AN INEQUALITY FOR THE NUMBER OF ELEMENTS IN A SUM OF TWO SETS OF LATTICE POINTS

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For a fixed positive integer \( n \), let \( Q \) be the set of all \( n \)-dimensional lattice points \((x_1, \ldots, x_n)\) with each \( x_i \) a nonnegative integer and at least one \( x_i \) positive. A finite nonempty subset \( R \) of \( Q \) is called a fundamental set if for every \((r_1, \ldots, r_n)\) in \( R \), all vectors \((x_1, \ldots, x_n)\) of \( Q \) with \( x_i \leq r_i \), \( i=1, \ldots, n \), are also in \( R \). If \( A \) is any subset of \( Q \) and \( R \) is any fundamental set, let \( A(R) \) denote the number of vectors in \( A \cap R \). Finally, if \( A \) is any proper subset of \( Q \), let the density of \( A \) be the quantity

\[
\alpha = \frac{A(R)}{Q(R) + 1} ,
\]

taken over all fundamental sets \( R \) for which \( A(R) < Q(R) \). Then the theorem proved in this paper can be stated as follows.

**Theorem.** Let \( A \) and \( B \) be subsets of \( Q \), let \( C \) be the set of all vectors of the form \( a, b, \) or \( a+b \) where \( a \in A \) and \( b \in B \), let \( \alpha \) be the density of \( A \), and let \( R \) be any fundamental set such that (1) there exists at least one vector in \( R \) which is not in \( C \), and (2) for each \( b \) in \( B \cap R \) (if any) there exists \( g \) in \( R \) but not in \( C \) such that \( g - b \) is in \( Q \). Then

\[
C(R) \geq \alpha(Q(R) + 1) + B(R) .
\]

It will be seen that for \( n = 1 \) this theorem implies a result of H. B. Mann [2].

Let \( A \) and \( B \) be sets of positive integers, and for any positive integer \( x \) denote by \( A(x) \) the number of integers in \( A \) which are not greater than \( x \). Let the modified density (or Erdős density) of \( A \) be the quantity

\[
\alpha = \frac{A(x)}{x + 1}
\]

where \( k \) is the smallest positive integer not in \( A \). If \( C = A + B \) is the set of all integers of the form \( a, b, \) or \( a+b \), where \( a \) is in \( A \) and \( b \) is in \( B \), and if \( x \) is a positive integer not in \( C \), then Mann has shown [2] that

\[
C(x) \geq \alpha x + B(x) .
\]

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(Actually, Mann's work is sufficient to establish $C(x) \geq a(x + 1) + B(x)$.) We will show that this theorem, with somewhat weaker hypotheses, can be extended to certain sets of $n$-dimensional lattice points.

Let $Q$ be the set of all lattice points $x = (x_1, \ldots, x_n)$ for which each component is a nonnegative integer and at least one component is positive. Define the sum of subsets of $Q$ in the same manner as was done for sets of positive integers, addition of lattice points being done componentwise, and for any subsets $A$ and $B$ of $Q$ let $A - B$ denote the set of all elements of $A$ which are not in $B$. If $A$ and $S$ are subsets of $Q$ and $S$ is finite let $A(S)$ be the number of elements in $A \cap S$. Let $\omega_i$ be that element of $Q$ for which the $i$th component is 1 and the others are 0.

**Definition 1.** A finite nonempty subset $R$ of $Q$ will be called a fundamental set if whenever $r = (r_1, \ldots, r_n)$ is in $R$ then all vectors $x = (x_1, \ldots, x_n)$ of $Q$ such that $x_i \geq r_i$, $i = 1, \ldots, n$, are also in $R$.

**Definition 2.** Let $A$ be any proper subset of $Q$. Then the density of $A$ is the quantity

$$\alpha = \text{glb} \frac{A(R)}{Q(R) + 1},$$

taken over all fundamental sets $R$ for which $A(R) < Q(R)$.

