REMARKS ON SIMPLE EXTENDED LIE ALGEBRAS

Arthur Argyle Sagle
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ARTHUR A. SAGLE

We continue the discussion of finite dimensional simple extended Lie algebras over an algebraically closed field \( F \) of characteristic zero with nondegenerate form \( (x, y) = \text{trace } R_x R_y \) where \( R_x \) (or \( R(x) \)) denotes the mapping \( A \rightarrow A: a \rightarrow ax \); for brevity we call such an algebra a simple el-algebra. The main result of this paper is that those simple el-algebras which are not Lie or Malcev algebras probably cannot be analyzed by the usual desirable Lie-type methods.

First if we assume the simple el-algebra \( A \) has a diagonalizable Cartan subalgebra \([3]\) such that for any weight space \( A(N, \alpha) \) of \( N \) in \( A \) we have \( A(N, \alpha)^2 = 0 \) or \( A(N, \alpha)^2 \subset A(N, \beta) \) for some weight \( \beta \) (which is a function of \( \alpha \)), then \( A \) is a Lie or Malcev algebra. Thus if one attempts to remedy the situation that \( A(N, \alpha)^2 \) is difficult to locate by the rather desirable above assumptions and tries to construct a multiplication table for a new simple el-algebra, then actually nothing new is obtained. Next we show that if the derivation algebra \( D(A) \) is used to analyze a simple el-algebra, using \([1, \text{page 54}]\) or possibly Lie module theory, then again a difficult situation is encountered: If \( A \) is simple el-algebra, then \( A \) is not a simple Lie or Malcev algebra if and only if there exists a nonzero element \( a \in A \) such that for every derivation \( D \in D(A) \) we have \( aD = 0 \). The element \( a \in A \) reflects the structure of \( A \) and so it appears that the structure of \( A \) is not accurately reflected in its derivation algebra.

The proofs of the above results use the following lemma.

**Lemma 1.1.** If \( A \) is a simple el-algebra, then \( A \) is a Lie or 7-dimensional Malcev algebra if and only if \( u(x) = \text{trace } R_x \) is the zero linear functional.

**Proof.** A linearization of the defining identities of an extended Lie algebra

\[
xy = -yx \quad \text{and} \quad J(xy, x, y) = 0
\]

where \( J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y \) yields

\[
J(wx, y, z) + J(yz, w, x) = J(wy, z, x) + J(zx, w, y)
\]

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(1.3) \[ wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) = 3[J(wx, y, z) + J(yz, w, x)] \]

for all \( w, x, y, z \in A \). From (1.2) we obtain by operating on \( w \) that

\[ (xz, y) - (x, zy) = \text{trace } R(xz)R(y) - \text{trace } R(x)R(zy) \]

(1.4) \[ = \text{trace } R(xz \cdot y + x \cdot zy) \]

Now if \( u(x) = 0 \) for all \( x \in A \), then from (1.4) we see \((x, y)\) is a nondegenerate invariant form and from [3], \( A \) is a simple Lie or 7-dimensional Malcev algebra. Conversely, from the identities for these algebras [2] we see that \( u(x) = 0 \) for all \( x \in A \).

We continue the use of the notation in [3] for sets and algebraic operations.

2. On the construction. We shall first investigate the assumption that a simple el-algebra \( A \) has a diagonalizable Cartan subalgebra \( N \) [3]. That is, \( N \) is a nilpotent Lie subalgebra of \( A \) such that for all \( m, n \in N \),

\( R_{mn} = [R_m, R_n] = R_mR_n - R_nR_m \);

furthermore, decomposing \( A \) into its weight spaces relative to \( R(N) = \{R_n: n \in N\} \) we have [1; 3]

\[ A = A(N, 0) \oplus \sum_{\alpha \neq 0} A(N, \alpha) \]

where, since \( R(N) \) is diagonalizable,

\[ A(N, \lambda) = \{x \in A: xR_n = \lambda(n)x\} \]

is the weight space of \( N \) corresponding to the weight \( \lambda \) and, since \( N \) is Cartan [3],

\[ N = A(N, 0) \]

Since we are using a fixed Cartan subalgebra we use the notation \( A_{\alpha} \) or \( A(\sigma) \) for \( A(N, \sigma) \) and the convention \( A(\sigma) = 0 \) if \( \sigma \) is not a weight of \( N \) in \( A \). From [3] we have the identities

(2.1) \[ A_{\alpha}A_{\beta} \subset A_{\alpha + \beta} \text{ if } \alpha \neq \beta \]

(2.2) \[ J(A_{\alpha}, A_{\beta}, A_{\gamma}) = 0 \text{ if } \alpha \neq \beta \neq \gamma \neq \alpha \]

and \( J(A_{\alpha}, A_{\beta}, N) = 0 \) if \( \alpha \neq \beta \).

