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We continue the discussion of finite dimensional simple extended Lie algebras over an algebraically closed field F of characteristic zero with nondegenerate form $(x, y) =$ trace R_xR_y where R_x (or $R(x)$) denotes the mapping $A \to A: a \to ax$; for brevity we call such an algebra a *simple el-algebra*. The main result of this paper is that those simple el-algebras which are not Lie or Malcev algebras probably cannot be analyzed by the usual desirable Lie-type methods.

First if we assume the simple el-algebra [3] A has a diagonalizable Cartan subalgebra [3] such that for any weight space $A(N, \alpha)$ of N in A we have $A(N, \alpha)^2 = 0$ or $A(N, \alpha)^2 \subset A(N, \beta)$ for some weight β (which is a function of α), then A is a Lie or Malcev algebra. Thus if one attempts to remedy the situation that $A(N, \alpha)^2$ is difficult to locate by the rather desirable above assumptions and tries to construct a multiplication table for a new simple el-algebra, then actually nothing new is obtained. Next we show that if the derivation algebra $D(A)$ is used to analyze a simple el-algebra, using [1, page 54] or possibly Lie module theory, then again a difficult situation is encountered: If A is simple el-algebra, then A is not a simple Lie or Malcev algebra if and only if there exists a nonzero element $a \in A$ such that for every derivation $D \in D(A)$ we have $aD = 0$. The element $a \in A$ reflects the structure of A and so it appears that the structure of A is not accurately reflected in its derivation albgebra.

The proofs of the above results use the following lemma.

LEMMA 1.1. *If A is a simple el-algebra, then A is a Lie or 7-dimensional Malcev algebra if and only if $u(x) =$ trace R_x is the zero linear functional.*

Proof. A linearization of the defining identities of an extended Lie algebra

$$xy = -yx \qquad \text{and} \qquad J(xy, x, y) = 0$$

where $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$ yields

$$J(wx, y, z) + J(yz, w, x) = J(wy, z, x) + J(zx, w, y) \tag{1.2}$$

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$$(1.3)\qquad wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) = 3[J(wx, y, z) + J(yz, w, x)]$$

for all $w, x, y, z \in A$. From (1.2) we obtain by operating on w that

$$(1.4)\qquad \begin{aligned}(xz, y) - (x, zy) &= \operatorname{trace} R(xz)R(y) - \operatorname{trace} R(x)R(zy) \\ &= \operatorname{trace} R(xz\cdot y + x\cdot zy) \\ &= u(xz\cdot y + x\cdot zy) .\end{aligned}$$

Now if $u(x) = 0$ for all $x \in A$, then from (1.4) we see (x, y) is a nondegenerate invariant form and from [3], A is a simple Lie or 7-dimensional Malcev algebra. Conversely, from the identities for these algebras [2] we see that $u(x) = 0$ for all $x \in A$.

We continue the use of the notation in [3] for sets and algebraic operations.

2. On the construction. We shall first investigate the assumption that a simple el-algebra A has a diagonalizable Cartan subalgebra N [3]. That is, N is a nilpotent Lie subalgebra of A such that for all $m, n \in N$,

$$R_{mn} = [R_m, R_n] \equiv R_mR_n - R_nR_m ;$$

furthermore, decomposing A into its weight spaces relative to $R(N) = \{R_n : n \in N\}$ we have [1; 3]

$$A = A(N, 0) \oplus \sum_{\alpha\neq 0} A(N, \alpha)$$

where, since $R(N)$ is diagonalizable,

$$A(N, \lambda) = \{x \in A : xR_n = \lambda(n)x\}$$

is the weight space of N corresponding to the weight λ and, since N is Cartan [3],

$$N = A(N, 0) .$$

Since we are using a fixed Cartan subalgebra we use the notation A_σ or $A(\sigma)$ for $A(N, \sigma)$ and the convention $A(\sigma) = 0$ if σ is not a weight of N in A. From [3] we have the identities