2. Extension of Mann's result. The theorem to be proved can now be stated as follows.

**Theorem.** Let $A$ and $B$ be subsets of $Q$, let $C = A + B$, and let $\alpha$ be the density of $A$. Let $R$ be any fundamental set such that for each $b$ in $B \cap R$ there exists $g$ in $R - C$ such that $g - b$ is in $Q$, and $Q(R - C) \geq 1$. Then

$$C(R) \geq \alpha[Q(R) + 1] + B(R).$$

**Proof.** Let the elements of $Q$ be ordered so that $(x_1, \ldots, x_n) > (y_1, \ldots, y_n)$ if $x_1 > y_1$, or if $x_1 = y_1, \ldots, x_k = y_k, x_{k+1} > y_{k+1}$. Consider a nonempty set $S = R' - R''$, where $R'$ and $R''$ are fundamental sets, and let $\delta_i = (\delta_{i1}, \ldots, \delta_{in}), \ldots, \delta_u = (\delta_{u1}, \ldots, \delta_{un})$ be all the vectors of $S$ such that for each $i = 1, \ldots, n$ and for each $j = 1, \ldots, u$ we have either (1) $\delta_i - \omega_i$ is in $R''$, or (2) $\delta_i - \omega_i = 0 = (0, \ldots, 0)$, or (3) $\delta_{ij} = 0$. There must be at least one such vector in $S$, for $S$ is a nonempty finite set, and hence has a least element (in our ordering). This least element will satisfy the given conditions. Also, it is easily seen that if $(s_1, \ldots, s_n)$ is any vector in $S$ then for at least one of the $\delta_i$ we have $\delta_{ij} \leq s_i$, $i = 1, \ldots, n$.

From this it follows that if for each $j = 1, \ldots, u$ we let
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\[ S_j = \{ s = (s_1, \cdots, s_n) \mid s \in S, s_i \geq \delta_{ji}, i = 1, \cdots, n \}, \]

then \( S = S_1 \cup \cdots \cup S_n \). Also, let \( S'_j = \{ s - \delta_j \mid s \in S_j, s \neq \delta_j \} \) and let \( S' = S'_1 \cup \cdots \cup S'_n \). Each \( S'_j \), and therefore also \( S' \), is either a fundamental set or is empty.

**Lemma 1.** \( Q(S') + 1 \leq Q(S) \).

*Proof of Lemma 1.* The lemma is obvious if \( n = 1 \), since then \( u = 1 \) also. Hence assume \( n \geq 2 \). Let \( \lambda_1 \) be a mapping defined so that

\[ S_j \lambda_1 = \{ s - \delta_{ji}\omega_i \mid s \in S_j, j = 1, \cdots, u \}, \]
\[ S_{\lambda_1} = S_1 \lambda_1 \cup \cdots \cup S_n \lambda_1. \]

Partition \( S \) into sets \( T_{e_2 \cdots e_n} \) such that

\[ T_{e_2 \cdots e_n} = \{ s = (x_1, e_2, \cdots, e_n) \mid s \in S \}, \]

and let

\[ T_{e_2 \cdots e_n} \lambda_1 = \{ s = (x_1, e_2, \cdots, e_n) \mid s \in S \lambda_1 \}, \]
\[ k_{e_2 \cdots e_n} = \max \left( \max_{1 \leq j \leq u} \{ x_j - \delta_{ji}(x_1, e_2, \cdots, e_n) \in S_j \} \right). \]

Then \( Q(T_{e_2 \cdots e_n} \lambda_1) = k_{e_2 \cdots e_n} + 1 \) or \( Q(T_{e_2 \cdots e_n} \lambda_1) + 1 = k_{e_2 \cdots e_n} + 1 \), according as \( 0 \in T_{e_2 \cdots e_n} \lambda_1 \) or \( 0 \in T_{e_2 \cdots e_n} \lambda_1 \), and \( k_{e_2 \cdots e_n} + 1 \leq Q(T_{e_2 \cdots e_n}) \).

Hence \( Q(S \lambda_1) \leq Q(S) \), and \( Q(S \lambda_1) + 1 \leq Q(S) \) if \( 0 \in S \lambda_1 \).

