CERTAIN ALGEBRAS OF DEGREE ONE

FRANK JAMES KOSIER
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In this note the following is proved: Suppose $R$ is a finite-dimensional algebra over an algebraically closed field $F$ of characteristic 0 whose associator satisfies $4(y, x, x) = 4(x, y, x) + [[y, x], x]$ and $(x, x, x) = 0$. If $R$ is simple and non-nil then $R$ is isomorphic to $F$.

We call it Theorem B, and prove it below.

In [3] nonassociative algebras satisfying identities of degree three were studied and it was shown that relative to quasi-equivalence any algebra satisfying such an identity (subject to some rather weak additional hypotheses) must in fact satisfy at least some one of seven particular identities; each of degree three. In this note we concern ourselves with one of these seven residual cases; namely the identity

$$4(y, x, x) = 4(x, y, x) + [[y, x], x]$$

where the associator $(x, y, z)$ is defined by $(x, y, z) = (xy)z - x(yz)$ and the commutator $[x, y]$ by $[x, y] = xy - yx$ for elements $x, y, z$ of the algebra.

Throughout the remainder of this note $R$ will be a ring of characteristic not two or three which satisfies (1) in addition to the following identity:

$$(2) (x, x, x) = 0.$$ 

The following result was established in [3]:

**Theorem A.** Suppose $R$ has an idempotent $e \neq 0, 1$. Then $R$ is not simple.

This reduces the study of simple rings to the consideration of rings whose only nonzero idempotent is the identity element.

**Ideals and Simple rings.** A well-known consequence of (2) is

$$(3) (x, x, y) + (x, y, x) + (y, x, x) = 0.$$ 

We define $x \circ y = xy + yx$ and proceed to simplify (1). We rewrite (1) as

$$(4) 4yx^2 = 4x \circ yx - 3(xy)x - x(xy) - (yx)x + x(yx)$$

and (3) as

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\[2yx^3 = x^2 \circ y + (xy)x + (yx)x - x(yx) - x(xy)\].

Adding (4) and (5) we obtain

\[6yx^3 = x^2 \circ y + 4x \circ yx - 2x \circ xy\].

Finally we add and subtract \(2x \circ yx\) to the right-hand member of (6) giving us

\[6yx^3 = 6x \circ yx - 2x \circ (x \circ y) + x^2 \circ y\].

Replacing \(x\) by \(x_1 + x_2\) in (7) and then using (7) to simplify the result we find

\[6y(x_1 \circ x_2) = 6x_1 \circ yx_3 + 6x_2 \circ yx_1 - 2x_1 \circ (x_2 \circ y) - 2x_2 \circ (x_1 \circ y) + (x_1 \circ x_2) \circ y\].

We define the ring \(R^+\) to be the same additive group as \(R\) but the multiplication in \(R^+\) is given by \((x, y) = 1/2x \circ y\). We set \((x, y, z)^+ = (x \circ y) \circ z - x \circ (y \circ z)\) and note that \(R^+\) is associative if and only if \((x, y, z)^+ = 0\) for all \(x, y, z \in R\).

**Lemma 1.** Let \(L\) be the additive group generated by all \((x, y, z)^+\) where \(x, y, z \in R\). Then \(L\) is a left ideal of \(R\).

**Proof.** First of all we consider \(y[(x_1 \circ x_2) \circ x_3]\). Then (8) (with \(x_i\) replaced by \(x_1 \circ x_i\) and \(x_j\) by \(x_i\)) becomes

\[6y[(x_1 \circ x_2) \circ x_3] = 6(x_1 \circ x_2) \circ yx_3 + 6x_2 \circ y(x_1 \circ x_3) - 2(x_1 \circ x_2) \circ (x_3 \circ y) - 2x_3 \circ [(x_1 \circ x_2) \circ y] + [(x_1 \circ x_2) \circ x_3] \circ y\].

We use (8) to rewrite the second term of the right-hand member of (9) as:

\[6x_5 \circ y(x_1 \circ x_3) = 6x_5 \circ (x_1 \circ yx_3) + 6x_2 \circ (x_1 \circ yx_1) - 2x_5 \circ [x_1 \circ (x_2 \circ y)] - 2x_3 \circ [x_1 \circ (x_2 \circ y)] + x_3 \circ [(x_1 \circ x_2) \circ y]\].

A substitution of this into (9) results in

\[6y[(x_1 \circ x_2) \circ x_3] = 6(x_2 \circ x_3) \circ yx_3 + 6x_3 \circ (x_1 \circ yx_3) + 6x_3 \circ (x_2 \circ yx_1) - 2x_5 \circ [x_1 \circ (x_2 \circ y)] - 2x_3 \circ [x_1 \circ (x_2 \circ y)] - 2(x_1 \circ x_2) \circ (x_3 \circ y) + (x_2, x_1 \circ x_2, y)^+\].

