{P}$,

(c) $\mathscr{A} \subseteqq \mathscr{Q} \subseteqq \mathscr{P}$.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics

Vol. 15, No. 2 October, 1965