Let \( K \) denote the kernel of the linear functional \( u : x \rightarrow \text{trace } R_x \), then we have
\[(2.3) \quad (\alpha + \beta)(n)(x, y) = (\alpha - \beta)(n)u(xy) \]
if \(n \in N, x \in A_{\alpha}, y \in A_{\beta}\)

\[(2.4) \quad (A_{\alpha}, A_{\beta}) = 0 \quad \text{if} \quad \alpha \neq 0 \quad \text{and} \quad \beta \neq 0 \quad \text{and} \quad \alpha \neq -\beta\]

\[(2.5) \quad A_{\alpha}A_{\beta} \subset K \quad \text{if} \quad \alpha \neq 0 \quad \text{and} \quad \beta \neq 0 \quad \text{and} \quad \alpha \neq \beta .\]

For (2.3), let \(n \in N\), then \(xn = \alpha(n)x, yn = \beta(n)y\) and using (1.4) we have

\[(\alpha(n) + \beta(n))(x, y) = (xn, y) - (x, ny) = u(xn \cdot y + x \cdot ny) = (\alpha(n) - \beta(n))u(xy).\]

For (2.4) and (2.5), let \(x \in A_{\alpha}, y \in A_{\beta}\) and first assume \(\alpha \neq 0\) and \(\beta \neq 0\), \(\pm \alpha\). If \(xy = 0\) for all \(x, y\) as above, then the results follow from (2.3). So assume \(0 \neq xy \in A(\alpha)A(\beta) \subset A(\alpha + \beta)\), then \(\alpha + \beta\) is a weight of \(N\) in \(A\). Let \(z \in A(\alpha + \beta)\), then since \(\alpha \neq \alpha + \beta \neq \beta \neq \alpha\) we use (2.2) to obtain \(J(x, y, z) \in J(A(\alpha), A(\beta), A(\alpha + \beta)) = 0\). Therefore

\[zR(xy) = z \cdot y + yz \cdot x \in A(2\alpha + \beta)A(\beta) + A(\alpha + 2\beta)A(\alpha) \subset A(2(\alpha + \beta)) .\]

Using this result and (2.1) we see that for any weight \(\gamma\),

\[A(\gamma)R(xy) \subset A(\gamma + (\alpha + \beta)) \neq A(\gamma)\]

and therefore the matrix for \(R(xy)\) has zeros on its diagonal so that \(u(xy) = \text{trace } R(xy) = 0\). Next we relax the assumptions on \(\beta\), use the above result and (2.3) to see that (2.4) and (2.5) now follow.

Now we shall start using the hypothesis that if \(\alpha\) is any weight of \(N\) in \(A\), then \(A_{\alpha}^2 = 0\) or there exists a weight \(\pi(\alpha)\) such that \(A_{\alpha}^2 \subset A_{\pi(\alpha)}\). Thus we are assuming that if \(A_{\alpha}^2 \neq 0\), then there exists a weight \(\pi(\alpha)\) such that for each \(x, y \in A_{\alpha}, xy \in A_{\pi(\alpha)}\); that is, \(\pi\) is a function of the weight and not a function of the particular elements used in forming the products. Using this assumption we shall show that for any weight \(\alpha, A_{\alpha} \subset K(= \text{kernel of } u)\) and therefore by Lemma 1.1 conclude that \(A\) is Lie or Malcev.

