THE GENERALIZED GIBBS PHENOMENON FOR REGULAR HAUSDORFF MEANS

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One says that the means \( \sigma_n(x) \), of the Fourier series of a function \( f(x) \), exhibit the (generalized) Gibbs phenomenon at the point \( x = x_0 \) if the interval between the upper and lower limit of \( \sigma_n(x) \), as \( n \to \infty \) and \( x \to x_0 \) independently, contains points outside the interval between the upper and lower limits of \( f(x) \) as \( x \to x_0 \). Theorem. In order that the Hausdorff summability method given by \( g(t) \) not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that \( 1 - g(t) \) be positive definite.

A new inequality which must be satisfied by \( g(t) \), whenever \( 1 - g(t) \) is positive definite, is \( \Re \int_0^1 (1 - zt)^n dg(t) \geq 0 \) where \( z = 1 - e^{iz} \).

This generalized definition of Gibbs phenomenon is an extension of the classical one, and is due to Kuttner [4].

Whereas originally the phenomenon was investigated for functions having a simple discontinuity at the point considered, he includes any Lebesgue integrable function. Kuttner proved the following:

**Theorem.** In order that a given K-method [3, P. 56] not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that the kernel \( K_n(x) \) be bounded below.

Here \( K_n(x) \) are the means of the series \( \frac{1}{2} + \cos x + \cos 2x + \cdots \). For regular Hausdorff means [10] (which, being triangular, are K-methods) the kernel takes the form

\[
K_n(x) = \text{Im} \frac{e^{iz/2}}{2 \sin x/2} \int_0^1 (1 - t + te^{iz})^n dg(t)
\]

where \( g(t) \) is of bounded variation in \( 0 \leq t \leq 1 \), \( g(0+) = g(0) = 0 \), and \( g(1) = 1 \). We find it useful to let \( g(t) \) be normalized in \( 0 \leq t \leq 1 \), and to define it outside this interval by \( g(t) = 1 \) for \( t > 1 \) and \( g(-t) = g(t) \).

**Theorem.** In order that the Hausdorff summability method given by \( g(t) \) not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that \( 1 - g(t) \) be positive definite.

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(For the status of the corresponding problem for the classical Gibbs phenomenon, see [8], [7], and [6].

**Proof of necessity.** We shall show that $K_n(x)$ is not bounded below if $1 - g(t)$ is not positive definite. Since $|1 - t + te^{iz}| \leq 1$, Im $\int_0^1 (1 - t + te^{iz})^n dg(t)$ is bounded, and it suffices to consider

$$h_n(x) = \text{Im} \cot x/2 \int_0^1 (1 - t + te^{iz})^n dg(t) .$$

Let $1 - t + te^{iz} = R e^{i\alpha}$. Therefore $R \cos \alpha = 1 - t + t \cos x$, $R \sin \alpha = t \sin x$, $R^2 = 1 - 2t(1 - t)(1 - \cos x)$, and

$$\tan (x/2) h_n(x) = \text{Im} \int_0^1 R^n e^{inz} dg(t) = \int_0^1 R^n \sin n\alpha \, dg(t) .$$

We now choose a sequence of $n$ and $x$ so that $nx \rightarrow A < \infty$, $A$ to be specified later, as $n \rightarrow \infty$ and $x \rightarrow 0$. Szász [8] shows that

$$1 - R^2 = \lambda n(1 - R^2)$$

where $0 < \lambda < 1$, and

$$\sin n\alpha - \sin nx = 2 \cos n(\alpha + tx)/2 \cdot \sin (ntx) .$$

Since $1 - R^2 < x^2$ and $nx \rightarrow A$, it follows that

$$\tan (x/2) \cdot h_n(x) = \int_0^1 \sin nx \, dg(t) + 0(x)$$

$$= nx \int_0^1 \cos nx \cdot (1 - g(t)) \, dt + 0(x) .$$

The last equality is obtained by integrating by parts.

According to the way the definition of $g(t)$ was extended,

$$\tan (x/2) \cdot h_n(x) = (nx/2) \int_{-\infty}^{\infty} e^{inz} (1 - g(t)) \, dt + 0(x) .$$

Since $1 - g(t)$ belongs to $L^1(-\infty, \infty)$ and is of bounded variation in $(-\infty, \infty)$, it follows from Bochner's theorem [2] that its Fourier transform is not always nonnegative. Consequently, there is an $A_0 > 0$ for which

$$\int_{-\infty}^{\infty} e^{i4\alpha t} (1 - g(t)) \, dt = -B < 0 .$$

Then let $A = A_0$ and obtain $\tan (x/2) \cdot h_n(x) \rightarrow -A_0B/2$. (Taking the limit under the integral sign is permitted by "bounded convergence".) This implies that $h_n(x) \rightarrow -\infty$, and completes this part of the proof.