Now define mappings \( \lambda_2, \cdots, \lambda_n \) such that

\[ S_j \lambda_1 \cdots \lambda_{i-1} \lambda_i = \{ s - \delta_{ji}\omega_i \mid s \in S_j \lambda_1 \cdots \lambda_{i-1} \}, \]

\( i = 2, \cdots, n \), and obtain as above

\[ Q(S \lambda_1 \cdots \lambda_i) + \theta_i \leq Q(S \lambda_1 \cdots \lambda_{i-1}) + \theta_{i-1} \leq Q(S) , \]

where \( \theta_i = 0 \) or \( 1 \) according as \( 0 \in S \lambda_1 \cdots \lambda_i \) or \( 0 \in S \lambda_1 \cdots \lambda_i \). This establishes the lemma.

**Definition 3.** A set \( S \) will be said to be of type \( I \) if

1. \( S \) is a fundamental set,
2. \( Q(S - C) \geq 1 \), and
3. for all \( b \) in \( B \cap S \) (if any) and all \( g \) in \( S - C \) we have \( g - b \) contained in \( Q \).

**Definition 4.** A set \( S \) will be said to be of type \( II \) if

1. there exist fundamental sets \( R', R'' \) such that \( S = R' - R'' \),
2. \( B(S) \geq 1 \) and \( Q(S - C) \geq 1 \), and
3. for all \( b \) in \( B \cap S \) and \( g \) in \( S - C \) we have \( g - b \) contained in \( Q \).
Lemma 2. If \( S \) is any set of type II then
\[
C(S) \geq \alpha Q(S) + B(S).
\]

Proof of Lemma 2. Define the sets \( S'_j \) and \( S' \) as above. Let \( b = (b_1, \ldots, b_n) \) be the largest vector such that

1. \( b \) is in \( B \cap S \), and
2. \( b_1 + \cdots + b_n = \max \{x_1 + \cdots + x_n \mid (x_1, \ldots, x_n) \in B \cap S \} \).

Likewise, let \( g = (g_1, \ldots, g_n) \) be the largest vector such that

1. \( g \) is in \( S - C \), and
2. \( g_1 + \cdots + g_n = \max \{y_1 + \cdots + y_n \mid (y_1, \ldots, y_n) \in S - C \} \).

Let \( B(S) = \rho \geq 1 \), \( Q(S - C) = \sigma \geq 1 \), \( Q(S' - A) = \tau \). The set \( \{g - x \mid x \in B \cap S \} \) contains \( \rho \) elements of \( Q \) (Definition 4, part 3), none of which is in \( A \). We show that these are in \( S' \): If \( x = (x_1, \ldots, x_n) \) is in \( B \cap S \) then \( x \) is in \( S_j \) for some \( j \) such that \( 1 \leq j \leq n \). Hence \( b_j \leq x_i \leq g_i \) for all \( i = 1, \ldots, n \), and \( g \) is in \( S_j \). \( 0 \neq g - x = (g - \delta_j) - (x - \delta_j) \). But \( g - \delta_j \) is in \( S'_j \) and \( S'_j \) is a fundamental set. Hence \( g - x \) is in \( S'_j \), therefore in \( S' \).

Likewise, the (possibly empty) set \( \{y - b \mid y \in S - C, y \neq g \} \) contains \( \sigma - 1 \) elements, all of which are in \( S' - A \). We must show that the two sets are disjoint. Hence suppose that for some \( y \neq g \) and, therefore, \( x \neq b \), we have
\[
g - x = y - b.
\]

Equating the \( i \)th components and transposing gives the \( n \) equations
\[
\begin{align*}
g_1 + b_1 &= y_1 + x_1 \\
g_2 + b_2 &= y_2 + x_2 \\
&\vdots \\
g_n + b_n &= y_n + x_n
\end{align*}
\]
and
\[
g_1 + \cdots + g_n + b_1 + \cdots + b_n = y_1 + \cdots + y_n + x_1 + \cdots + x_n.
\]

Because of the way in which \( g \) and \( b \) were chosen, this implies
\[
g_1 + \cdots + g_n = y_1 + \cdots + y_n \quad \text{and} \quad b_1 + \cdots + b_n = x_1 + \cdots + x_n.
\]

