

Pacific Journal of Mathematics

MONOTONE APPROXIMATION

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O. SHISHA

How close can one approximate a monotone function by a monotone polynomial of degree $\leq n$, or a convex function by a convex polynomial of degree $\leq n$? This leads to the following general question. Let k and n be given, and suppose a real function f satisfies $f^{(k)}(x) \geq 0$ throughout a closed, finite interval $[a, b]$. How close can one approximate f on $[a, b]$ by a polynomial of degree $\leq n$ whose k th derivative, too, is ≥ 0 there? We give an answer to the question.

2. THEOREM 1. Let k and p be integers, $1 \leq k \leq p$, and let a real function f satisfy throughout $[a, b]$

$$f^{(k)}(x) \geq 0, \\ |f^{(p)}(x_2) - f^{(p)}(x_1)| \leq \lambda |x_2 - x_1|,$$

λ being a constant. Then for every integer $n (\geq p)$ there exists a real polynomial $Q_n(x)$ of degree¹ $\leq n$ such that

- (a) $Q_n^{(k)}(x) \geq 0$ throughout $[a, b]$,
 (b) $\text{Max}_{a \leq x \leq b} |f(x) - Q_n(x)| \leq 2\lambda \left(\frac{\pi}{4}\right)^{p-k+1} (b-a)^{p+1} \left[k! \prod_{\nu=k}^p (n+1-\nu) \right]^{-1}$.

3. To prove Theorem 1, we begin by quoting the following result of J. Favard [2] and N. Ahiezer and M. Krein [1] which strengthens a previous result of D. Jackson.

THEOREM 2. (Favard, Ahiezer-Krein) Let f (with period 2π) map the reals into the reals, and satisfy for every real x_1, x_2

$$(1) \quad |f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|,$$

λ being a constant. Then for $n = 0, 1, 2, \dots$, there exists a trigonometric polynomial $T_n(x) \equiv \sum_{\nu=0}^n a_\nu^{(n)} \cos \nu x + b_\nu^{(n)} \sin \nu x$ such that $\max_{0 \leq x \leq 2\pi} |f(x) - T_n(x)| \leq \lambda(\pi/2)[1/(n+1)]$.

From Theorem 2 one obtains by the method of [3], pp. 13-14 the following

THEOREM 3. Let f be a real function satisfying (1) throughout $[a, b]$, λ being a constant. Then for $n = 0, 1, 2, \dots$, there exists a

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¹ By degree of a polynomial we mean its exact degree. (The degree of the polynomial 0 is -1).

polynomial $P_n(x)$ of degree $\leq n$ such that

$$\max_{a \leq x \leq b} |f(x) - P_n(x)| \leq \lambda \frac{\pi}{4} \frac{b-a}{n+1}.$$

For future use, we make the following simple observation. (Compare [3], p. 16).

LEMMA. *Let f be a real function, continuous in $[a, b]$ and differentiable in (a, b) . Let n be an integer (≥ 0), $q_{n-1}(x)$ a real polynomial of degree $\leq n-1$, and let ε be such that $|f'(x) - q_{n-1}(x)| \leq \varepsilon$ throughout (a, b) . Then there exists a polynomial $P_n(x)$ of degree $\leq n$ such that*

$$(2) \quad \max_{a \leq x \leq b} |f(x) - P_n(x)| \leq \varepsilon \frac{\pi}{4} \frac{b-a}{n+1}.$$

To prove the lemma, set $r(x) \equiv f(x) - \int_a^x q_{n-1}(t) dt$. Throughout (a, b) , $|r'(x)| \leq \varepsilon$, and therefore, throughout $[a, b]$, $|r(x_2) - r(x_1)| \leq \varepsilon |x_2 - x_1|$. By Theorem 3, there exists a polynomial $\pi_n(x)$ of degree $\leq n$ such that $\max_{a \leq x \leq b} |r(x) - \pi_n(x)| \leq \varepsilon(\pi/4)(b-a)/(n+1)$. Setting $P_n(x) \equiv \pi_n(x) + \int_a^x q_{n-1}(t) dt$, we obtain (2).