$$(2.1)\qquad A_\alpha A_\beta \subset A_{\alpha+\beta} \quad \text{if } \alpha \neq \beta$$

$$(2.2)\qquad J(A_\alpha, A_\beta, A_\gamma) = 0 \quad \text{if } \alpha \neq \beta \neq \gamma \neq \alpha$$

$$\text{and } J(A_\alpha, A_\beta, N) = 0 \quad \text{if } \alpha \neq \beta .$$

Let K denote the kernel of the linear functional $u : x \to \operatorname{trace} R_x$, then we have

$$(\alpha+\beta)(n)(x, y) = (\alpha-\beta)(n)u(xy) \quad \text{if } n \in N, x \in A_\alpha, y \in A_\beta \tag{2.3}$$

$$(A_\alpha, A_\beta) = 0 \quad \text{if } \alpha \neq 0 \text{ and } \beta \neq 0 \text{ and } \alpha \neq -\beta \tag{2.4}$$

$$A_\alpha A_\beta \subset K \quad \text{if } \alpha \neq 0 \text{ and } \beta \neq 0 \text{ and } \alpha \neq \beta . \tag{2.5}$$

For (2.3), let $n \in N$, then $xn = \alpha(n)x$, $yn = \beta(n)y$ and using (1.4) we have

$$(\alpha(n)+\beta(n))(x, y) = (xn, y) - (x, ny)$$
$$= u(xn \cdot y + x \cdot ny) = (\alpha(n) - \beta(n))u(xy) .$$

For (2.4) and (2.5), let $x \in A_\alpha$, $y \in A_\beta$ and first assume $\alpha \neq 0$ and $\beta \neq 0, \pm\alpha$. If $xy = 0$ for all x, y as above, then the results follow from (2.3). So assume $0 \neq xy \in A(\alpha)A(\beta) \subset A(\alpha+\beta)$, then $\alpha+\beta$ is a weight of N in A. Let $z \in A(\alpha+\beta)$, then since $\alpha \neq \alpha+\beta \neq \beta \neq \alpha$ we use (2.2) to obtain $J(x, y, z) \in J(A(\alpha), A(\beta), A(\alpha+\beta)) = 0$. Therefore

$$zR(xy) = zx \cdot y + yz \cdot x \in A(2\alpha+\beta)A(\beta)$$
$$+ A(\alpha+2\beta)A(\alpha) \subset A(2(\alpha+\beta)) .$$

Using this result and (2.1) we see that for any weight γ,

$$A(\gamma)R(xy) \subset A(\gamma+(\alpha+\beta)) \neq A(\gamma)$$

and therefore the matrix for $R(xy)$ has zeros on its diagonal so that $u(xy) = \text{trace}\, R(xy) = 0$. Next we relax the assumptions on β, use the above result and (2.3) to see that (2.4) and (2.5) now follow.

Now we shall start using the hypothesis that if α is any weight of N in A, then $A_\alpha^2 = 0$ or there exists a weight $\pi(\alpha)$ such that $A_\alpha^2 \subset A_{\pi(\alpha)}$. Thus we are assuming that if $A_\alpha^2 \neq 0$, then there exists a weight $\pi(\alpha)$ such that for each $x, y \in A_\alpha$, $xy \in A_{\pi(\alpha)}$; that is, π is a function of the weight and not a function of the particular elements used in forming the products. Using this assumption we shall show that for any weight α, $A_\alpha \subset K$(=kernel of u) and therefore by Lemma 1.1 conclude that A is Lie or Malcev.

First for $\alpha = 0$ we have $A_0^2 = A_0N = 0$. So assume $\alpha \neq 0$. If $xy = 0$ for all $x, y \in A_\alpha$, then using (2.1) we see that for any $x \in A_\alpha$, $u(x) = \text{trace}\, R_x = 0$ and therefore $A_\alpha \subset K$. So next we consider $0 \neq A_\alpha^2 \subset A_{\pi(\alpha)}$ where $\alpha \neq 0$.