Therefore \( g > y \) and \( b > x \), and at least one of the \( n \) equations of (N) must fail to hold. We now have
\[
\tau \geq \sigma - 1 + \rho,
\]
\[
Q(S) - \sigma \geq Q(S) - \tau - 1 + \rho,
\]
\[
Q(S) - \sigma \geq Q(S') - \tau + Q(S) - Q(S') - 1 + \rho.
\]
We recall that \( Q(S) - Q(S') - 1 \geq 0 \), and that \( S' \) is a fundamental set. Hence

\[
C(S) \geq A(S') + Q(S) - Q(S') - 1 + B(S)
\]

\[
\geq \alpha[Q(S') + 1] + \alpha[Q(S) - Q(S') - 1] + B(S)
\]

\[
= \alpha Q(S) + B(S).
\]

**Lemma 3.** If \( S \) is any set of type I then

\[
C(S) \geq \alpha[Q(S) + 1] + B(S).
\]

**Proof of Lemma 3.** (i) Suppose \( B(S) = 0 \). Then

\[
C(S) = A(S) \geq \alpha[Q(S) + 1] + B(S).
\]

(ii) Suppose \( B(S) \geq 1 \). Define \( b \) and \( g \) as in the proof of Lemma 2. Let \( B(S) = \rho \), \( Q(S - C) = \sigma \), \( Q(S - A) = \tau \). Again the two sets \( \{g - x \mid x \in B \cap S\} \) and \( \{y - b \mid y \in S - C, y \neq g\} \) give \( \sigma - 1 + \rho \) elements not in \( A \), which now will be in \( S \). Also \( g \) is in \( S - C \), hence is in \( S - A \), but is in neither of the two sets above. This implies that

\[
\tau \geq \sigma + \rho,
\]

\[
Q(S) - \sigma \geq Q(S) - \tau + \rho,
\]

\[
C(S) \geq A(S) + B(S) \geq \alpha[Q(S) + 1] + B(S).
\]

We can now return to the proof of the theorem. Let \( R \) be any fundamental set satisfying the hypotheses of the theorem. We will use induction on the number of elements in \( R - C \).

(i) Let \( Q(R - C) = 1 \). Then \( R \) is a set of type I, and we may apply Lemma 3.

(ii) Assume the the theorem holds for any fundamental set \( R' \) satisfying the hypotheses of the theorem and such that \( Q(R' - C) < k \), \( k \geq 2 \), and let \( Q(R - C) = k \). If \( B(R) = 0 \) then \( R \) is of type I, so assume \( B(R) \geq 1 \).

Let \( g_1, g_2, \ldots, g_k \) be the \( k \) vectors in \( R - C \), \( T_j = \{x \mid x = g_j \text{ or } g_j - x \in Q\}, j = 1, \ldots, k \). If \( b \in T_j \) for all \( j = 1, \ldots, k \) and all \( b \in B \cap R \) then again \( R \) is of type I, so assume (by re-numbering, if necessary) that \( B(R - T_j) > 0 \). Let \( J \) be the maximum \( j \) such that \( B(R - (T_1 \cup \cdots \cup T_j)) > 0 \). Then \( b \in B \) and \( b \in R - (T_1 \cup \cdots \cup T_j) \) implies \( b \in T_{j+1} \). We observe that \( J < k \), since \( b \in R - (T_1 \cup \cdots \cup T_k) \) would imply that there does not exist \( g \) in \( R - C \) such that \( g - b \) is in \( Q \), contrary to hypothesis. Also, \( g_{j+1} \in T_{j+1} \cup \cdots \cup T_k \).

Let \( W_0 = T_1 \cup \cdots \cup T_j \). If \( R - W_0 \) is not of type II, there exists \( b \in B \cap T_{j+1} \), and a subscript \( i \) such that \( i > J + 1 \), \( b \in T_{i} \). Let \( W_i = W_{i-1} \cup T_{i} \). If \( R - W_i \) is not of type II, we may repeat the above
to form $W_2 = W_1 \cup T_a$, and so on. Eventually we must obtain a set
$W_m$ such that $R - W_m$ is of type II, $m \geq 0$.

But $W_m$ is a fundamental set satisfying the hypotheses of the
theorem, and $Q(W_m - C) < k$ since $g_{j+1} \notin W_m$. Hence

$$C(W_m) \geq \alpha [Q(W_m) + 1] + B(W_m).$$

Also,

$$C(R - W_m) \geq \alpha Q(R - W_m) + B(R - W_m).$$

Adding, we obtain

$$C(R) \geq \alpha [Q(R) + 1] + B(R).$$

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