From Theorem 3 and the Lemma one gets readily (cf. [3], pp. 16-17) the following

THEOREM 4. *Let f be a real function satisfying throughout $[a, b]$, for some constant integer $p(\geq 0)$ and some constant λ ,*

$$|f^{(p)}(x_2) - f^{(p)}(x_1)| \leq \lambda |x_2 - x_1|.$$

Then for every integer $n(\geq p)$ there exists a polynomial $P_n(x)$ of degree $\leq n$ such that

$$\max_{a \leq x \leq b} |f(x) - P_n(x)| \leq \lambda \left[\frac{\pi}{4} (b-a) \right]^{p+1} \left[\prod_{\nu=0}^p (n+1-\nu) \right]^{-1}.$$

3. Proof of Theorem 1. Let n be an integer $\geq p$. Set $f_n(x) \equiv f^{(k)}(x) + \lambda[(\pi/4)(b-a)]^{p-k+1} [\prod_{\nu=k}^p (n+1-\nu)]^{-1}$. Then throughout $[a, b]$, $|f_n^{(p-k)}(x_2) - f_n^{(p-k)}(x_1)| \leq \lambda |x_2 - x_1|$. By Theorem 4, there exists a real polynomial $P_{n-k}(x)$ of degree $\leq n-k$ such that

$$\max_{a \leq x \leq b} |f_n(x) - P_{n-k}(x)| \leq \lambda \left[\frac{\pi}{4} (b-a) \right]^{p-k+1} \left[\prod_{\nu=k}^p (n+1-\nu) \right]^{-1}.$$

So, throughout $[a, b]$, $P_{n-k}(x) \geq f^{(k)}(x) \geq 0$. Let

$$Q_n(x) \equiv \left[\sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(a)}{\nu!} (x-a)^\nu \right] + \int_a^{t_{k+1}} \int_a^{t_k} \cdots \int_a^{t_2} P_{n-k}(t_1) dt_1 dt_2 \cdots dt_k$$

(t_{k+1} being here and below, x). Then $Q_n(x)$ is a real polynomial of degree $\leq n$, and $Q_n^{(k)}(x) = P_{n-k}(x) \geq 0$ throughout $[a, b]$. Furthermore, throughout that interval, we have

$$f(x) = \left[\sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(a)}{\nu!} (x-a)^\nu \right] + \int_a^{t_{k+1}} \int_a^{t_k} \cdots \int_a^{t_2} f^{(k)}(t_1) dt_1 \cdots dt_k,$$

and therefore

$$\begin{aligned} |f(x) - Q_n(x)| &\leq \int_a^{t_{k+1}} \int_a^{t_k} \cdots \int_a^{t_2} |f^{(k)}(t_1) - P_{n-k}(t_1)| dt_1 \cdots dt_k \\ &\leq 2\lambda \left[\frac{\pi}{4} (b-a) \right]^{p-k+1} \left[\prod_{\nu=k}^p (n+1-\nu) \right]^{-1} \frac{(x-a)^k}{k!} \\ &\leq 2\lambda \left(\frac{\pi}{4} \right)^{p-k+1} (b-a)^{p+1} \left[k! \prod_{\nu=k}^p (n+1-\nu) \right]^{-1}. \end{aligned}$$

4. The following Theorem 5 deals with a somewhat more general situation than that of Theorem 1.

THEOREM 5. *Let k and p be integers, $1 \leq k \leq p$, and let a real function f satisfy throughout $[a, b]$*

$$\begin{aligned} f^{(k)}(x) &\geq 0, \\ |f^{(p)}(x)| &\leq M, \end{aligned}$$

M being a constant. Let $\omega(x)$ be the modulus of continuity of $f^{(p)}$ in $[a, b]$. Then for every integer $n (\geq p)$ there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$\begin{aligned} \text{(a)} \quad &Q_n^{(k)}(x) \geq 0 \text{ throughout } [a, b], \\ \text{(b)} \quad &\max_{a \leq x \leq b} |f(x) - Q_n(x)| \\ &\leq 2 \left(1 + \frac{\pi}{4} \right) \left(\frac{\pi}{4} \right)^{p-k} (b-a)^p \left[k! \prod_{\nu=k}^{p-1} (n+1-\nu) \right]^{-1} \omega \left(\frac{b-a}{n-p+1} \right) \end{aligned}$$

(an ‘‘empty’’ product means always 1).