LEMMA 2.6. *If $\alpha \neq 0$ and $0 \neq A_\alpha^2 \subset A_{\pi(\alpha)}$, then $\pi(\alpha) \neq 0$.*

COROLLARY 2.7. $N = \sum_{\alpha \neq 0} A(\alpha)A(-\alpha) \subset K$.

Suppose Lemma 2.6 has been proven, then to prove the corollary we first note $\sum_{\alpha\neq 0} A(\alpha)A(-\alpha) \subset A(0) = N$. Next set $B = \sum_{\alpha\neq 0} A(\alpha)(-\alpha) \oplus \sum_{\alpha\neq 0} A(\alpha)$; we shall show B is an ideal of A. For any weight $\beta \neq 0$,

$$BA(\beta) \subset (\textstyle\sum_{\alpha\neq 0} A(\alpha)A(-\alpha))A(\beta) + A(\beta)^2 + A(\beta)A(-\beta) + \sum_{\alpha\neq 0, \pm\beta} A(\alpha+\beta) .$$

Then using $A(\beta)^2 = 0$ or $A(\beta)^2 \subset A(\pi(\beta))$, where from Lemma 2.6 $\pi(\beta) \neq 0$, we see that $BA(\beta) \subset B$. For $\beta = 0$ we note that

$$(\textstyle\sum_{\alpha\neq 0} A(\alpha)A(-\alpha))A(0) \subset A(0)N = 0$$

and use (2.1) to obtain $BA(0) \subset B$. Thus $BA \subset B$ so that B is an ideal of A and since A is simple, $B = 0$ or $B = A$. If $B = 0$, then $A_\alpha = 0$ for each $\alpha \neq 0$ and $A = A_0 = N$ so that $A^2 = A_0 N = 0$, a contradiction. Thus $B = A$ and from this $N = \sum_{\alpha\neq 0} A(\alpha)A(-\alpha) \subset K$, using (2.5).

For Lemma 2.6 assume $\pi(\alpha) = 0$ and let $x, y \in A_\alpha$, then $xy \in A_\alpha^2 \subset A_0 = N$. We shall show for any weight β that $\beta(xy) = 0$, then for any $z \in A_\beta$ we have $z(xy) = zR(xy) = \beta(xy)z = 0$. Therefore $(xy)F$ is an ideal of A which must be zero and so $A_\alpha^2 = 0$, a contradiction. For $x, y \in A_\alpha$ we have from the defining identity

$$0 = J(xy, x, y) = (xy\cdot x)y + (y\cdot xy)x$$

which implies, since $xy \in N$, $2\alpha(xy)xy = 0$. From this and the fact that α is a linear functional on N we have $2\alpha(xy)^2 = 0$ and so $\alpha(xy) = 0$. Thus for $\beta = 0, \alpha$ we have $\beta(xy) = 0$ so we now assume $\beta \neq 0, \alpha$ and let $z \in A_\beta$, $n \in N$, then using (2.1) and (2.2) we obtain

$$\begin{aligned} J(zx, y, n) + J(yn, z, x) &= \alpha(n)J(y, z, x) \\ &= -\alpha(n)\beta(xy)z + \alpha(n)(yz\cdot x + zx\cdot y) \end{aligned}$$

and

$$\begin{aligned} J(zn, x, y) + J(xy, z, n) &= \beta(n)J(z, x, y) \\ &= -\beta(n)\beta(xy)z + \beta(n)(yz\cdot x + zx\cdot y) . \end{aligned}$$

We combine these equations by using (1.2) to obtain

$$\alpha(n)(-\beta(xy)z + zx\cdot y + yz\cdot x) = \beta(n)(-\beta(xy)z + zx\cdot y + yz\cdot x) .$$

From this equality we obtain, since $\beta(n) \neq \alpha(n)$ for some n, that

$$\beta(xy)z = zx\cdot y + yz\cdot x \in A(2\alpha + \beta) .$$

But since $\beta(xy)z \in A(\beta)$ we have

$$\beta(xy)z \in A(\beta) \cap A(2\alpha + \beta) = 0 .$$

Thus if $z \neq 0$, $\beta(xy) = 0$ and this proves the lemma.