First for \(\alpha = 0\) we have \(A_{\alpha}^2 = A_{\alpha}N = 0\). So assume \(\alpha \neq 0\). If \(xy = 0\) for all \(x, y \in A_{\alpha}\), then using (2.1) we see that for any \(x \in A_{\alpha}\), \(u(x) = \text{trace } R_{\alpha} = 0\) and therefore \(A_{\alpha} \subset K\). So next we consider \(0 \neq A_{\alpha}^2 \subset A_{\pi(\alpha)}\) where \(\alpha \neq 0\).

**Lemma 2.6.** If \(\alpha \neq 0\) and \(0 \neq A_{\alpha}^2 \subset A_{\pi(\alpha)}\), then \(\pi(\alpha) \neq 0\).

**Corollary 2.7.** \(N = \sum_{\alpha \neq 0} A(\alpha)A(-\alpha) \subset K\).
Suppose Lemma 2.6 has been proven, then to prove the corollary we first note \( \sum_{\alpha \neq 0} A(\alpha) A(- \alpha) \cap A(0) = N \). Next set \( B = \sum_{\alpha \neq 0} A(\alpha)(- \alpha) \oplus \sum_{\alpha \neq 0} A(\alpha) \); we shall show \( B \) is an ideal of \( A \). For any weight \( \beta \neq 0 \),

\[
BA(\beta) \subset (\sum_{\alpha \neq 0} A(\alpha) A(- \alpha)) A(\beta) + A(\beta)^2
\]

\[
+ A(\beta) A(- \beta) + \sum_{\alpha \neq 0, \pm \beta} A(\alpha + \beta).
\]

Then using \( A(\beta)^2 = 0 \) or \( A(\beta)^2 \subset A(\pi(\beta)) \), where from Lemma 2.6 \( \pi(\beta) \neq 0 \), we see that \( BA(\beta) \subset B \). For \( \beta = 0 \) we note that

\[
(\sum_{\alpha \neq 0} A(\alpha) A(- \alpha)) A(0) \subset A(0) N = 0
\]

and use (2.1) to obtain \( BA(0) \subset B \). Thus \( BA \subset B \) so that \( B \) is an ideal of \( A \) and since \( A \) is simple, \( B = 0 \) or \( B = A \). If \( B = 0 \), then \( A_\alpha = 0 \) for each \( \alpha \neq 0 \) and \( A = A_0 = N \) so that \( A^2 = A_0 N = 0 \), a contradiction. Thus \( B = A \) and from this \( N = \sum_{\alpha \neq 0} A(\alpha) A(- \alpha) \subset K \), using (2.5).

For Lemma 2.6 assume \( \pi(\alpha) = 0 \) and let \( x, y \in A_\alpha \), then \( xy \in A_\alpha \subset A_0 = N \). We shall show for any weight \( \beta \) that \( \beta(xy) = 0 \), then for any \( z \in A_\beta \) we have \( z(xy) = zR(xy) = \beta(xy)z = 0 \). Therefore \((xy)F\) is an ideal of \( A \) which must be zero and so \( A_\alpha = 0 \), a contradiction. For \( x, y \in A_\alpha \) we have from the defining identity

\[
0 = J(xy, x, y) = (xy \cdot x)y + (y \cdot xy)x
\]

which implies, since \( xy \in N, 2\alpha(xy)xy = 0 \). From this and the fact that \( \alpha \) is a linear functional on \( N \) we have \( 2\alpha(xy)^2 = 0 \) and so \( \alpha(xy) = 0 \). Thus for \( \beta = 0, \alpha \) we have \( \beta(xy) = 0 \) so we now assume \( \beta \neq 0, \alpha \) and let \( z \in A_\beta, n \in N \), then using (2.1) and (2.2) we obtain

\[
J(zx, y, n) + J(yn, z, x) = \alpha(n) J(y, z, x)
= - \alpha(n) \beta(xy) z + \alpha(n) (yz \cdot x + zx \cdot y)
\]

and

\[
J(zn, x, y) + J(xy, z, n) = \beta(n) J(z, x, y)
= - \beta(n) \beta(xy) z + \beta(n) (yz \cdot x + zx \cdot y).
\]

We combine these equations by using (1.2) to obtain

\[
\alpha(n)(- \beta(xy) z + zx \cdot y + yz \cdot x) = \beta(n)(- \beta(xy) z + zx \cdot y + yz \cdot x).
\]

From this equality we obtain, since \( \beta(n) \neq \alpha(n) \) for some \( n \), that

\[
\beta(xy) z = zx \cdot y + yz \cdot x \in A(2\alpha + \beta)\,.
\]

But since \( \beta(xy) z \in A(\beta) \) we have

\[
\beta(xy) z \in A(\beta) \cap A(2\alpha + \beta) = 0\,.
\]
Thus if \( z \neq 0 \), \( \beta(xy) = 0 \) and this proves the lemma.

