CERTAIN ALGEBRAS OF DEGREE ONE

FRANK JAMES KOSIER
CERTAIN ALGEBRAS OF DEGREE ONE

FRANK KOSIER

In this note the following is proved: Suppose $R$ is a finite-dimensional algebra over an algebraically closed field $F$ of characteristic 0 whose associator satisfies $4(y, x, x) = 4(x, y, x) + [[y, x], x]$ and $(x, x, x) = 0$. If $R$ is simple and non-nil then $R$ is isomorphic to $F$.

We call it Theorem B, and prove it below.

In [3] nonassociative algebras satisfying identities of degree three were studied and it was shown that relative to quasi-equivalence any algebra satisfying such an identity (subject to some rather weak additional hypotheses) must in fact satisfy at least some one of seven particular identities; each of degree three. In this note we concern ourselves with one of these seven residual cases; namely the identity

(1) \[ 4(y, x, x) = 4(x, y, x) + [[y, x], x] \]

where the associator $(x, y, z)$ is defined by $(x, y, z) = (xy)z - x(yz)$ and the commutator $[x, y]$ by $[x, y] = xy - yx$ for elements $x, y, z$ of the algebra.

Throughout the remainder of this note $R$ will be a ring of characteristic not two or three which satisfies (1) in addition to the following identity:

(2) \( (x, x, x) = 0 \).

The following result was established in [3]:

THEOREM A. Suppose $R$ has an idempotent $e \neq 0, 1$. Then $R$ is not simple.

This reduces the study of simple rings to the consideration of rings whose only nonzero idempotent is the identity element.

Ideals and Simple rings. A well-known consequence of (2) is

(3) \( (x, x, y) + (x, y, x) + (y, x, x) = 0 \).

We define $x \circ y = xy + yx$ and proceed to simplify (1). We rewrite (1) as

(4) \[ 4yx^2 = 4x \circ yx - 3(xy)x - x(xy) - (yx)x + x(yx) \]

and (3) as

Received April 6, 1964.
Adding (4) and (5) we obtain

\[ 6yx^2 = x^2y + 4x \circ yx - 2x \circ xy. \]

Finally we add and subtract \( 2x \circ yx \) to the right-hand member of (6) giving us

\[ 6yx^2 = 6x \circ yx - 2x \circ (x \circ y) + x^3 \circ y. \]

Replacing \( x \) by \( x_1 + x_2 \) in (7) and then using (7) to simplify the result we find

\[ 6y(x_1 \circ x_2) = 6x_1 \circ yx_2 + 6x_2 \circ yx_1 - 2x_1 \circ (x_2 \circ y) \]
\[ -2x_2 \circ (x_1 \circ y) + (x_1 \circ x_2) \circ y. \]

We define the ring \( R^+ \) to be the same additive group as \( R \) but the multiplication in \( R^+ \) is given by \( (x, y) = 1/2x \circ y \). We set \( (x, y, z)^+ = (x \circ y) \circ z - x \circ (y \circ z) \) and note that \( R^+ \) is associative if and only if \( (x, y, z)^+ = 0 \) for all \( x, y, z \in R \).

**Lemma 1.** Let \( L \) be the additive group generated by all \( (x, y, z)^+ \) where \( x, y, z \in R \). Then \( L \) is a left ideal of \( R \).

**Proof.** First of all we consider \( y[(x_1 \circ x_2) \circ x_3] \). Then (8) (with \( x_1 \) replaced by \( x_1 \circ x_2 \) and \( x_2 \) by \( x_3 \)) becomes

\[ 6y[(x_1 \circ x_2) \circ x_3] = 6(x_1 \circ x_2) \circ yx_3 + 6x_3 \circ y(x_1 \circ x_2) - 2(x_1 \circ x_2) \circ (x_3 \circ y) \]
\[ -2x_3 \circ [(x_1 \circ x_2) \circ y] + [(x_1 \circ x_2) \circ x_3] \circ y. \]

We use (8) to rewrite the second term of the right-hand member of (9) as:

\[ 6x_3 \circ y(x_1 \circ x_2) = 6x_3 \circ (x_1 \circ yx_2) + 6x_3 \circ (x_2 \circ yx_1) - 2x_3 \circ [x_1 \circ (x_2 \circ y)] \]
\[ -2x_3 \circ [x_2 \circ (x_1 \circ y)] + x_3 \circ [(x_1 \circ x_2) \circ y]. \]