Thus far we have considered for $\alpha \neq 0$: (1) $A_\alpha^2 = 0$ which implies $A_\alpha \subset K$; (2) $A_\alpha^2 \neq 0$ which implies $\pi(\alpha) \neq 0$ and consequently $N = A_0 = \sum_{\alpha\neq 0} A(\alpha)A(-\alpha) \subset K$. So we next investigate (2) more closely and note that it suffices to consider $0 \neq A_\alpha^2 \subset A_{\pi(\alpha)}$ where $\pi(\alpha) = \alpha$. For if $\pi(\alpha) \neq \alpha$, then using (2.1) we see that the matrix of R_x for any $x \in A_\alpha$ has zeros on its diagonal and therefore $u(x) = 0$ so that $A_\alpha \subset K$ which is what we eventually want to show for any weight α.

Thus we are considering $0 \neq A_\alpha^2 \subset A_\alpha$. Since (x, y) is nondegenerate and $A_\alpha^2 \neq 0$, there exists a weight β so that

$$(A_\alpha^2,\ A_\beta) \neq 0 .$$

But since $A_\alpha^2 \subset A_\alpha$ this means $(A_\alpha, A_\beta) \neq 0$ and from (2.4) and the assumption that $\alpha \neq 0$ we conclude $\beta = 0$ or $\beta = -\alpha$. We shall consider these two cases and show that the situation $0 \neq A_\alpha^2 \subset A_\alpha$ actually does not exist so that we may conclude that for any weight α, $A_\alpha \subset K$.

Case $\beta = 0$. Let $x, y \in A_\alpha$, $n \in A_0$ and $xy \in A_\alpha$, then using $(A_\alpha, A_\alpha) = 0$ (from (2.3)) we have

$$\begin{aligned}(xy, n) &= (xy, n) - (x, yn)\\ &= u(xy\cdot n + x\cdot yn)\\ &= u(\alpha(n)xy + \alpha(n)xy)\\ &= 2\alpha(n)u(xy) .\end{aligned} \tag{2.8}$$

However from (2.3) and $xy \in A_\alpha$ we have

$$\begin{aligned}\alpha(n)(xy, n) &= (\alpha + 0)(n)(xy, n)\\ &= (\alpha - 0)(n)u(xy\cdot n)\\ &= \alpha(n)^2u(xy) .\end{aligned} \tag{2.9}$$

From (2.8) we also have $\alpha(n)(xy, n) = 2\alpha(n)^2u(xy)$ and therefore from

$$\alpha(n)^2u(xy) = 0 \qquad \text{for all } n \in N,\ x, y \in A_\alpha . \tag{2.9}$$

Now there exists $x, y \in A_\alpha$ so that $u(xy) \neq 0$, otherwise from (2.8) we would have $(A_\alpha^2, A_0) = 0$, contrary to our assumption for case $\beta = 0$. But from the previous equation this implies $\alpha(n) = 0$ for all $n \in N$, contradicting the assumption $\alpha \neq 0$. Thus case $\beta = 0$ does not exist.

Case $\beta = -\alpha$. That is, $\alpha \neq 0$, $A_\alpha^2 \subset A_\alpha$ and $(A_\alpha^2, A_\beta) \neq 0$ with $\beta = -\alpha$; in particular we are assuming $-\alpha$ is a weight. We shall show in this case that the dimension of A_α is one and therefore $A_\alpha^2 = 0$, a contradiction; thus case $\beta = -\alpha$ does not exist. So assume the dimension of A_α is greater than one and let $x, y \in A_\alpha$, $z \in A_{-\alpha}$ and $n \in N$, then using $xy \in A_\alpha$ and (2.2) we have

$$J(ny, z, x) + J(zx, n, y) = -\alpha(n)J(y, z, x)$$

and $$J(nz, x, y) + J(xy, n, z) = \alpha(n)J(z, x, y) .$$

Applying (1.2) to these equations we have, since $\alpha \neq 0$,

$$\begin{aligned} 0 = J(y, z, x) &= yz \cdot x + zx \cdot y + xy \cdot z \\ &= xy \cdot z - \alpha(yz)x - \alpha(zx)y . \end{aligned}$$