Thus far we have considered for \( \alpha \neq 0 \): (1) \( A_\alpha^2 = 0 \) which implies \( A_\alpha \subset K \); (2) \( A_\alpha^2 \neq 0 \) which implies \( \pi(\alpha) \neq 0 \) and consequently \( N = A_\alpha = \sum_{\alpha \neq 0} A(\alpha)A(-\alpha) \subset K \). So we next investigate (2) more closely and note that it suffices to consider \( 0 \neq A_\alpha^2 \subset A_{\pi(\alpha)} \) where \( \pi(\alpha) = \alpha \). For if \( \pi(\alpha) \neq \alpha \), then using (2.1) we see that the matrix of \( R_x \) for any \( x \in A_\alpha \) has zeros on its diagonal and therefore \( u(x) = 0 \) so that \( A_\alpha \subset K \) which is what we eventually want to show for any weight \( \alpha \).

Thus we are considering \( 0 \neq A_\alpha^2 \subset A_\alpha \). Since \((x, y)\) is nondegenerate and \( A_\alpha^2 \neq 0 \), there exists a weight \( \beta \) so that

\[
(A_\alpha^2, A_\beta) \neq 0.
\]

But since \( A_\alpha^2 \subset A_\alpha \) this means \((A_\alpha, A_\beta) \neq 0\) and from (2.4) and the assumption that \( \alpha \neq 0 \) we conclude \( \beta = 0 \) or \( \beta = -\alpha \). We shall consider these two cases and show that the situation \( 0 \neq A_\alpha^2 \subset A_\alpha \) actually does not exist so that we may conclude that for any weight \( \alpha, A_\alpha \subset K \).

Case \( \beta = 0 \). Let \( x, y \in A_\alpha, n \in A_\alpha \) and \( xy \in A_\alpha \), then using \((A_\alpha, A_\alpha) = 0 \) (from (2.3)) we have

\[
(xy, n) = (xy, n) - (x, yn)
= u(xy \cdot n + x \cdot yn)
= u(\alpha(n)xy + \alpha(n)xy)
= 2\alpha(n)u(xy).
\]

(2.8)

However from (2.3) and \( xy \in A_\alpha \) we have

\[
\alpha(n)(xy, n) = (\alpha + 0)(n)(xy, n)
= (\alpha - 0)(n)u(xy \cdot n)
= \alpha(n)^2u(xy).
\]

(2.9)

From (2.8) we also have \( \alpha(n)(xy, n) = 2\alpha(n)^2u(xy) \) and therefore from (2.9)

\[
\alpha(n)^2u(xy) = 0 \text{ for all } n \in N, x, y \in A_\alpha.
\]

Now there exists \( x, y \in A_\alpha \) so that \( u(xy) \neq 0 \), otherwise from (2.8) we would have \((A_\alpha^2, A_\alpha) = 0\), contrary to our assumption for case \( \beta = 0 \). But from the previous equation this implies \( \alpha(n) = 0 \) for all \( n \in N \), contradicting the assumption \( \alpha \neq 0 \). Thus case \( \beta = 0 \) does not exist.

Case \( \beta = -\alpha \). That is, \( \alpha \neq 0, A_\alpha^2 \subset A_\alpha \) and \((A_\alpha^2, A_\beta) \neq 0 \) with \( \beta = -\alpha \); in particular we are assuming \(-\alpha\) is a weight. We shall show in this case that the dimension of \( A_\alpha \) is one and therefore \( A_\alpha^2 = 0 \), a contradiction; thus case \( \beta = -\alpha \) does not exist. So assume the dimension of \( A_\alpha \) is greater than one and let \( x, y \in A_\alpha, z \in A_{-\alpha} \) and \( n \in N \), then using \( xy \in A_\alpha \) and (2.2) we have

\[
J(nx, z, x) + J(zx, n, y) = -\alpha(n)J(y, z, x).
\]
and 
\[ J(nz, x, y) + J(xy, n, z) = a(n)J(z, x, y). \]