If we interchange \(x_1\) and \(x_3\) in (10) we obtain

\[6y[(x_1 \circ x_2) \circ x_1] = 6(x_5 \circ x_3) \circ yx_1 + 6x_1 \circ (x_3 \circ yx_3) + 6x_1 \circ (x_2 \circ yx_1)

- 2x_1 \circ [x_1 \circ (x_5 \circ y)] - 2x_3 \circ [x_1 \circ (x_2 \circ y)]
- 2(x_3 \circ x_3) \circ (x_1 \circ y) + (x_1, x_5 \circ x_2, y)^+.\]
Then subtracting (11) from (10) yields
\[
6y(x_1, x_2, x_3) = 6(x_1, x_2, yx_3) + 6(x_1, yx_2, x_3) + 6(yx_1, x_3, x_2) \\
- 2(x_1, x_2 \circ y, x_3) + 2(x_1, x_2, x_1 \circ y) \\
- 2(x_1, x_2 \circ x_3, y) + (x_3, x_1 \circ x_2, y) + (x_1, x_3 \circ x_2, y).
\]
Thus \(yL \subseteq L\) and \(L\) is a left ideal of \(R\).

**Theorem 1.** \(L + LR\) is an ideal (two-sided) of \(R\).

**Proof.** As is immediate from Lemma 1 it suffices to show that \(R(LR) + (LR)R \subseteq L + LR\). Suppose \(x_1, x_2 \in R, y \in L\). Then \((x_1, x_2, y)^+ \in L\) so that \((x_1 \circ x_2) \circ y = x_1 \circ (x_2 \circ y) \in L\). But \((x_1 \circ x_2) \circ y\) and \(x_1 \circ x_2 y\) belong to \(L + LR\). Hence, \(x_1 \circ y x_2 \in L + LR\). Next we interchange \(x_1\) and \(y\) in (8), obtaining
\[
6x_2(x_1 \circ y) = 6x_2 \circ x_2 y + 6y \circ x_2 x_1 - 2x_2 \circ (x_1 \circ y) \\
- 2y \circ (x_1 \circ x_2) + (x_1 \circ y) \circ x_2.
\]
But Lemma 1 along with the preceding remarks implies that each term of the right-hand member belongs to \(L + LR\). Hence, \(x_1 \circ y x_2 \in L + LR\). Therefore \(R(LR) + (LR)R \subseteq L + LR\) and \(L + LR\) is an ideal of \(R\).

**Theorem 2.** \(L\) is an ideal of \(L + LR\).

**Proof.** Since \(L\) is a left ideal of \(R\) we need only show that \(L(LR) \subseteq L\). Suppose \(x_1, x_2 \in L, y \in R\). Then (8) implies
\[
2x_1(x_2 y) + 2x_2(x_1 y) = (x_1 \circ x_2) y \in L .
\]
Considering that \((x_1, x_2, y)^+\) and \((x_1, y, x_2)^+\) belong to \(L\) we find
\[
(x_1 \circ x_2) y = x_1(x_2 y) \in L .
\]
and
\[
x_1(x_2 y) - x_2(x_1 y) \in L .
\]
Adding (13) and (14) we obtain \(x_1(x_2 y) + 2x_2(x_1 y) \in L\). This along with (15) implies that \(x_2(x_1 y) \in L\) or \(L(LR) \subseteq L\), as was to be shown.

**Corollary.** If \(R\) is simple then either \(R^+\) is associative or \(L = R\).

**Proof.** If \(R\) is simple then either \(L + LR = 0\) or \(L + LR = R\). In the first instance \(L = 0\) so that \(R\) is associative while in the second
$L$ is an ideal of $R$. Hence, either $L = 0$ or $L = R$.

Now suppose $R$ is a simple finite-dimensional algebra over an algebraically closed field $F$ of characteristic 0. Then $R$ is power-associative [3, Lemma 2] so that if $R$ is nonnil, $R$ must possess a nonzero idempotent $e$. By Theorem A of the Introduction we must, in fact, have $e = 1$, the identity of $R$. A result of Albert's [1] states that $R = F 1 + N$ where all the elements of $N$ are nilpotent and $N$ is an ideal of $R^+$. From this it is immediate that $(x, y, z)^+ \in N$ for all $x, y, z \in R$ so that $L \subseteq N \cong R$. Hence, $L = 0$ and $R^+$ is associative. But then $R$ satisfies

$$2(y, x, x) = 2(x, x, y) + [[y, x], x]$$

(See [3]). Subtracting this relation from (1) we have

$$2(y, x, x) = 4(x, y, x) - 2(x, x, y)$$

which along with (3) implies that $(x, y, x) = 0$. Hence, $R$ is flexible and the results of Theorem B follow from [2].

BIBLIOGRAPHY


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