Therefore since $xy \cdot z \in A_0$ and $x, y \in A_\alpha$ we have $xy \cdot z = 0$ and $\alpha(yz)x + \alpha(zx)y = 0$. But since we have assumed the dimension of $A_\alpha > 1$ and x, y are arbitrary in A_α we have $\alpha(zx) = 0$ for any $z \in A_{-\alpha}$; for just choose $0 \neq x$ arbitrary in A_α and y to be linearly independent of x, then for any $z \in A$, $\alpha(yz)x + \alpha(zx)y = 0$ which yields the result.

Next we shall show $\beta(zx) = 0$ for any weight β of N and any $z \in A(-\alpha)$, $x \in A(\alpha)$. If $\beta = q\alpha$ where q is a rational number, the results follow. Next suppose $\beta \neq q\alpha$ and let $M = \sum_k A(\beta + k\alpha)$, $k = 0, \pm 1, \pm 2, \cdots$. Using (2.1) and $\beta \neq q\alpha$ we see that M is $R_x -$, $R_z -$, and $R(xz)$-invariant and for any $y = \sum_k y_k \in M$ where $y_k \in A(\beta + k\alpha)$ we have

$$J(y, x, z) = \sum_k J(y_k, x, z) = 0 ,$$

using (2.2). Thus $y([R_x, R_z] - R(xz)) = 0$; that is, on M we have $R(xz) = [R_x, R_z]$ so that

$$\text{(2.10)} \qquad \operatorname{trace}_M R(xz) = 0 ,$$

where trace_M denotes the trace function restricted to M. However calculating the $\operatorname{trace}_M R(xz)$ from the matrix of $R(xz)$ on M we see that

$$\begin{aligned} \operatorname{trace}_M R(xz) &= \sum_k N_k(\beta + k\alpha)(xz), \qquad N_k = \dim A(\beta + k\alpha) \\ &= (\sum_k N_k)\beta(xz) + (\sum_h kN_k)\alpha(xz) \\ &= (\sum_k N_k)\beta(xz), \text{ since } \alpha(xz) = 0 . \end{aligned}$$

This equation and (2.10) imply $\beta(xz) = 0$. Thus for any weight β and any $y \in A_\beta$ we have $yR(xz) = \beta(xz)y = 0$ which implies $R(xz) = 0$ and therefore $xz = 0$ i.e. $A(\alpha)A(-\alpha) = 0$. We use this fact to obtain a contradiction to $(A^2(\alpha), A(-\alpha)) \neq 0$. So let $x, y \in A(\alpha)$, $z \in A(-\alpha)$, then using (1.4) we have

$$\begin{aligned} (xy, z) &= (x, yz) + u(xy \cdot z + x \cdot yz) \\ &= u(xy \cdot z), \text{ using } yz \in A(\alpha)A(-\alpha) = 0 \\ &= 0, \text{ using } xy \in A(\alpha) \text{ and } A(\alpha)A(-\alpha) = 0 . \end{aligned}$$

This contradiction shows case $\beta = -\alpha$ does not exist and so from previous remarks we have for any weight α, $A_\alpha \subset K$ which proves

THEOREM 2.11. *Let A be a simple el-algebra satisfying the*

following conditions
(1) *there exists a Cartan subalgebra N of A so that $R(N) = \{R_n : n \in N\}$ acts diagonally in A*
(2) *if $A = \sum_\alpha A(N, \alpha)$ is the weight space decomposition of A relative to $R(N)$ where N is the subalgebra of* (1), *then $A(N, \alpha)^2 = 0$ or $A(N, \alpha)^2 \subset A(N, \pi(\alpha))$ for some weight $\pi(\alpha)$.*
Then A is a Lie or 7-dimensional Malcev algebra.