Applying (1.2) to these equations we have, since \( \alpha \neq 0 \),
\[ 0 = J(y, z, x) = yz \cdot x + zx \cdot y + xy \cdot z \\
= xy \cdot z - \alpha(yz)x - \alpha(zx)y. \]

Therefore since \( xy \cdot z \in A_0 \) and \( x, y \in A_\alpha \) we have \( xy \cdot z = 0 \) and \( \alpha(yz)x + \alpha(zx)y = 0 \). But since we have assumed the dimension of \( A_\alpha > 1 \) and \( x, y \) are arbitrary in \( A_\alpha \) we have \( \alpha(zx) = 0 \) for any \( z \in A_{-\alpha} \); for just choose \( 0 \neq x \) arbitrary in \( A_\alpha \) and \( y \) to be linearly independent of \( x \), then for any \( z \in A \), \( \alpha(yz)x + \alpha(zx)y = 0 \) which yields the result.

Next we shall show \( \beta(zx) = 0 \) for any weight \( \beta \) of \( N \) and any \( z \in A(-\alpha), x \in A(\alpha) \). If \( \beta = q\alpha \) where \( q \) is a rational number, the results follow. Next suppose \( \beta \neq q\alpha \) and let \( M = \sum_k A(\beta + k\alpha) \), \( k = 0, \pm 1, \pm 2, \ldots \). Using (2.1) and \( \beta \neq q\alpha \) we see that \( M \) is \( R_x - \), \( R_x - \), and \( R(xz) \)-invariant and for any \( y = \sum_k y_k \in M \) where \( y_k \in A(\beta + k\alpha) \) we have
\[ J(y, x, z) = \sum_k J(y_k, x, z) = 0, \]
using (2.2). Thus \( y([R_x, R_z] - R(xz)) = 0 \); that is, on \( M \) we have \( R(xz) = [R_x, R_z] \) so that
\[ (2.10) \quad \text{trace}_M R(xz) = 0, \]
where \( \text{trace}_M \) denotes the trace function restricted to \( M \). However calculating the \( \text{trace}_M R(xz) \) from the matrix of \( R(xz) \) on \( M \) we see that
\[
\text{trace}_M R(xz) = \sum_k N_k(\beta + k\alpha)(xz), \quad N_k = \dim A(\beta + k\alpha) \\
= (\sum_k N_k)\beta(xz) + (\sum_k kN_k)\alpha(xz) \\
- (\sum_k N_k)\beta(zx), \text{ since } \alpha(xz) = 0.
\]

This equation and (2.10) imply \( \beta(xz) = 0 \). Thus for any weight \( \beta \) and any \( y \in A_\beta \) we have \( yR(xz) = \beta(xz)y = 0 \) which implies \( R(xz) = 0 \) and therefore \( xz = 0 \) i.e. \( A(\alpha)A(-\alpha) = 0 \). We use this fact to obtain a contradiction to \( (A(\alpha), A(-\alpha)) \neq 0 \). So let \( x, y \in A(\alpha), z \in A(-\alpha) \), then using (1.4) we have
\[
(xy, z) = (x, yz) + u(xy \cdot z + x \cdot yz) \\
= u(xy \cdot z), \text{ using } yz \in A(\alpha)A(-\alpha) = 0 \\
= 0, \text{ using } xy \in A(\alpha) \text{ and } A(\alpha)A(-\alpha) = 0.
\]

This contradiction shows case \( \beta = -\alpha \) does not exist and so from previous remarks we have for any weight \( \alpha, A_\alpha \subset K \) which proves

**Theorem 2.11.** Let \( A \) be a simple el-algebra satisfying the
following conditions
(1) there exists a Cartan subalgebra $N$ of $A$ so that $R(N) = \{R_x : x \in N\}$ acts diagonally in $A$
(2) if $A = \sum \alpha A(N, \alpha)$ is the weight space decomposition of $A$ relative to $R(N)$ where $N$ is the subalgebra of (1), then $A(N, \alpha)^2 = 0$ or $A(N, \alpha)^2 \subset A(N, \pi(\alpha))$ for some weight $\pi(\alpha)$.