A substitution of this into (9) results in

\[ 6y[(x_1 \circ x_2) \circ x_3] = 6x_3 \circ (x_1 \circ yx_2) + 6x_3 \circ (x_2 \circ yx_1) + 6x_3 \circ (x_1 \circ yx_2) \]
\[ -2x_3 \circ [x_1 \circ (x_2 \circ y)] - 2x_3 \circ [x_2 \circ (x_1 \circ y)] \]
\[ -2(x_1 \circ x_2) \circ (x_3 \circ y) + (x_1 \circ x_2, x_3)^+. \]

If we interchange \( x_1 \) and \( x_2 \) in (10) we obtain

\[ 6y[(x_3 \circ x_2) \circ x_1] = 6x_3 \circ (x_2 \circ yx_1) + 6x_1 \circ (x_3 \circ yx_2) + 6x_1 \circ (x_2 \circ yx_3) \]
\[ -2x_1 \circ [x_3 \circ (x_2 \circ y)] - 2x_1 \circ [x_2 \circ (x_3 \circ y)] \]
\[ -2(x_3 \circ x_2) \circ (x_1 \circ y) + (x_1, x_3 \circ x_2, y)^+. \]
Then subtracting (11) from (10) yields
\[
6y(x_1, x_2, x_3)^+ = 6(x_1, x_2, yx_3)^+ + 6(x_1, yx_2, x_3)^+ + 6(yx_1, x_2, x_3)^+
- 2(x_1, x_2, y, x_3)^+ + 2(x_3, x_2, x_1 \circ y)^+
- 2(x_1, x_2 \circ x_3, y)^+ + (x_2, x_1 \circ x_2, y) + (x_1, x_3 \circ x_2, y)^+ .
\]

Thus \(yL \subseteq L\) and \(L\) is a left ideal of \(R\).

**Theorem 1.** \(L + LR\) is an ideal (two-sided) of \(R\).

**Proof.** As is immediate from Lemma 1 it suffices to show that \(R(LR) + (LR)R \subseteq L + LR\). Suppose \(x_1, x_2 \in R, y \in L\). Then \((x_1, x_2, y)^+ \in L\) so that \((x_1 \circ x_3) \circ y - x_1 \circ (x_2 \circ y) \in L\). But \((x_1 \circ x_2) \circ y\) and \(x_1 \circ x_2 y\) belong to \(L + LR\). Hence, \(x_1 \circ yx_2 \in L + LR\). Next we interchange \(x_2\) and \(y\) in (8), obtaining
\[
6x_2(x_1 \circ y) = 6x_1 \circ x_2 y + 6y \circ x_2 x_1 - 2x_1 \circ (x_2 \circ y)
- 2y \circ (x_1 \circ x_2) + (x_1 \circ y) \circ x_2 .
\]

But Lemma 1 along with the preceding remarks implies that each term of the right-hand member belongs to \(L + LR\). Hence, \(x_2(x_1 \circ y) \in L + LR\) but \(x_2(x_1 \circ y) \in R(RL) \subseteq L\) so that \(x_2(yx_1) \in L + LR\). Thus, we must also have \((yx_1)x_2 \in L + LR\), since \(x_3 \circ yx_1 \in L + LR\). Therefore \(R(LR) + (LR)R \subseteq L + LR\) and \(L + LR\) is an ideal of \(R\).

**Theorem 2.** \(L\) is an ideal of \(L + LR\).

**Proof.** Since \(L\) is a left ideal of \(R\) we need only show that \(L(LR) \subseteq L\). Suppose \(x_1, x_2 \in L, y \in R\). Then (8) implies
\[
2x_1(x_2 y) + 2x_2(x_1 y) - (x_1 \circ x_2) y \in L .
\]

Considering that \((x_1, x_2, y)^+\) and \((x_1, y, x_2)^+\) belong to \(L\) we find
\[
(x_1 \circ x_3) y - x_1(x_2 y) \in L .
\]

and
\[
x_2(x_1 y) - x_1(x_2 y) \in L .
\]

Adding (13) and (14) we obtain \(x_1(x_2 y) + 2x_2(x_1 y) \in L\). This along with (15) implies that \(x_2(x_1 y) \in L\) or \(L(LR) \subseteq L\), as was to be shown.