**Proof of sufficiency.** We shall show now that $K_n(x)$ is not only bounded below when $1 - g(t)$ is positive definite but is, in fact, positive for all $n$ and $x$. 


\[ K_n(x) = \text{Im} \frac{e^{ix/2}}{2 \sin x/2} \int_0^1 [1 - (1 - e^{ix})t^n] \, dg(t). \]

Let \( z = 1 - e^{ix} \). Then

\[ K_n(x) = \text{Im} \frac{iz}{4 \sin^2 x/2} \int_0^1 (1 - zt^n) \, dg(t) \]
\[ = \text{Re} \frac{z}{4 \sin^2 x/2} \int_0^1 (1 - zt^n) \, dg(t). \]

Let \( f_n(t) = (1 - zt)^{n+1} \) in \( 0 \leq t \leq 1 \) and \( (1 + \bar{z}t)^{n+1} \) in \( -1 \leq t < 0 \). Therefore

\[ K_n(x) = \frac{-1}{8(n+1) \sin^2 x/2} \int_{-1}^1 f_n'(t) \, dg(t). \]

It suffices to show that

\[ \int_{-1}^1 f_n'(t) \, dg(t) \leq 0. \]

Let \( G(t) = f_0(t) e^{-ixt} \). Since \( G(-t) = G(1 - t) \) for \( 0 \leq t \leq 1 \), \( f_0(t) \) may be defined for \( t, |t| > 1 \), so that \( G(t) \) will be periodic of period 1.

Now

\[ \int_{-1}^1 G(t) e^{-2\pi ik t} \, dt = 2 \int_0^1 [1 - (1 - e^{ix})t] e^{-ixt} e^{-2\pi ik t} \, dt \]
\[ = \frac{4(1 - \cos x)}{(x + 2\pi k)^2}. \]

Consequently

\[ G(t) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi ik t} \] where each \( C_k \geq 0 \).

\( G(t) \), therefore, is positive definite, and since the product of two positive definite functions is positive definite, it follows that \( f_0(t) \) is positive definite. Also, each \( f_n(t) \) is positive definite if it is defined for \( t, |t| > 1 \), by

\[ f_n(t) = e^{i(n+1)xt} [G(t)]^{n+1}. \]

Therefore

\[ f_n(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos \lambda_k t + ib_k \sin \lambda_k t) \]

where \( a_k \geq 0 \) and \( \lambda_k > 0, \ k = 1, 2, \cdots \), and

\[ f_n'(t) = \sum_{k=1}^{\infty} (-a_k \lambda_k \sin \lambda_k t + ib_k \lambda_k \cos \lambda_k t) \]

so that
\[
\int_{-1}^{1} f''_n(t) dg(t) = \sum_{k=1}^{\infty} \int_{-1}^{1} \left( -a_k \lambda_k \sin \lambda_k t + ib_k \lambda_k \cos \lambda_k t \right) dg(t).
\]

Since \( f''_n(t) \) is of bounded variation, its Fourier series is boundedly convergent [9, P. 408] and the order of summation and integration may be interchanged [1, P. 74].

\[
\int_{-1}^{1} \cos At \, dg(t) = 0 \text{ since } g(t) \text{ is even, and}
\]

\[
\int_{-1}^{1} \sin At \, dg(t) = A \int_{-1}^{1} \cos At(1 - g(t)) \, dt
\]
\[
= A \int_{-\infty}^{\infty} e^{iAt}(1 - g(t)) \, dt
\]

which is positive for positive \( A \) [2, P. 26] since \( 1 - g(t) \) is positive definite and belongs to \( L^1(-\infty, \infty) \). Finally

\[
\int_{-1}^{1} f''_n(t) \, dg(t) \leq 0
\]

and the theorem is proved.

This result, about positive kernels, may be compared with Kuttner's result in [5].

It is worth noting that we have proved

\[
\text{Re} \, z \int_{0}^{1} (1 - zt)^n \, dg(t) \geq 0,
\]

where \( z = 1 - e^{ix} \), whenever \( 1 - g(t) \) is positive definite. This provides some new inequalities which must be satisfied by a class of positive definite functions which is encountered quite often. For example, when \( n = 1 \) and \( x = \pi \), we obtain

\[
\int_{0}^{1} (1 - 2t) \, dg(t) \geq 0.
\]

**REFERENCES**

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