Then $A$ is a Lie or 7-dimensional Malcev algebra.

3. On derivations. Again let $A$ be a simple el-algebra. To use the derivation algebra $D(A)$ in the analysis of $A$ we first locate the derivations of $A$ as follows.

**Theorem 3.1.** Every derivation of $A$ is inner, that is, $D(A)$ is contained in the Lie transformation algebra $L(A)$ which is the smallest Lie algebra containing $R(A) = \{R_x : x \in A\}$ [4].

**Proof.** Since $A$ is simple it contains no nontrivial $L(A)$-invariant subspaces and so $L(A)$ is irreducible in $A$. This implies $L(A) = C \oplus L(A)'$ where $C$ is the center of $L(A)$ and $L(A)' = [L(A), L(A)]$ is semisimple [1; Th. 2.11]. Furthermore $C = 0$ or $C = FI$; for if $S$ is a linear transformation in $C$, then since $F$ is algebraically closed $S$ has a characteristic root $\lambda$ in $F$. Using the fact $[R(A), S] = 0$ we see \{x \in A : xS = \lambda x\} is a nonzero ideal of $A$ and therefore equals $A$. From this the results concerning $C$ follow.

Now let $D \in D(A)$, then we have $[R_x, D] = R(xD)$ for all $x \in A$ and this together with the Jacobi identity imply $[L(A)', D] \subset L(A)'$. Thus the mapping

$L(A)' \rightarrow L(A)' : X' \rightarrow [X', D]$ all $X' \in L[A]'$

is a derivation of $L(A)'$. Since $L(A)'$ is semisimple every derivation of $L(A)'$ is inner and therefore there exists $D' \in L(A)'$ so that $[X', D] = [X', D']$ all $X' \in L(A)'$ [1; Th. 3.6]. But for any $X = aI + X' \in L(A)$ where $a \in F$ (if $C \neq 0$) we have $[X, D] = [X, D']$. Thus if $T = D - D'$ we have in particular that $[R(A), T] = 0$. Again since $F$ is algebraically closed $T$ has a characteristic root $\mu$ and we see that \{x \in A : xT = \mu x\} is a nonzero ideal in $A$. This implies either $T = 0$ in which case $D = D'$ or $T = \mu I$ in which case $D = \mu I + D'$. Now in this latter case we note $D' \in L(A)'$ so that trace $D' = 0$ and since $(x, y) = \text{trace } R_x R_y$ is nondegenerate we have from $[R_x, D] = R(xD)$ that $(xD, y) + (x, yD) = 0$ so that $D$ is skewsymmetric and also trace $D = 0$. From these facts on trace and $D = \mu I + D'$ we conclude $D = D' \in L(A)$ in both cases.

Even though we know all derivations of a simple el-algebra are inner, their exact form has not yet been determined. However the
following is not too difficult to prove: If \( A \) is a simple el-algebra, then 
\( A \) is a Lie algebra if and only if there exists an element \( x \in A \) so that 
\( R_x \) is a nonzero derivation of \( A \). Next we have

**Theorem 3.2.** If \( A \) is a simple el-algebra, then \( A \) is not a Lie 
or 7-dimensional Malcev algebra if and only if there exists a nonzero 
element \( a \in A \) such that for every derivation \( D \) of \( A \) we have \( aD = 0 \).

**Proof.** If \( A \) is a Lie or 7-dimensional Malcev algebra then the 
conclusion is well known [2]. Conversely, if \( A \) is not Lie or 7-dimen-
sional Malcev, then since \( (x, y) = \text{trace } R_x R_y \) is nondegenerate we use 
Lemma 1.1 to obtain a nonzero element \( a \in A \) so that for all \( x \in A \), 
\( u(x) = (x, a) \). But for any derivation \( D \) we have \( R(xD) = [R_x, D] \) and 
\( (xD, y) + (x, yD) = 0 \) so that in particular we have for any \( x \in A \), 
\( (aD, x) = -(a, xD) = -u(xD) = -\text{trace } R(xD) = 0 \). Thus since \( (x, y) \) 
is nondegenerate \( aD = 0 \).

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