**Corollary.** If \(R\) is simple then either \(R^+\) is associative or \(L = R\).

**Proof.** If \(R\) is simple then either \(L + LR = 0\) or \(L + LR = R\). In the first instance \(L = 0\) so that \(R\) is associative while in the second
L is an ideal of R. Hence, either \( L = 0 \) or \( L = R \).

Now suppose \( R \) is a simple finite-dimensional algebra over an algebraically closed field \( F \) of characteristic 0. Then \( R \) is power-associative [3, Lemma 2] so that if \( R \) is nonnil, \( R \) must possess a nonzero idempotent \( e \). By Theorem A of the Introduction we must, in fact, have \( e = 1 \), the identity of \( R \). A result of Albert's [1] states that \( R = F \ 1 + N \) where all the elements of \( N \) are nilpotent and \( N \) is an ideal of \( R^+ \). From this it is immediate that \( (x, y, z)^+ \in N \) for all \( x, y, z \in R \) so that \( L \subseteq N \subseteq R \). Hence, \( L = 0 \) and \( R^+ \) is associative. But then \( R \) satisfies

\[
2(y, x, x) = 2(x, x, y) + [[y, x], x]
\]

(See [3]). Subtracting this relation from (1) we have

\[
2(y, x, x) = 4(x, y, x) - 2(x, x, y)
\]

which along with (3) implies that \( (x, y, x) = 0 \). Hence, \( R \) is flexible and the results of Theorem B follow from [2].

**BIBLIOGRAPHY**


UNIVERSITY OF WISCONSIN AND
SYRACUSE UNIVERSITY
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
Patrick Robert Ahern, *On the generalized F. and M. Riesz theorem* ................. 373
A. A. Albert, *On exceptional Jordan division algebras* ............................... 377
J. A. Anderson and G. H. Fullerton, *On a class of Cauchy exponential series* .......................................................................................................................... 405
Allan Clark, *Hopf algebras over Dedekind domains and torsion in H-spaces* ... 419
John Dauns and D. V. Widder, *Convolution transforms whose inversion functions have complex roots* ................................................................. 427
Ronald George Douglas, *Contractive projections on an L_1 space* ................. 443
Robert E. Edwards, *Changing signs of Fourier coefficients* .......................... 463
Ramesh Anand Gangolli, *Sample functions of certain differential processes on symmetric spaces* .............................................................. 477
Robert William Gilmer, Jr., *Some containment relations between classes of ideals of a commutative ring* ................................................................. 497
Basil Gordon, *A generalization of the coset decomposition of a finite group* ......................................................................................................................... 503
Teruo Ikebe, *On the phase-shift formula for the scattering operator* ............ 511
Makoto Ishida, *On algebraic homogeneous spaces* ........................................ 525
Donald William Kahn, *Maps which induce the zero map on homotopy* ......... 537
Frank James Kosier, *Certain algebras of degree one* ........................................ 541
Betty Kvarda, *An inequality for the number of elements in a sum of two sets of lattice points* ................................................................. 545
Jonah Mann and Donald J. Newman, *The generalized Gibbs phenomenon for regular Hausdorff means* ................................................................. 551
Charles Alan McCarthy, *The nilpotent part of a spectral operator. II* .......... 557
Donald Steven Passman, *Isomorphic groups and group rings* ....................... 561
R. N. Pederson, *Laplace’s method for two parameters* ................................... 585
Tom Stephen Pitcher, *A more general property than domination for sets of probability measures* ................................................................. 597
Arthur Argyle Sagle, *Remarks on simple extended Lie algebras* .................... 613
Arthur Argyle Sagle, *On simple extended Lie algebras over fields of characteristic zero* .......................................................................................... 621
Tôru Saitô, *Proper ordered inverse semigroups* ............................................. 649
Oved Shisha, *Monotone approximation* .......................................................... 667
Indranand Sinha, *Reduction of sets of matrices to a triangular form* ............. 673
Raymond Earl Smithson, *Some general properties of multi-valued functions* .................................................................................................................. 681
Johannes Stuelpnagel, *Euclidean fiberings of solvmanifolds* ......................... 705
Richard Steven Varga, *Minimal Gerschgorin sets* .......................................... 719
James Juei-Chin Yeh, *Convolution in Fourier-Wiener transform* .................... 731