

# Pacific Journal of Mathematics

**THE GENERALIZED GIBBS PHENOMENON FOR REGULAR  
HAUSDORFF MEANS**

JONAH MANN AND DONALD J. NEWMAN

## THE GENERALIZED GIBBS PHENOMENON FOR REGULAR HAUSDORFF MEANS

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One says that the means  $\sigma_n(x)$ , of the Fourier series of a function  $f(x)$ , exhibit the (generalized) Gibbs phenomenon at the point  $x = x_0$  if the interval between the upper and lower limit of  $\sigma_n(x)$ , as  $n \rightarrow \infty$  and  $x \rightarrow x_0$  independently, contains points outside the interval between the upper and lower limits of  $f(x)$  as  $x \rightarrow x_0$ . *Theorem.* In order that the Hausdorff summability method given by  $g(t)$  not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that  $1 - g(t)$  be positive definite. A new inequality which must be satisfied by  $g(t)$ , whenever  $1 - g(t)$  is positive definite, is  $\operatorname{Re} z \int_0^1 (1 - zt)^n dg(t) \geq 0$  where  $z = 1 - e^{ix}$ .

This generalized definition of Gibbs phenomenon is an extension of the classical one, and is due to Kuttner [4].

Whereas originally the phenomenon was investigated for functions having a simple discontinuity at the point considered, he includes any Lebesgue integrable function. Kuttner proved the following:

**THEOREM.** *In order that a given  $K$ -method [3, P. 56] not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that the kernel  $K_n(x)$  be bounded below.*

Here  $K_n(x)$  are the means of the series  $1/2 + \cos x + \cos 2x + \dots$ . For regular Hausdorff means [10] (which, being triangular, are  $K$ -methods) the kernel takes the form

$$K_n(x) = \operatorname{Im} \frac{e^{ix/2}}{2 \sin x/2} \int_0^1 (1 - t + te^{ix})^n dg(t)$$

where  $g(t)$  is of bounded variation in  $0 \leq t \leq 1$ ,  $g(0+) = g(0) = 0$ , and  $g(1) = 1$ . We find it useful to let  $g(t)$  be normalized in  $0 \leq t \leq 1$ , and to define it outside this interval by  $g(t) = 1$  for  $t > 1$  and  $g(-t) = g(t)$ .

**THEOREM.** *In order that the Hausdorff summability method given by  $g(t)$  not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that  $1 - g(t)$  be positive definite.*

(For the status of the corresponding problem for the classical Gibbs phenomenon, see [8], [7], and [6].)

*Proof of necessity.* We shall show that  $K_n(x)$  is not bounded below if  $1 - g(t)$  is not positive definite. Since  $|1 - t + te^{ix}| \leq 1$ ,  $\text{Im} \int_0^1 (1 - t + te^{ix})^n dg(t)$  is bounded, and it suffices to consider

$$h_n(x) = \text{Im} \cot x/2 \int_0^1 (1 - t + te^{ix})^n dg(t).$$

Let  $1 - t + te^{ix} = Re^{i\alpha}$ . Therefore  $R \cos \alpha = 1 - t + t \cos x$ ,  $R \sin \alpha = t \sin x$ ,  $R^2 = 1 - 2t(1 - t)(1 - \cos x)$ , and

$$\tan(x/2)h_n(x) = \text{Im} \int_0^1 R^n e^{in\alpha} dg(t) = \int_0^1 R^n \sin n\alpha dg(t).$$

We now choose a sequence of  $n$  and  $x$  so that  $nx \rightarrow A < \infty$ ,  $A$  to be specified later, as  $n \rightarrow \infty$  and  $x \rightarrow 0$ . Szász [8] shows that  $1 - R^n = \lambda n(1 - R^2)$  where  $0 < \lambda < 1$ , and

$$\sin n\alpha - \sin ntx = 2 \cos n(\alpha + tx)/2 \cdot \sin 0(ntx^3).$$

Since  $1 - R^2 < x^2$  and  $nx \rightarrow A$ , it follows that

$$\begin{aligned} \tan(x/2) \cdot h_n(x) &= \int_0^1 \sin ntx dg(t) + 0(x) \\ &= nx \int_0^1 \cos ntx \cdot (1 - g(t)) dt + 0(x). \end{aligned}$$

The last equality is obtained by integrating by parts.