Theorem 5 is proved by means of the following Theorem 6, in the same way that Theorem 1 was proved by means of Theorem 4.

THEOREM 6. *Let f be a real function having a bounded p th ($p \geq 0$) derivative throughout $[a, b]$. Let $\omega(x)$ be as in Theorem 5. Then for every integer $n (\geq p)$ there exists a polynomial $P_n(x)$ of degree $\leq n$ such that throughout $[a, b]$*

$$|f(x) - P_n(x)| \leq \left(1 + \frac{\pi}{4}\right) \left[\frac{\pi}{4}(b-a)\right]^p \left[\prod_{\nu=0}^{p-1} (n+1-\nu)\right]^{-1} \omega\left(\frac{b-a}{n-p+1}\right).$$

5. Theorem 6 follows from Theorem 3 by Jackson's method ([3], pp. 15-18). For the reader's convenience we hereby prove Theorem 6 in full. We do it by induction on p . Suppose first $p = 0$. Let n be an integer (≥ 0). Let $\phi(x)$ be the function whose graph is obtained by joining successively the points $(\xi_\nu, f(\xi_\nu))$ ($\nu = 0, 1, \dots, n+1$) of the x, y plane, where $\xi_\nu = a + [(b-a)/(n+1)]\nu$. For $\nu = 1, 2, \dots, n+1$ we have $|\phi(\xi_\nu) - \phi(\xi_{\nu-1})| \leq \omega[(b-a)/(n+1)]$. Hence, if $a \leq x_1 < x_2 \leq b$, then

$$\frac{|\phi(x_2) - \phi(x_1)|}{x_2 - x_1} \leq \frac{n+1}{b-a} \omega\left(\frac{b-a}{n+1}\right).$$

By Theorem 3, there exists a polynomial $P_n(x)$ of degree $\leq n$ such that throughout $[a, b]$

$$|\phi(x) - P_n(x)| \leq \frac{n+1}{b-a} \omega\left(\frac{b-a}{n+1}\right) \frac{\pi}{4} \frac{b-a}{n+1} = \frac{\pi}{4} \omega\left(\frac{b-a}{n+1}\right).$$

Clearly, for every $x \in [a, b]$, $|f(x) - \phi(x)| \leq \omega[(b-a)/(n+1)]$. Therefore, throughout $[a, b]$, $|f(x) - P_n(x)| \leq [1 + (\pi/4)]\omega[(b-a)/(n+1)]$. This proves Theorem 6 when $p = 0$. Suppose the theorem was proved for some $p-1$ (≥ 0). We shall prove it for p . Let n be an integer ($\geq p$). By our hypothesis there exists a polynomial $P_{n-1}(x)$ of degree $\leq n-1$ such that throughout $[a, b]$

$$\begin{aligned} & |f'(x) - P_{n-1}(x)| \\ & \leq \left(1 + \frac{\pi}{4}\right) \left[\frac{\pi}{4}(b-a)\right]^{p-1} \left[\prod_{\nu=1}^{p-1} (n+1-\nu)\right]^{-1} \omega\left(\frac{b-a}{n-p+1}\right). \end{aligned}$$

By the lemma, there exists a polynomial $P_n(x)$ of degree $\leq n$, such that

$$\begin{aligned} & \max_{a \leq x \leq b} |f(x) - P_n(x)| \\ & \leq \left(1 + \frac{\pi}{4}\right) \left[\frac{\pi}{4}(b-a)\right]^p \left[\prod_{\nu=0}^{p-1} (n+1-\nu)\right]^{-1} \omega\left(\frac{b-a}{n-p+1}\right). \end{aligned}$$

This completes the proof.

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