3. **On derivations.** Again let A be a simple el-algebra. To use the derivation algebra $D(A)$ in the analysis of A we first locate the derivations of A as follows.

THEOREM 3.1. *Every derivation of A is inner, that is, $D(A)$ is contained in the Lie trasformation algebra $L(A)$ which is the smallest Lie algebra containing $R(A) = \{R_x : x \in A\}$* [4].

Proof. Since A is simple it contains no nontrivial $L(A)$-invariant subspaces and so $L(A)$ is irreducible in A. This implies $L(A) = C \oplus L(A)'$ where C is the center of $L(A)$ and $L(A)' = [L(A), L(A)]$ is semi-simple [1; Th. 2.11]. Furthermore $C = 0$ or $C = FI$; for if S is a linear transformation in C, then since F is algebraically closed S has a characteristic root λ in F. Using the fact $[R(A), S] = 0$ we see $\{x \in A : xS = \lambda x\}$ is a nonzero ideal of A and therefore equals A. From this the results concerning C follow.

Now let $D \in D(A)$, then we have $[R_x, D] = R(xD)$ for all $x \in A$ and this together with the Jacobi identity imply $[L(A)', D] \subset L(A)'$. Thus the mapping

$$L(A)' \to L(A)' : X' \to [X', D] \qquad \text{all } X' \in L[A]'$$

is a derivation of $L(A)'$. Since $L(A)'$ is semi-simple every derivation of $L(A)'$ is inner and therefore there exists $D' \in L(A)'$ so that $[X', D] = [X', D']$ all $X' \in L(A)'$ [1; Th. 3.6]. But for any $X = aI + X' \in L(A)$ where $a \in F$ (if $C \neq 0$) we have $[X, D] = [X, D']$. Thus if $T = D - D'$ we have in particular that $[R(A), T] = 0$. Again since F is algebraically closed T has a characteristic root μ and we see that $\{x \in A : xT = \mu x\}$ is a nonzero ideal in A. This implies either $T = 0$ in which case $D = D'$ or $T = \mu I$ in which case $D = \mu I + D'$. Now in this latter case we note $D' \in L(A)'$ so that trace $D' = 0$ and since $(x, y) =$ trace $R_x R_y$ is nondegenerate we have from $[R_x, D] = R(xD)$ that $(xD, y) + (x, yD) = 0$ so that D is skewsymmetric and also trace $D = 0$. From these facts on trace and $D = \mu I + D'$ we conclude $D = D' \in L(A)$ in both cases.

Even though we know all derivations of a simple el-algebra are inner, their exact form has not yet been determined. However the following is not too difficult to prove: If A is a simple el-algebra, then A is a Lie algebra if and only if there exists an element $x \in A$ so that R_x is a nonzero derivation of A. Next we have

THEOREM 3.2. *If A is a simple el-algebra, then A is not a Lie or 7-dimensional Malcev algebra if and only if there exists a nonzero element $a \in A$ such that for every derivation D of A we have $aD = 0$.*

Proof. If A is a Lie or 7-dimensional Malcev algebra then the conclusion is well known [2]. Conversely, if A is not Lie or 7-dimensional Malcev, then since $(x, y) = \text{trace } R_x R_y$ is nondegenerate we use Lemma 1.1 to obtain a nonzero element $a \in A$ so that for all $x \in A$, $u(x) = (x, a)$. But for any derivation D we have $R(xD) = [R_x, D]$ and $(xD, y) + (x, yD) = 0$ so that in particular we have for any $x \in A$, $(aD, x) = -(a, xD) = -u(xD) = -\text{trace } R(xD) = 0$. Thus since (x, y) is nondegenerate $aD = 0$.

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