According to the way the definition of  $g(t)$  was extended,

$$\tan(x/2) \cdot h_n(x) = (nx/2) \int_{-\infty}^{\infty} e^{inx t} (1 - g(t)) dt + 0(x).$$

Since  $1 - g(t)$  belongs to  $L^1(-\infty, \infty)$  and is of bounded variation in  $(-\infty, \infty)$ , it follows from Bochner's theorem [2] that its Fourier transform is not always nonnegative. Consequently, there is an  $A_0 > 0$  for which

$$\int_{-\infty}^{\infty} e^{iA_0 t} (1 - g(t)) dt = -B < 0.$$

Then let  $A = A_0$  and obtain  $\tan(x/2) \cdot h_n(x) \rightarrow -A_0 B/2$ . (Taking the limit under the integral sign is permitted by "bounded convergence".) This implies that  $h_n(x) \rightarrow -\infty$ , and completes this part of the proof.

*Proof of sufficiency.* We shall show now that  $K_n(x)$  is not only bounded below when  $1 - g(t)$  is positive definite but is, in fact, positive for all  $n$  and  $x$ .

$$K_n(x) = \text{Im} \frac{e^{ix/2}}{2 \sin x/2} \int_0^1 [1 - (1 - e^{ix})t]^n dg(t) .$$

Let  $z = 1 - e^{ix}$ . Then

$$\begin{aligned} K_n(x) &= \text{Im} \frac{iz}{4 \sin^2 x/2} \int_0^1 (1 - zt)^n dg(t) \\ &= \text{Re} \frac{z}{4 \sin^2 x/2} \int_0^1 (1 - zt)^n dg(t) . \end{aligned}$$

Let  $f_n(t) = (1 - zt)^{n+1}$  in  $0 \leq t \leq 1$  and  $(1 + \bar{z}t)^{n+1}$  in  $-1 \leq t < 0$ . Therefore

$$K_n(x) = \frac{-1}{8(n + 1) \sin^2 x/2} \int_{-1}^1 f'_n(t) dg(t) .$$

It suffices to show that

$$\int_{-1}^1 f'_n(t) dg(t) \leq 0 .$$

Let  $G(t) = f_o(t)e^{-ixt}$ . Since  $G(-t) = G(1 - t)$  for  $0 \leq t \leq 1$ ,  $f_o(t)$  may be defined for  $t, |t| > 1$ , so that  $G(t)$  will be periodic of period 1.

Now

$$\begin{aligned} \int_{-1}^1 G(t)e^{-2\pi ikt} dt &= 2 \int_0^1 [1 - (1 - e^{ix})t] e^{-ixt} e^{-2\pi ikt} dt \\ &= \frac{4(1 - \cos x)}{(x + 2\pi k)^2} . \end{aligned}$$

Consequently

$$G(t) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi ikt} \text{ where each } C_k \geq 0 .$$

$G(t)$ , therefore, is positive definite, and since the product of two positive definite functions is positive definite, it follows that  $f_o(t)$  is positive definite. Also, each  $f_n(t)$  is positive definite if it is defined for  $t, |t| > 1$ , by

$$f_n(t) = e^{i(n+1)xt} [G(t)]^{n+1} .$$

Therefore

$$f_n(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos \lambda_k t + ib_k \sin \lambda_k t)$$

where  $a_k \geq 0$  and  $\lambda_k > 0, k = 1, 2, \dots$ , and

$$f'_n(t) = \sum_{k=1}^{\infty} (-a_k \lambda_k \sin \lambda_k t + ib_k \lambda_k \cos \lambda_k t) ,$$

so that

$$\int_{-1}^1 f'_n(t) dg(t) = \sum_{k=1}^{\infty} \int_{-1}^1 (-a_k \lambda_k \sin \lambda_k t + i b_k \lambda_k \cos \lambda_k t) dg(t) .$$

Since  $f'_n(t)$  is of bounded variation, its Fourier series is boundedly convergent [9, P. 408] and the order of summation and integration may be interchanged [1, P. 74].

$$\begin{aligned} \int_{-1}^1 \cos At dg(t) &= 0 \text{ since } g(t) \text{ is even, and} \\ \int_{-1}^1 \sin At dg(t) &= A \int_{-1}^1 \cos At(1 - g(t)) dt \\ &= A \int_{-\infty}^{\infty} e^{iAt}(1 - g(t)) dt \end{aligned}$$

which is positive for positive  $A$  [2, P. 26] since  $1 - g(t)$  is positive definite and belongs to  $L^1(-\infty, \infty)$ . Finally

$$\int_{-1}^1 f'_n(t) dg(t) \leq 0$$

and the theorem is proved.

This result, about positive kernels, may be compared with Kuttner's result in [5].

It is worth noting that we have proved

$$\operatorname{Re} z \int_0^1 (1 - zt)^n dg(t) \geq 0 ,$$

where  $z = 1 - e^{ix}$ , whenever  $1 - g(t)$  is positive definite. This provides some new inequalities which must be satisfied by a class of positive definite functions which is encountered quite often. For example, when  $n = 1$  and  $x = \pi$ , we obtain

$$\int_0^1 (1 - 2t) dg(t) \geq 0 .$$

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